Sample problems for the final exam — (3 pages!) — Math 310

1. Consider bases $U = [\mathbf{u}_1, \mathbf{u}_2]$ and $V = [\mathbf{v}_1, \mathbf{v}_2]$ with the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \text{ and } \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

   (a) What are the transition matrices A and B for the basis change to the standard basis $[\mathbf{e}_1, \mathbf{e}_2]$ and from U and V respectively?

   (b) Use A and B to find the transition matrix C for the basis change from U to V.

2. Consider $V = \text{Span}(1 + \cos x - x,\ 1 + \cos x,\ x)$, a subspace of $C(-\infty, \infty)$.

   (a) Find a basis of V:

   (b) What is the dimension of V?

3. Consider the following chemical reaction: x_1 molecules of benzene burn with the help of x_2 molecules of oxygen resulting in x_3 molecules of carbon dioxide and x_4 molecules of water, i.e.

$$x_1 C_6H_6 + x_2 O_2 \to x_3 CO_2 + x_4 H_2O$$

   Balance the equation above by solving a system of linear equation.

4. Find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix}$$

5. Given a system of linear equations

$$\begin{array}{rcrcl} x & + & y & = & c_1 \\ x & + & 2y & = & c_2 \\ x & - & y & = & c_3 \end{array}$$

   (a) reduce the system to row-echelon form. For what values does the system have a solution?

   (b) find the best approximate solution for $c_1 = 1$, $c_2 = 2$, $c_3 = 1$.

   (c) find the best approximate solution for $c_1 = 2$, $c_2 = 3$, $c_3 = 0$.

6. The subspace V of $\mathbb{R}^3$ is spanned by

$$\mathbf{u}_1 = (2, 0, 1)^T,\ \mathbf{u}_2 = (1, 2, 3)^T,\ \mathbf{u}_3 = (5, 2, 5)^T.$$

   Find a basis for its orthogonal complement $V^\perp$.

7. Let $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ be the standard basis of $\mathbb{R}^3$ and

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

   The linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$L(\mathbf{v}_1) = \mathbf{v}_1,\ L(\mathbf{v}_2) = 2\mathbf{v}_2,\ L(\mathbf{v}_3) = 3\mathbf{v}_3.$$

(a) Find the coordinates of $\mathbf{e}_1$ with respect to the basis $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. Express $L(\mathbf{e}_1)$ in terms of the standard basis.

(b) What is the first column of the matrix representing L with respect to the standard basis?

(c) What is the matrix representing L with respect to the basis $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$?

8. Let $L : P_3 \to P_3$ be the linear operator defined by

$$L(p) = p' + p$$

(a) Find the matrix A representing L in the standard basis $[1, x, x^2]$.

(b) Find the matrix B representing L in the basis $[1, x+1, x^2+1]$.

(c) Are A and B similar matrices? Explain.

(d) Find a matrix T such that $B = T^{-1}AT$.

9. Given two vectors $\mathbf{u} = (1, 2, 3)^T$ and $\mathbf{v} = (3, -1, 2)^T$,

(a) Find $||\mathbf{u}||$, $||\mathbf{v}||$ and $\langle \mathbf{u}, \mathbf{v} \rangle$.

(b) Find the vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

(c) What is the angle between these vectors?

10. Consider $\mathbf{p} = (3, 3, 3)$ and the plane $x - y + 3z = 0$.

(a) Find a unit normal vector to the plane.

(b) Find the distance from the point $\mathbf{p}$ to the plane.

(c) Find the projection of $\mathbf{p}$ onto the plane.

11. Find the equation of the line $y = c_0 + c_1 x$ which is a least squares best fit to the points $(1, 2), (2, 1), (4, -3), (5, -4)$.

12. Find the equation of the best quadratic fit, $y = c_0 + c_1 x + c_2 x^2$, to the points $(1, 2), (2, 1), (4, -3), (5, -4)$.

13. Find an orthonormal basis of the subspace of $\mathbb{R}^3$ of vectors orthogonal to $\mathbf{v} = (4, 3, -3)^T$.

14. Let V be the subspace of $\mathbb{R}^3$ spanned by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

(a) Find a basis for the orthogonal space $V^\perp$.

(b) What are the dimensions of V and $V^\perp$?

(c) Find a basis for V.

(d) Use the Gram-Schmidt method to give an orthonormal basis for V.

(e) Use your answer to (a) to extend the orthonormal basis for V found in (d) to the orthonormal basis of $\mathbb{R}^3$.

15. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 4y \\ -2x - 3y \end{pmatrix}.$$

Find the matrix representing L in the new basis

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

16. Find the eigenvalues and the corresponding eigenvectors for

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

17. The functions $x(t)$ and $y(t)$ satisfy the system of linear differential equations

$$\begin{array}{rcrcr} x'(t) & = & 3x & + & 4y \\ y'(t) & = & -2x & - & 3y \end{array}$$

and the initial conditions $x(0) = 1$, $y(0) = 2$. Find these functions.

18. The functions $x(t)$ and $y(t)$ satisfy the system of linear differential equations

$$\begin{array}{rcrcr} x'(t) & = & 3x & + & 5y \\ y'(t) & = & 2x & + & 3y \end{array}$$

and the initial conditions $x(0) = 1$, $y(0) = 2$. Find these functions.

19. Find the eigenvalues and the corresponding eigenvectors for

$$A = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}$$

20. The eigenvalues of the symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \end{pmatrix}$$

are $\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 9$.

Find an orthonormal basis consisting of eigenvectors of the matrix A.

21. Find the symmetric matrix representing the quadratic form

$$q(x, y) = x^2 + 2y^2 + 2z^2 + xy - 2xz + 3yz.$$

22. Given the matrix

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 9 \end{pmatrix}$$

(a) Show that A is positive definite.

(b) Find a lower triangular matrix L and a diagonal matrix D such that $A = L^T DL$.

(c) Find the Cholesky decomposition of A, i.e. find a lower triangular matrix L such that $A = L^T L$.