Associativity of addition of natural numbers from Peano axioms

We want to prove the following statement:

$$a, b, c \in \mathbf{Z}^+ \Rightarrow (a+b)+c = a+(b+c) \quad (1)$$

We shall use the Peano definition of the set of natural numbers i.e. defined as a set $\mathbf{N}$ with a distinguished element $1 \in \mathbf{N}$ and an injective map $s : \mathbf{N} \to \mathbf{N}$ such that 1 does not belong to $s(\mathbf{N})$ and such that the following induction axiom (ind) is satisfied:

(ind) If $A \subseteq \mathbf{N}$ such that $1 \in \mathbf{N}$ and $a \in A \Rightarrow s(a) \in A$ then $A = \mathbf{N}$.

In such a set $\mathbf{N}$ one defines addition of $b \in \mathbf{N}$ to $a \in \mathbf{N}$ so that the following two properties are satisfied:

(i) $a + 1 = s(a)$
(ii) If $a + b$ is defined, then $a + (b + 1)$ is defined as $s(a + b)$.

Note that the set B of $b \in \mathbf{N}$ for which $a + b$ is defined contains 1 and by (ii) if $b \in B$ then $b + 1 \in B$. Therefore, by (ind) $B = \mathbf{N}$ and hence $a + b$ is defined for all $a, b \in \mathbf{N}$.

We shall prove the identity (1) using induction over c i.e. we show that the set C *defined* as subset of $\mathbf{N}$ consisting of $c \in \mathbf{N}$ for which (1) is true is equal to the set $\mathbf{N}$ (this is the same as to say that (1) is true for any $c \in \mathbf{N}$).

Claim 1: $1 \in C$ i.e. associativity takes place for $c = 1$ i.e.

$$(a+b)+1 = a+(b+1)$$

Indeed, LHS by part (i) of definition of addition is $s(a+b)$ and RHS is $s(a+b)$ by part (ii). Hence both sides are the values of function s at $a + b$ and hence are equal (by definition of a function).

Claim 2: if $(a + b) + c = a + (b + c)$ then $(a + b) + (c + 1) = a + (b + (c + 1))$.

. Indeed

$$(a+b)+(c+1) = s((a+b)+c) \quad \text{(by (ii) in definition of addition)}$$

$$= s(a+(b+c)) \quad \text{(by assumption of Claim 2)}$$

$$= a+((b+c)+1) \quad \text{(by (ii) of definition of addition)}$$

$$= a+(b+(c+1)) \quad \text{(by Claim 1 i.e. associativity of addition when last summand equals 1)}$$

QED.