Exam (1) Math 310 Anton Leykin Friday 17 Oct 2003

NAME:

question	1	2	3	4	5	6	total
points							
maximum	20	15	15	15	15	20	100

1. A system of linear equations is given by the following augmented matrix:

$$\left(\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 5 & -3 & 0 \\ 1 & 2 & 1 & 1 & 4 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 7 & 2 \end{array}\right).$$

Find the reduced row echelon form:

solution

$$= \left(\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & 1 & -1 & 6 & 3 \\ 0 & 0 & 0 & 1 & 1 & 7 & 2 \end{array}\right)$$

row2 := row2 − row1

$$= \left(\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 7 & 2 \end{array}\right).$$

row2 := row2 − row3

Specify the lead variables: x_1, x_3, x_4, the free variables: x_2, x_5, x_6

Write the solution set using parameters :

solution

$$\mathbf{x} = \begin{pmatrix} -2\alpha - 5\beta + 3\gamma \\ \alpha \\ 2\beta + \gamma + 1 \\ -\beta - 7\gamma + 2 \\ \beta \\ \gamma \end{pmatrix}$$

2. Consider the following chemical reaction: x_1 molecules of benzene burn with the help of x_2 molecules of oxygen resulting in x_3 molecules of carbon dioxide and x_4 molecules of water, i.e.

$$x_1 C_6 H_6 + x_2 O_2 \rightarrow x_3 CO_2 + x_4 H_2 O$$

Balance the equation above by solving a system of linear equation.

✓ solution

For each chemical element balance the left and the right hand side:
C: $6x_1 = x_3$
H: $6x_1 = 2x_4$
O: $2x_2 = 2x_3 + x_4$
This system of homogeneous linear equations can be presented by the augmented matrix

$$\left(\begin{array}{cccc|c} 6 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right),$$

which can be reduced to

$$\left(\begin{array}{cccc|c} 3 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -5 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right).$$

Therefore the solution of the system is $\mathbf{x} = (\frac{1}{3}\alpha, \frac{5}{2}\alpha, 2\alpha, \alpha)^T$, which for $\alpha = 6$ equals $(2, 15, 12, 6)^T$.

3. Compute the determinant of $A = \begin{pmatrix} 4 & 0 & 4 & 3 \\ 0 & 0 & 2 & 2 \\ 4 & 3 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

✓ solution

One of the many possible ways to solve this problem is to make use of row operations:

$$\det A = \begin{vmatrix} 4 & 0 & 4 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

row3 := row3 − row1

$$= -\begin{vmatrix} 4 & 0 & 4 & 3 \\ 0 & 3 & -3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

swap row2 and row3, this changes the sign of the determinant

$$= -4 \cdot 3 \cdot 2 \cdot 1 = -24$$

the determinant of a triangular matrix equals the product of the diagonal elements

4. Consider the matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$.

   (a) Give the adjoint of A:

   solution

$$\operatorname{adj}(A) = \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}.$$

   (b) Give the inverse of A:

   solution

$$A^{-1} = \frac{1}{\det \operatorname{adj}(A)} A = \begin{pmatrix} 1/2 & -1 \\ -1/2 & 2 \end{pmatrix}.$$

5. Consider $V = \operatorname{Span}(1 + x - x^2,\ 1 + x^2,\ 2 + x)$, a subspace of P_3.

   (a) Find a basis of V:

   solution

   Since $2 + x = (1 + x - x^2) + (1 + x^2)$, the subspace V is spanned by $\{1 + x - x^2, 1 + x^2\}$. This set of two elements is linearly independent and, therefore, is a basis for V.

   (b) What is the dimension of V?

   solution

   $\dim V = 2$ by definition, since the size of the basis that we found is 2.

6. Consider bases $U = [\mathbf{u}_1, \mathbf{u}_2]$ and $V = [\mathbf{v}_1, \mathbf{v}_2]$ with the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \quad \text{and } \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(a) Compute the transition matrix for the basis change from U to V.

solution

Solving two vector equations

$$\begin{aligned} \mathbf{u}_1 &= s_{11}\mathbf{v}_1 + s_{21}\mathbf{v}_2 \\ \mathbf{u}_2 &= s_{12}\mathbf{v}_1 + s_{22}\mathbf{v}_2 \end{aligned}$$

produces the entries for the transition matrix $S = \begin{pmatrix} 0 & 2 \\ 1 & -2 \end{pmatrix}$

(b) What are the coordinates of $\mathbf{x} = 2\mathbf{u}_1 + 3\mathbf{u}_2$ with respect to the basis V?

solution

The coordinates of $\mathbf{x}$ with respect to U are $(2,3)^T$. The new coordinates with respect to V are equal to

$$S\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$$

(c) What are the coordinates of $\mathbf{y} = 2\mathbf{v}_1 + 3\mathbf{v}_2$ with respect to the basis U?

solution

First, to get the transition matrix from V to U compute the inverse of S:

$$S^{-1} = \begin{pmatrix} 1 & 1 \\ 1/2 & 0 \end{pmatrix}.$$

The coordinates of $\mathbf{y}$ with respect to V are $(2,3)^T$. The new coordinates with respect to U are equal to

$$S^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$