1. In order to balance the following chemical reaction

$$x_1C_{10}H_{16} + x_2Cl_2 \rightarrow x_3C + x_4HCl,$$

set up a system of linear equations for the variables x_i.

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(a) Write the augmented matrix representing this linear system.

✓ solution

$$\left(\begin{array}{cccc|c} 10 & 0 & -1 & 0 & 0 \\ 16 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right)$$

(b) Find the solution set of the system.

✓ solution

$$\mathbf{x} = \alpha \begin{pmatrix} 1/16 \\ 1/2 \\ 10/16 \\ 1 \end{pmatrix}$$

(c) Specialize the solution so that the values of the variables x_i are positive integers.

✓ solution

For $\alpha = 16$ the solution is $\mathbf{x} = \begin{pmatrix} 1 \\ 8 \\ 10 \\ 16 \end{pmatrix}$.

2. Given a system of linear equations

$$\begin{array}{rcrcl} x & + & y & = & c_1 \\ 2x & + & 3y & = & c_2 \\ x & - & y & = & c_3 \end{array}$$

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(a) Reduce the system to row-echelon form.

✓ solution

$$\left(\begin{array}{cc|l} 1 & 1 & c_1 \\ 0 & 1 & c_2 - 2c_1 \\ 0 & 0 & -5c_1 + 2c_2 + c_3 \end{array}\right)$$

(b) For what values does the system have a solution? Explain.

✓ solution

Only if $-5c_1 + 2c_2 + c_3 = 0$ the system is consistent.

(c) Find the best approximate solution for $c_1 = 1$, $c_2 = 2$, $c_3 = 1$.

✓ solution

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(Note: since the given values satisfy the equation in (b), the best approximate solution is the exact solution)

3. Find a basis of the subspace of $\mathbb{R}^4$ of vectors orthogonal to $(1, 3, -1, 0)^T$ and $(1, 3, 0, 1)^T$.

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✓ solution

$$\left[\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right]$$

4. Given a matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{pmatrix},$$

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Find the eigenvalues of A.

✓ solution

$\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$

5. Consider $\mathbf{p} = (3, 2, 1)$ and the plane $x + 2y + 2z = 0$.

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(a) Find a unit normal vector to the plane.

✓ solution

$$\mathbf{n} = \tfrac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

(b) Find the distance from the point $\mathbf{p}$ to the plane.

✓ solution

$d = 3$

(c) Find the projection of $\mathbf{p}$ onto the plane.

✓ solution

$$\mathbf{p} - d\mathbf{n} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

6. Find the symmetric matrix representing the quadratic form

$$q(x, y) = x^2 - 3y^2 + 2z^2 + 2xy - 2xz + 4yz.$$

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$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & 2 \end{pmatrix}$$

✓ solution

7. Let $L_1 : P_3 \to P_3$ and $L_2 : P_3 \to P_3$ be the linear operators defined by

$$L_1(p) = p'' + 2p', \; L_2(p) = p''(0) + 2p'(0),$$

where $p(x) \in P_3$.

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(a) Find the matrix representing L_1 in the standard basis $[1, x, x^2]$.

✓ solution

$$\begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Find the matrix representing L_2 in the standard basis $[1, x, x^2]$.

✓ solution

$$\begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

8. Use Gram-Schmidt orthogonalization to find an orthonormal basis for the space spanned by the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ 3 \\ -2 \\ 1 \end{pmatrix}.$$

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✓ solution

$$\left[\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right]$$

9. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

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(a) Give all eigenvalues and corresponding eigenvectors of A.

✓ solution

$$\lambda_1 = 1, \; \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_1 = 2,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_1 = -1,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

(b) Give the general form of the solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$.

✓ solution

$\mathbf{x}(t) = c_1 e^t \mathbf{v}_1 + c_2 e^{2t} \mathbf{v}_2 + c_3 e^{-t} \mathbf{v}_3$

(c) If $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, what is then the solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$?

✓ solution

$c_1 = c_2 = 1,\ c_3 = -1$

$$\mathbf{x} = \begin{pmatrix} e^t + e^{2t} - e^{-t} \\ e^{2t} \\ -2e^{-t} \end{pmatrix}$$