Math 210, **Solutions for the Second Hour Exam**, Fall 2005

1. (15 points) Let $F(x,y,z) = 3x^2 - 4y^2 + z^2$. Find the equation of the plane tangent to the level surface $F(x,y,z) = 12$ at the point $(-3,2,1)$.

The gradient is $\nabla F = \langle 6x, -8y, 2z\rangle$, *at the point* $(-3,2,1)$ *it is* $\nabla F = \langle -18, -16, 2\rangle$. *The gradient is normal to the level surface, therefore, the folowing is an equation of the tangent plane at the given point:*

$$-18(x+3) - 16(y-2) + 2(z-1) = 0,$$

or after simplification

$$-9x - 8y + z - 12 = 0.$$

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2. (20 points) Let $f(x,y,z) = x^2 - xy + xyz$.

(a) Find the rate of change of f at the point $(1,1,1)$ in the direction of the unit vector $\mathbf{v} = (\frac{1}{\sqrt{3}})\langle 1,-1,1\rangle$.

Compute the gradient: $\nabla f(x,y,z) = \langle 2x - y + yz, -x + xz, xy\rangle$; $\nabla f(1,1,1) = \langle 2,0,1\rangle$. *The rate of change equals* $D_{\boldsymbol{v}} f(1,1,1) = \nabla f(1,1,1)\cdot \boldsymbol{v} = \langle 1,-1,1\rangle \cdot \langle 2,0,1\rangle = (\frac{1}{\sqrt{3}})\sqrt{3} = \sqrt{3}$.

(b) Find the direction in which f increases most rapidly at the point $(1,1,1)$, and find the maximum rate of change of f at that point.

The direction is the direction of the gradient. The maximum rate equals $|\nabla f(1,1,1)| = \sqrt{5}$.

(c) Suppose that the function f gives the temperature at each point in space. A bug is flying around, with position function $\mathbf{p}(t) = \langle t, t^2, t^3\rangle$, carrying a thermometer in his pocket. Use the chain rule to find the rate of change of his temperature *with respect to time* t at the moment when his position is $(1,1,1)$.

We need $\frac{\partial}{\partial t} f(p(t))$ *at the point* $t = 1$, *since* $(1,1,1) = p(1)$. *By the chain rule,*

$$\frac{\partial}{\partial t} f(p(t)) = \nabla f(p(1)) \cdot \frac{\partial}{\partial t} p(1) = \langle 2,0,1\rangle \cdot \langle 1, 2t, 3t^2\rangle|_{t=1} = 5.$$

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3. (15 points) Find the critical points of the function $f(x,y) = x^4 + y^4 - 4xy + 2$ and classify them as maximum, minimum or saddle points.

Critical points are solutions to the system

$$\begin{cases} f_x &= 4x^3 - 4y = 0 \\ f_y &= 4y^3 - 4x = 0 \end{cases}$$

This gives us $y = x^3$ *and* $x = y^3 = x^9$. *The equation* $x - x^9 = x(1-x^8) = x(1-x^4)(1+x^4) = x(1-x)(1+x)(1+x^2)(1+x^4) = 0$ *is satisfied by* $x = -1, 0, 1$. *Therefore, the solutions are the points* $(0,0), (1,1), (-1,1)$.

The second derivative test: $D = f_{xx} f_{yy} - f_{xy}^2 = 12x^2 12y^2 - 4^2 = 144x^2y^2 - 16$.

For $(0,0)$ *the value of* D *is* $-16 < 0$, *hence, it is a saddlepoint. Both* $(1,1)$ *and* $(-1,-1)$ *are minima as* $D > 0$ *and* $f_{xx} > 0$.

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4. (15 points) Let $f(x, y, z) = 1 + x^2 + y^3 - z^5$. Suppose you were using the method of Lagrange multipliers to find the maximum value of the function f on the ellipsoid $x^2 + 2y^2 + 3z^2 = 2$.

(a) Write down the system of 4 algebraic equation in 4 unknowns that you would need to solve. **Do not try to solve these equations.**

$$\begin{cases} 2x &= 2x\lambda \\ 3y^2 &= 4y\lambda \\ -5z^4 &= 6z\lambda \\ x^2 + 2y^2 + 3z^2 &= 2 \end{cases}$$

(b) State how you would find the maximum value, given the list of solutions to the equations in part (a).

Take the maximum of the values of $f(x, y, z)$ for all the solutions (x, y, z, λ) of the system.

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5. (20 points) Change the order of integration to compute the iterated integral

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dxdy.$$

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dxdy = \int_0^1 \int_0^{x^2} e^{x^3}\, dydx = \int_0^1 x^2 e^{x^3}\, dx = (1/3)(e-1)$$

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6. (15 points) Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$.

Integrating over the region D inside the circle $4 = x^2 + y^2$ in polar coordinates gives:

$$\int\int_D \sqrt{1 + (2x)^2 + (2y)^2} dA = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta =$$

...substitute $u = 1 + 4r^2$...

$$= \int_0^{2\pi} \int_1^{17} (1/8)\sqrt{u}\, du\, d\theta = (\pi/6)\left[u^{3/2}\right]_1^{17} = \frac{\pi}{6}(17^{3/2} - 1).$$