MCS 471, **Second Hour Exam**, 3 Apr 2006, **NAME:**

Calculators are allowed; write computational results with precision of 5 significant digits, unless stated otherwise.

1. (20 points)

(a) Determine the missing values in the table of divided differences:

| | | | | |
|---|---|---|---|---|
| $x_0 = 0$ | $f[x_0] = 1$ | $f[x_0, x_1] = 2$ | $f[x_0, x_1, x_2] = \mathbf{-1}$ | $f[x_0, x_1, x_2, x_3] = \frac{1}{3}$ |
| $x_1 = 1$ | $f[x_1] = 3$ | $f[x_1, x_2] = \mathbf{0}$ | $f[x_0, x_1, x_2] = 0$ | |
| $x_2 = 2$ | $f[x_2] = 3$ | $f[x_2, x_3] = 0$ | | |
| $x_3 = 3$ | $f[x_3] = \mathbf{3}$ | | | |

(b) Use a part of the table to construct the polynomial $P(x)$ interpolating $f(x)$ at **three** points $x = 0, 1, 2$.

$f(x) = 1 + (x - 0) * 2 + (x - 0)(x - 1) * (-1) = -x^2 + 3x + 1$

(c) Now construct Lagrange interpolating polynomial $Q(x)$ using the same three points.

$f(x) = \frac{(x-1)(x-2)}{2)} + 3\frac{x(x-2)}{-1} + 3\frac{(x-1)x}{2} = -x^2 + 3x + 1$

(d) Is $P(x) = Q(x)$? Do the two methods always lead to the same result? Explain.

*Yes. Yes, always; the polynomial of degree $d$ interpolating $d + 1$ points is unique.*

2. (20 points) The table below gives information about the tolls charged at the main toll booth at the end of New Jersey Turnpike. The distance $x_i$ is the number of miles from the $i$-th exit to the main toll booth. The corresponding toll charge $= y_i$.

| $i$ | 1 | 2 | 3 |
|---|---|---|---|
| $x_i$ | 12 | 25 | 33 |
| $f_i$ | 0.45 | 0.70 | 0.95 |

Assuming the toll charge should depend linearly on the distance, what should it be for the exit 4, which is 43 miles away from the main toll booth? (Round the result of the linear regression to the nearest cent.)

*The least-squares approximation technique leads to the following results:*

*approximating function:* $y \approx 0.023368 + 0.15475x$

*answer:* $y(43) \approx 1.16$

3. (15 points) (a) For the function $f(x) = \ln x$ approximate the derivative $f'(x)$ at the point $x = 1$ by three methods: forward, backward, and central differences with the step $h = 0.1$.

*forward:* 0.95310

*backward:* 1.0536

*central:* 1.00335

(b) What method gives the best results?

*central differences*

(c) The error $E(h) =$ (approximation at $h$) $- f'(1)$ is of order $O(h^d)$. What is $d$ for each of the three methods?

*backward/forward:* $d = 1$; *central:* $d = 2$

4. (25 points) The function $f(x)$ is sampled on the interval $[0, 1]$:

| $x$ | 0. | .125 | .250 | .375 | .500 | .625 | .750 | .875 | 1. |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0. | .125 | .247 | .366 | .479 | .585 | .682 | .768 | .841 |

(a) Use this data and the composite trapezoidal method to compute the approximation $I(h)$ of $\int_0^1 f(x)\,dx$ for the step sizes $h = 0.5, 0.25, 0.125$.

$I(0.5) = 0.44975$

$I(0.25) = 0.45712$

$I(0.125) = 0.45906$

(b) The error of the method $E(h) = \int_0^1 f(x)\,dx - I(h)$ is of order $O(h^d)$. What is $d$?

*2*

(c) Using the results of (a) perform Romberg integration.

| $h$ | $I_2 = I(h)$ | $I_4 =$ *4th order* | $I_6 =$ *6th order* |
|---|---|---|---|
| 0.5 | 0.44975 | 0.45958 | 0.45971 |
| 0.25 | 0.45712 | 0.45970 | |
| 0.125 | 0.45906 | | |

*Given a method $I_n$ of order $O(h^n)$ a method of order $O(h^{n+2})$ is produced by extrapolation:*

$$I_{n+2} = a + \frac{1}{2^n - 1}(a - b),$$

*where $a$ is a more precise and $b$ is a less precise approximations given by the method $I_n$.*

(d) What is the order of error of (c)? Explain.

*6*

5. (20 points) Let $f(x)$ be a function on $[-1, 1]$. Consider Gaussian quadrature formula for two points $t_1$, $t_2$ with the weights $w_1, w_2$:

$$\int_{-1}^{1} f(x)\,dx = w_1 f(t_1) + w_2 f(t_2)$$

(a) What is the maximal $d$, such that the formula is exact for polynomials of degree at most $d$? Explain why.

$d = 3$, *since a polynomial of degree 3 has 4 coefficients, which matches the number of parameters in the quadrature formula.*

(b) Assuming it is exact for $f(x) = 1$, $f(x) = x$, and $f(x) = x^2$, $f(x) = x^3$ write out the system of equations for $t_1, t_2, w_1, w_2$. (Do not solve!)

$$\begin{aligned}
\int_{-1}^{1} 1\,dx = 2 &= w_1 + w_2 \\
\int_{-1}^{1} x\,dx = 0 &= w_1 t_1 + w_2 t_2 \\
\int_{-1}^{1} x^2\,dx = \frac{2}{3} &= w_1 t_1^2 + w_2 t_2^2 \\
\int_{-1}^{1} x^3\,dx = 0 &= w_1 t_1^3 + w_2 t_2^3
\end{aligned}$$

(c) The solution to (b) is

$$w_1 = w_2 = 1, t_1 = -0.57735, t_2 = 0.57735$$

Compute the approximate value of $\int_{-1}^{1} e^x \, dx$ using the quadrature formula.

$$\int_{-1}^{1} e^x \, dx \approx 2.3427$$

(d) What is the error of computation in (c)?
*The value that we get analytically is* 2.3504.

$$\textit{absolute error} = 2.3504 - 2.3427 \approx 0.0077$$