ALBANESE VARIETIES OF ABELIAN COVERS

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To the memory of Shreeram Abhyankar.

Abstract. We show that the Albanese variety of an abelian cover of the projective plane is isogenous to a product of isogeny components of abelian varieties associated with singularities of the ramification locus provided certain conditions are met. In particular Albanese varieties of abelian covers of $\mathbb{P}^2$ ramified over arrangements of lines and uniformized by the unit ball in $\mathbb{C}^2$ are isogenous to a product of Jacobians of Fermat curves. Periodicity of the sequence of (semi-abelian) Albanese varieties of unramified cyclic covers of complements to a plane singular curve is shown.

1. Introduction

Albanese varieties of cyclic branched covers of $\mathbb{P}^2$ ramified over singular curves are rather special. If singularities of the ramification locus are no worse than ordinary nodes and cusps then (cf. [8]) the Albanese variety of a cyclic cover is isogenous to a product of elliptic curves E_0 with j-invariant zero. More generally, in [26] it was shown that the Albanese variety of a cyclic cover with ramification locus having *arbitrary* singularities is isogenous to a product of isogeny components of local Albanese varieties i.e. the abelian varieties canonically associated with the local singularities of the ramification locus. In particular, Albanese varieties of cyclic covers are isogenous to a product of Jacobians of curves.

In this paper we shall describe Albanese varieties of *abelian* covers of $\mathbb{P}^2$. The main result is that the class of abelian varieties which are Albanese varieties of ramified abelian covers (with possible non reduced ramification locus) is also built from the isogeny components of local Albanese varieties, provided some conditions on fundamental group of the complement to ramification locus are met (cf. 2.2). Also, in abelian case one needs to allow local Albanese varieties of non reduced singularities having the same reduced structure as the germs of the singularities of ramification locus of the abelian cover.

One of the steps in our proof of this result involves a description of Jacobians of abelian covers of projective line having an independent interest. In this case we show that all isogeny components of Jacobians of abelian covers of $\mathbb{P}^1$ with arbitrary ramification are components of Jacobians of explicitly described cyclic covers. If the abelian cover is ramified only at three points and has the Galois group isomorphic to $\mathbb{Z}_n^2$ then it is biholomorphic to Fermat curve $x^n + y^n = z^n$. In this case, such results are going back to works of Gross, Rohrlich and Coleman (cf. [15],[9]) where isogeny components of Jacobians of Fermat curves were studied.

The proof of isogeny decomposition of abelian covers is constructive and, as an application, we obtain the isogeny classes of Albanese varieties of the abelian

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covers of $\mathbb{P}^2$, discovered by Hirzebruch (cf.[20]), having the unit ball as the universal cover. These Albanese varieties are isogenous to products of Jacobians of Fermat curves described explicitly. Another interesting abelian cover of $\mathbb{P}^2$ ramified over an arrangement of lines is the Fano surface of lines on the Fermat cubic threefold. The Albanese variety of this Fano surface (according to [7], this abelian variety is also the intermediate Jacobian of the Fermat cubic threefold) is isogenous to the product of five copies of E_0. This result was recently independently obtained in [29] and [6] (in [29] the *isomorphism* class of Albanese variety of Fano surfaces was found).

Another application considers the behavior of the Albanese varieties in the towers of cyclic and abelian covers. It is known for some time that Betti and Hodge numbers of cyclic (resp. abelian) covers are periodic (resp. polynomially periodic cf. [18]). It turns out that the sequence of isogeny classes of Albanese varieties of cyclic covers with given ramification locus is periodic but periodicity fails in abelian towers. Moreover, we show similar periodicity for sequence of semi-abelian varieties which are Albanese varieties of *quasi-projective surfaces* which are unramified covers of $\mathbb{P}^2 \setminus \mathcal{C}$.

The content of the paper is the following. In section 2 we recall several key definitions and results used later, in particular, the characteristic varieties, Albanese varieties in quasi-projective and local cases. Section 3 considers Jacobians of abelian covers of $\mathbb{P}^1$, and the main result is that isogeny components of such Jacobians are all the isogeny components of Jacobians of cyclic covers of $\mathbb{P}^1$. This section also contains calculation of multiplicities of characters of representation of the covering group on the space of holomorphic 1-forms. In the case of cyclic covers, such multiplicities were calculated in [2]. The main result of the paper, showing that Albanese varieties of abelian covers are isogenous to a product of isogeny components of local Albanese varieties of singularities, is proven in section 4. The case of covers ramified over arrangements of lines is considered in section 5. This includes, the already mentioned case of Fano surface (of lines) on the Fermat cubic threefold. The last section contains applications to calculation of Mordell-Weil ranks of isotrivial abelian varieties and periodicity properties of Albanese varieties in towers of abelian covers. Note that the prime field of all varieties, maps between them and function fields considered in this paper is $\mathbb{C}$.

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2. Preliminaries

2.1. **Characteristic varieties.** We recall the construction of invariants of the fundamental group of the complement playing the key role in description of the Albanese varieties of abelian covers. We follow [24] (cf. also [3]).

Let X be a quasi-projective smooth manifold such that $H_1(X, \mathbb{Z}) \neq 0$. The exact sequence

$$0 \to \pi_1(X)'/\pi_1(X)'' \to \pi_1(X)/\pi_1(X)'' \to \pi_1(X)/\pi_1(X)' \to 0 \tag{1}$$

(where G' denotes the commutator subgroup of a group G) can be used to define the action of $H_1(X, \mathbb{Z}) = \pi_1(X)/\pi_1(X)'$ on the left term in (1). This action allows to view $C(X) = \pi_1(X)'/\pi_1(X)'' \otimes \mathbb{C}$ as a $\mathbb{C}[H_1(X, \mathbb{Z})]$-module. Recall that the support of a module M over a commutative noetherian ring R is the sub-variety

$\mathrm{Supp}(M) \subset \mathrm{Spec}(R)$ consisting of the prime ideals $\wp$ for which the localization $M_\wp \neq 0$.

Definition 2.1. *The characteristic variety* $V_i(X)$ *is (the reduced) sub-variety of* $\mathrm{Spec}\mathbb{C}[H_1(X)]$ *which is the support* $\mathrm{Supp}(\Lambda^i(C(X)))$ *of the* i*-th exterior power of the module* $C(X)$*. The depth of* $\chi \in \mathrm{Spec}\mathbb{C}[H_1(X)]$ *is an integer given by*

$$d(\chi) = \{\max i | \chi \in V_i(X)\} \tag{2}$$

Using the canonical identification of $\mathrm{Spec}\mathbb{C}[H_1(X,\mathbb{Z})]$ and the torus of characters $\mathrm{Char}(\pi_1(X))$ one can interpret points of characteristic varieties as rank one local systems on X. This interpretation leads to the following alternative description of $V_i(X)$ (cf. [19], [24]))

$$V_i(X) \setminus 1 = \{\chi \in \mathrm{Char}\pi_1(X)|, \chi \neq 1, dim H^1(X,\chi) \geq i\} \tag{3}$$

It follows from [1] that if a smooth projective closure $\bar{X}$ of X satisfies[1] $H_1(\bar{X},\mathbb{Q}) = 0$ then each $V_i(X)$ is a finite union of translated subgroups of the affine torus $\mathrm{Char}(\pi_1(X))$ i.e. a finite union of subset of the form $\psi \cdot H$ where H is a subgroup of $\mathrm{Char}(\pi_1(X))$ and ψ is a character of $\pi_1(X)$. Moreover, such a character ψ can be chosen to have a finite order (cf. [25]). It also follows from [1] that each irreducible component $\mathcal{V}$ of characteristic variety having a dimension greater than one determines a holomorphic map: $\nu : X \to P$ where P is a hyperbolic curve (i.e. a curve with negative euler characteristic).

In the case when $X = \mathbb{P}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is a plane curve with arbitrary singularities, P is biholomorphic to $\mathbb{P}^1 \setminus D$ where D is a finite set.

Returning to the case when X is smooth quasi-projective, a component corresponding to a map $\nu : X \to P$ consists of the characters $\nu^*(\chi)$ where $\chi \in \mathrm{Char}(\pi_1(P))$ (here, for a map $\phi : X \to Y$ between topological spaces X, Y, we denote by ϕ^* the induced map $\mathrm{Char}(H_1(Y,\mathbb{Z})) = H^1(Y,\mathbb{C}^*) \to H^1(X,\mathbb{C}^*) = \mathrm{Char}(H_1(X,\mathbb{Z})))$. The map ν also induces homomorphisms $h^i(\nu*) : H^i(P,\chi) \to H^i(X,\nu^*(\chi))$ and $h_i(\nu*) : H_i(P,\chi) \to H_i(X,\nu^*(\chi))$. The maps $h^1(\nu^*)$ and $h_1(\nu^*)$ are isomorphisms for all but finitely many $\chi \in \mathrm{Char}(\pi_1(P))$ (cf.[1, Proof of Prop.1.7]).

At the intersection of components the depth of characters is bigger then the depth of generic character in either of the components i.e. the depth is jumping. More precisely, if $\chi \in V_k(X) \cap V_l(X)$ where both $V_k(X)$ and $V_l(X)$ have positive dimensions then the depth of χ is at least $k + l$ (cf. [4]). More precisely we shall use the following assumption on the characteristic variety at the points belonging to several components. In particular it includes an inequality on depth in the *the opposite* direction:

Condition 2.2. *(1) Let* $\chi \in \mathcal{V}_1 \cap \ldots \cap \mathcal{V}_s$ *and* $\chi = \nu_i^*(\chi_i)$ *for* $\chi_i \in \mathrm{Char}(P_i)$ *where* $\nu_i : X \to P_i$ *is the map corresponding to the component* $\mathcal{V}_i$*. Then:*

$$\bigoplus_i h_1(\nu_i) : H_1(X,\chi) \to \bigoplus H_1(P_i,\chi_i) \tag{4}$$

is injective. In particular, the depth of each character χ *in the intersection of several positive dimensional irreducible components* $\mathcal{V}_1, \ldots, \mathcal{V}_s$ *of the characteristic variety does not exceed the sum of the depths of the generic character in each component* $\mathcal{V}_i$*.*

[1]this condition is independent of a choice of smooth compactification $\bar{X}$

(2) If $\chi \in \mathcal{V}_i$ but $\chi \notin \mathcal{V}_i \cap \mathcal{V}_j, j \neq i$ then $h_1(\nu_i) : H_1(X, \chi) \to H_1(P_i, \chi_i)$ is an isomorphism.

This condition is satisfied in the examples considered in section 5.

2.2. **Abelian covers.** Given a surjection $\pi_\Gamma : \pi_1(X) \to \Gamma$ onto a finite group, there are a unique quasi-projective manifold $\widetilde{X}_\Gamma$ and a map $\tilde{\pi}_\Gamma : \widetilde{X}_\Gamma \to X$ which is an unramified cover with covering group Γ. The variety $\widetilde{X}_\Gamma$ is characterized by the property that Γ acts freely on $\widetilde{X}_\Gamma$ and $\widetilde{X}_\Gamma / \Gamma = X$. Let $\bar{X}_\Gamma$ denote a smooth model of a compactification of $\widetilde{X}_\Gamma$ such that $\tilde{\pi}_\Gamma$ extends to a regular map $\bar{\pi}_\Gamma : \bar{X}_\Gamma \to \bar{X}$ ($\bar{X}$ as above). The fundamental group X_Γ, being birational invariant, depends only on X and π_Γ.

Let $\mathcal{C} = \bar{X} \setminus X$ be the "divisor at infinity" and let $\tilde{\mathcal{C}} \subset \mathcal{C}$ be a divisor on $\bar{X}$ whose irreducible components are components of $\mathcal{C}$. If $\chi \in \mathrm{Char}(\pi_1(X))$ is trivial on the components of $\mathcal{C}$ not in $\tilde{\mathcal{C}}$ then χ is the pullback of a character of $\pi_1(\bar{X} \setminus \tilde{\mathcal{C}})$ via the inclusion $X \to \bar{X} \setminus \tilde{\mathcal{C}}$. We shall denote the corresponding character of $\pi_1(\bar{X} \setminus \mathcal{C})$ as χ as well but (since the depth of χ depends on the underlying space) corresponding depths will be denoted $d(\chi, \mathcal{C})$ and $d(\chi, \tilde{\mathcal{C}})$ respectively.

The homology groups of unramified and ramified covers can be found in terms of characteristic varieties as follows (cf. [24]).

Theorem 2.3. *1.(cf.* [24]*) With above notations:*

$$\mathrm{rk} H_1(\widetilde{X}_\Gamma, \mathbb{Q}) = \sum_{\chi \in Char \Gamma} d(\pi_\Gamma^*(\chi), \mathcal{C}) \tag{5}$$

2.(cf. [30]*) Let $I(\chi)$ be the collection of components of $\mathcal{C}$ such that $\chi(\gamma_{C_i}) \neq 1$ (γ_{C_i} is a meridian of the component C_i) and let $\mathcal{C}_\chi = \bigcup_{i \in I(\chi)} C_i$. Then*

$$\mathrm{rk} H_1(\bar{X}_\Gamma, \mathbb{Q}) = \sum_{\chi \in Char \Gamma} d(\pi_\Gamma^*(\chi), \mathcal{C}_{\pi_\Gamma^*(\chi)}) \tag{6}$$

The following special case of Theorem 2.3 will be used in section 3.

Corollary 2.4. *Let*

$$\pi_{\Gamma(a_{i_1}, ..., a_{i_l})} : \pi_1(\mathbb{P}^1 \setminus \{a_{i_1}, ..., a_{i_l}\}) \to H_1(\mathbb{P}^1 \setminus \{a_{i_1}, ..., a_{i_l}\}, \mathbb{Z}/n\mathbb{Z}), 0 \leq i_1, ...i_l, \leq k$$

be the composition of Hurewicz map with the reduction modulo n and let $X_n(a_{i_1}, ...a_{i_l})$ be the corresponding ramified abelian cover[2] of $\mathbb{P}^1$ with the covering group $\Gamma = H_1(\mathbb{P}^1 \setminus \{a_{i_1}, ..., a_{i_l}\}, \mathbb{Z}/n\mathbb{Z})$. Then

$$H^1(X_n(a_0, ..., a_k), \mathbb{C})_\chi = \oplus H^1(X_n(a_{i_1}, ..., a_{i_l}, \mathbb{C})_{\chi^r(a_{i_1}, ..., i_l)} \quad 3 \leq l \leq k, 0 \leq i_j \leq k \tag{7}$$

where the summation is over the characters $\chi^r(a_{i_1, ..., i_l})$ which are restricted in the sense that they do not take value 1 on a cycle which is the boundary of a small disk about any point $a_{i_1}, ..., a_{i_l}$.

[2]note that this is the universal cover for the covers having an abelian n-group as the covering group

2.3. **Albanese varieties of quasi-projective manifolds.** Let X be a smooth quasi-projective manifold and let $\bar{X}$ be a smooth compactification of X. Denote $\bar{X} \setminus X$ by $\mathcal{C}$ and assume in this section that $\mathcal{C}$ is a divisor with normal crossings. One associates to X a semi-abelian variety i.e. an extension:

$$0 \to T \to Alb(X) \to A \to 0 \tag{8}$$

where T is a torus and A is an abelian variety (the abelian part of $Alb(X)$) called the *Albanese variety* of X. Such a semi-abelian variety can be obtained as

$$H^0(\bar{X}, \Omega^1(\log(\mathcal{C}))^*/H_1(X,\mathbb{Z})$$

where embedding $H_1(X,\mathbb{Z}) \to H^0(\bar{X}, \Omega^1(\log(\mathcal{C}))^*$ is given by $\gamma \in H_1(X,\mathbb{Z}) \to (\omega \to \int_\gamma \omega)$ (and polarization of abelian part is coming from the Hodge form on $H_1(\bar{X},\mathbb{Z})$ given by $(\gamma_1,\gamma_2) = \int_{\bar{X}} \gamma_1^* \wedge \gamma_2^* \wedge h^{\dim X - 1}$ where $h \in H^2(\bar{X},\mathbb{Z})$ is the class of hyperplane section).

One can also view $AlbX$ as the semi-abelian part of the 1-motif associated to the (level one) mixed Hodge structure on $H_1(X,\mathbb{Z})$ (cf. [10], section 10.1). The abelian part of $Alb(X)$ is the Albanese variety of a smooth projective compactification of X. It clearly is independent of a choice of the latter.

In this paper we shall consider Albanese varieties of abelian covers of quasi-projective surfaces but note that the Albanese variety of an abelian covers of quasi-projective manifold of any dimension can be obtained as the Albanese variety of the corresponding abelian cover of a surface due to the following Lefschetz type result:

Proposition 2.5. *Let X be a quasi-projective manifold and $H \cap X$ a generic 2-dimension section by a linear space H. Then $\pi_1(X) = \pi_1(X \cap H)$.*

Let Γ be a finite quotient of these groups. Then the unramified Γ-covers $\tilde{X}_\Gamma$ and $\widetilde{(X \cap H)}_\Gamma$, corresponding to surjections of $\pi_1(X)$ and $\pi_1(X \cap H)$ onto Γ, have Albanese varieties which are isomorphic as semi-abelian varieties.

2.4. **Local Albanese varieties of plane curve singularities.** For details of the material of this section we refer to [26]. Let $f(x,y)$ be an analytic germ of a reduced isolated curve singularity in $\mathbb{C}^2$. One associates with it the Milnor fiber $M_f = B \cap f^{-1}(t)$ where B is a small ball in $\mathbb{C}^2$ centered at the singular point. The latter supports canonical level one limit Mixed Hodge structure on $H^1(M_f,\mathbb{Z})$ (cf. [31]). Again one can apply Deligne's construction [10, 10.3.1] which leads to the following.

Definition 2.6. *The local Albanese variety of a germ f is the abelian part of the 1-motif of the limit Mixed Hodge structure on $H^1(M_f,\mathbb{Z})$. Equivalently, this is quotient of $F^0 Gr^W_{-1} H_1(M_f \mathbb{C})/Im H_1(M_f,\mathbb{Z})$ where F and W are respectively the Hodge and weight filtrations. The canonical polarization is coming from the form induced by the intersection form of $H_1(M_f,\mathbb{Z})$ on $Gr^W_{-1} H_1(M_f,\mathbb{Z})$.*

The local Albanese has a description in terms of the Mixed Hodge structure on the cohomology of the link of the surface singularity associated to f.

Proposition 2.7. *(cf.* [26]*, Prop.3.1) Let $f(x,y)$ be a germ of a plane curve with Milnor fiber M_f and* [3] *for which the semi-simple part of monodromy has order N.*

[3]this assumption is a somewhat weaker than the one in [26] but the argument works in this case with no change

Let $L_{f,N}$ the the link of the corresponding surface singularity

$$z^N = f(x,y) \tag{9}$$

Then there is the isomorphism of the mixed Hodge structures:

$$Gr_3^W H^2(L_{f,N})(1) = Gr_1^W H^1(M_f) \tag{10}$$

where the mixed Hodge structure on the left is the Tate twist of the mixed Hodge structure constructed in [13] *and the one on the right is the mixed Hodge structure on vanishing cohomology constructed in* [31].

Below we shall use Albanese varieties for non reduced germs and those can be define using the abelian part of the 1-motif of mixed Hodge structure $Gr_3^W H^2(L_{f,N})(1)$.

Recall finally that the local Albanese can be described in terms of a resolution of the singularity (9).

Theorem 2.8. *(cf.* [26] *Theorem 3.11) Let $f(x,y)=0$ be a singularity let N be the order of the semi-simple part of its monodromy operator. The local Albanese variety of germ $f(x,y)=0$ is isogenous to the product of the Jacobians of the exceptional curves of positive genus for a resolution of the singularity (9).*

Example 2.9. Consider the non-reduced singularity

$$f(x,y) = x^{a_1}(x-y)^{a_2}y^{a_3} \quad a_1+a_2+a_3=n \tag{11}$$

having the ordinary triple point as the corresponding reduced germ. In this case, the local Albanese variety is isogeneous to the Jacobian of plane curve whose affine portion is given by

$$v^n = u^{a_1}(u-1)^{a_2} \tag{12}$$

Indeed, resolution of (11) can be achieved by a single blow up. The multiplicity of the exceptional curve is equal to n. It follows from A'Campo's formula that the characteristic polynomial of the monodromy is $(t^n-1)(t-1)$ and that the order of the monodromy operator acting on $Gr_1^W H^1(M_f)$ is equal to n. A resolution of n-fold cyclic cover of the surface singularity

$$z^n = x^{a_1}(x-y)^{a_2}y^{a_3} \tag{13}$$

can be obtained by resolving cyclic quotient singularities of the normalization of the pullback of this covering to the blow up of $\mathbb{C}^2$ resolving $f_{red}(x,y)=0$ (here f_{red} is corresponding reduced polynomial). This pull-back has as an open subset the surface given in $\mathbb{C}^3$ by equation: $w^n = u^n v^{a_1}(v-1)^{a_2}$. Such resolution of surface (13) has only one exceptional curve of positive genus and this exceptional curve is the n-fold cyclic cover of $\mathbb{P}^1$ ramified at 3 points. The monodromies of this n-cover around ramification points are multiplications by $exp(\frac{2\pi\sqrt{-1}a_i}{n}), i=1,2,3$. This allows to identify the exceptional curve with curve (12). It follows from the Theorem 2.8 that the local Albanese variety of singularity (11), as was claimed, is isogenous to the Jacobian of curve (12).

3. Jacobians of abelian covers of a line

The following will be used in the proof of the theorem 4.1.

Theorem 3.1. *Let X_n be the abelian cover of $\mathbb{P}^1$ ramified at $\mathcal{A} = \{a_0, a_1, ...a_k\} \subset \mathbb{P}^1$ corresponding to the surjection $\pi_1(\mathbb{P}^1 \setminus \mathcal{A}) \to H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}_n)$. Let $A_i \in \mathbb{N}, i = 0, ..., k$ be a collection of integers such that*

$$\sum_{i=0}^{i=k} A_i = 0 \ (mod\ n), 1 \le A_i < n \quad \gcd(n, A_0, ..., A_k) = 1 \tag{14}$$

Denote by $X_{n|A_0,....,A_k}$ a smooth model of the cyclic cover of $\mathbb{P}^1$ which affine portion is given by

$$y^n = (x - a_0)^{A_0} \cdot .... \cdot (x - a_k)^{A_k} \tag{15}$$

(by (14) this model is irreducible). Then the Jacobian of X_n is isogenous to the product of the isogeny components of the Jacobians of the curves $X_{n|A_0,...A_k}$.

Remark 3.2. If $k = 2$ then the curve X_n is biholomorphic to Fermat curve $x^n + y^n = z^n$ in $\mathbb{P}^2$, since as affine model of the abelian cover one can take the curve in $\mathbb{C}^3$ given by $x^n = u,\ y^n = 1 - u$, and the above theorem follows from the calculations in [15] containing explicit formulas for simple isogeny components of the Fermat curves.

Corollary 3.3. *Let X_Γ be a covering of $\mathbb{P}^1$ with abelian Galois group Γ ramified at $a_0, ..., a_k \in \mathbb{P}^1$. Then there exist a collection of curves, each being a cyclic covers (15) of $\mathbb{P}^1$, such that the Jacobian of X_Γ is isogenous to a product of isogeny components of Jacobians of the curves in this collection.*

Proof. Let $\pi_\Gamma : H_1(\mathbb{P}^1 \setminus \bigcup_{i=0}^{i=k} a_i, \mathbb{Z}) \to \Gamma$ be the surjection corresponding to the covering X_Γ, $\delta_i \in H_1(\mathbb{P}^1 \setminus \bigcup_{i=0}^{i=k} a_i, \mathbb{Z}), i = 0, ..., k$ be the boundary of a small disk about $a_i, i = 0, ..., k$ and let n_i be the order of the element $\pi_\Gamma(\delta_i) \in \Gamma$. Then for $n = lcm(n_0, ..., n_k)$ one has a surjection $H_1(\mathbb{P}^1 \setminus \bigcup_{i=0}^{i=k} a_i, \mathbb{Z}/n\mathbb{Z}) \to \Gamma$ and hence a dominant map $X_n \to X_\Gamma$. In particular the Jacobian of X_Γ is a quotient of the Jacobian of X_n and the claim follows. □

Proof of the theorem 3.1. We shall assume below that one of ramification points, say a_0, is the point of $\mathbb{P}^1$ at infinity.

A projective model of X_n can be obtained as the projective closure in $\mathbb{P}^{k+1}$ (which homogeneous coordinates we shall denote $x, z_1, ..., z_k, w$) of the complete intersection in $\mathbb{C}^{k+1}$ given by the equations:

$$z_1^n = x - a_1, ..., z_k^n = x - a_k \tag{16}$$

The Galois covering $X_n \to \mathbb{P}^1$ is given by the restriction on this complete intersection of the projection of $\mathbb{P}^{k+1}$ from the subspace $x = w = 0$.

For any $(A_0, A_1, ..., A_k)$ as above, consider the map

$$\Phi_{n|,A_0,....,A_k} : X_n \to X_{n|A_0,A_1,...A_k} \tag{17}$$

which in the chart $w \neq 0$ is the restriction on X_n of the map $\mathbb{C}^{k+1} \to \mathbb{C}^2$ given by:

$$\Phi_{A_1,...,A_k} : (z_1, ..., z_k, x) \to (y, x) = (z_1^{A_1} .... z_k^{A_k}, x) \tag{18}$$

The map $\Phi_{n|,A_0,....,A_k}$ is the map of the covering spaces of $\mathbb{P}^1$ corresponding to the surjection of the Galois groups

$$H_1(\mathbb{P}^1 \setminus \bigcup_{i=0}^{i=k} a_i, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$$

which is given by

$$(i_0, i_1, ..., i_k) \to \sum_j i_j A_j \mod n \tag{19}$$

The maps $\Phi_{n|A_0,....,A_k}$ induce the maps of Jacobians:

$$\bigoplus_{A_0,...,A_k, 0 \le A_i < n-1} (\Phi_{n|A_0,...,A_k})_* : Jac(X_n) \to \bigoplus Jac(X_{n|A_0,....,A_k}) \tag{20}$$

We claim that the kernel of a map (20) is finite. This clearly implies the Theorem 3.1. Finiteness for the kernel of morphism (20) will follow from surjectivity of the map of cotangent spaces at respective identities of Jacobians (20):

$$\bigoplus_{A_0,...,A_k} H^{1,0}(X_{n|A_0,....,A_k}, \mathbb{C}) \to H^{1,0}(X_n, \mathbb{C}) \tag{21}$$

For each $\chi \in \mathrm{Char}\mathbb{Z}/n\mathbb{Z}$ let $m^{1,0}_\chi(n|A_0, ..., A_k)$ (resp. $m^{0,1}_\chi(n|A_0, ..., A_k)$) denotes the dimension of isotypical summand of $H^{1,0}(X_{n|A_0,....,A_k}, \mathbb{C})$
(resp. $H^{0,1}(X_{n|A_0,....,A_k}, \mathbb{C})$) on which $\mathbb{Z}/n\mathbb{Z}$ acts via the character χ. Similarly $m^{1,0}_{\Phi^*_{n|A_0,....,A_k}(\chi)}(n)$ (resp. $m^{0,1}_{\Phi^*_{n|A_0,....,A_k}(\chi)}(n)$) will denote the dimension of the eigenspace of the pull back $\Phi^*_{n|A_0,....,A_k}(\chi) \in \mathrm{Char}H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z})$ for the action of the covering group of $X_n \to \mathbb{P}^1$ on $H^{1,0}(X_n)$ (resp. $H^{0,1}(X_n)$).

It follows from Theorem 2.3 (2), that the depth of χ considered as a character of $H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z})$ can be written as:

$$\begin{aligned} d(\chi) = m^{0,1}_\chi(n|A_0, ..., A_k) + m^{1,0}_\chi(n|A_0, ..., A_k) = \\ m^{0,1}_{\Phi^*_{n|A_0,....,A_k}(\chi)}(n) + m^{1,0}_{\Phi^*_{n|A_0,....,A_k}(\chi)}(n) \end{aligned} \tag{22}$$

Moreover, one has inequalities:

$$\begin{aligned} m^{0,1}_\chi(n|A_0, ..., A_k) \le m^{0,1}_{\Phi^*_{n|A_0,....,A_k}(\chi)}(n) \\ m^{1,0}_\chi(n|A_0, ..., A_k) \le m^{1,0}_{\Phi^*_{n|A_0,....,A_k}(\chi)}(n) \end{aligned} \tag{23}$$

Hence, in fact,

$$\begin{aligned} m^{0,1}_\chi(n|A_0, ..., A_k) = m^{0,1}_{\Phi^*_{n|A_0,....,A_k}(\chi)}(n) \\ m^{1,0}_\chi(n|A_0, ..., A_k) = m^{1,0}_{\Phi^*_{n|A_0,....,A_k}(\chi)}(n) \end{aligned} \tag{24}$$

Now let us fix $\chi \in \mathrm{Char}(H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z}))$, i.e. a character of the Galois group of the cover $X_n \to \mathbb{P}^1$, such that its value on the cycle $\delta_i \in H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z})$ corresponding to $a_i \in \mathbb{P}^1, i = 0, ..., m$ satisfies:

$$\chi(\delta_i) = exp\left(\frac{2\pi\sqrt{-1}j_i}{n}\right) \ne 1, (1 \le j_i < n) \tag{25}$$

and let $J = gcd(j_0, ..., j_k)$. The collection of integers $A_i = \frac{j_i}{J}$ satisfies condition (14). Denote by Γ_0 the cyclic group $\chi(H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z})) \subset \mathbb{C}^*$. Then χ can be considered as a character $\chi' \in \mathrm{Char}(\Gamma_0)$ and then $\chi = \pi^*(\chi')$ where π is projection of the abelian cover with covering group Γ onto the cyclic cover with the covering group Γ_0. It follows from (24) that any isotypical component in $H^{1,0}(X_n, \mathbb{C})_\chi$ is the image of the isotypical component of a cyclic covers and hence the map (21) is surjective which concludes the proof. □

We shall finish this section with an explicit formula for $dimH^0(X_n, \Omega^1_{X_n})_\chi$ i.e. the multiplicity of the isotypical component of the covering group of abelian cover acting on the space of holomorphic 1-forms.

Proposition 3.4. *Let the values of a character $\chi \in \mathrm{Char} H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z})$, $\chi \neq 1$, be given as in (25). Assume that $J = gcd(j_0, ...j_k) = 1$ and let $M = \sum_i (n - j_i)$. Then*

$$dimH^{1,0}(X_n)_\chi = \left[\frac{M}{n}\right] \tag{26}$$

Remark 3.5. If $J \neq 1$ then Prop. 3.4 yields an expression for the dimension of isotypical component corresponding to $\chi \in \mathrm{Char} H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z})$ as well. Indeed, this dimension coincides with the dimension of isotypical component for χ considered as the character of $H_1(\mathbb{P}^1 \setminus \mathcal{A}, \mathbb{Z}/(\frac{n}{J}\mathbb{Z}))$.

Proof of Prop. 3.4. The equations of the projective closure of the complete intersection (16) are

$$z_i^n = (x - a_i w)w^{n-1}, i = 1, ..., k \tag{27}$$

The only singularity of (27) occurs at $w = 0, z_i = 0, x = 1$. Near it (27) is a complete intersection locally given by $z_i^n = w^{n-1}\gamma_i$ where γ_i is a unit. It has n^{k-1} branches (corresponding to the orbits of the action $(z_1, ..., z_k) \to (\zeta z_1, ..., \zeta z_k), \zeta^n = 1$) each equivalent to $z_i = t^{n-1}, w = t^n$. Therefore (27) is a ramified cover of $\mathbb{P}^1$ with $k+1$ branching points $a_1, ..., a_k, \infty$ over which it has n^{k-1} preimages with ramification index n at each ramification point.

The space $H^0(\Omega^1_{X_n})$ (for a smooth model of (27)) is generated by the residues of $k+1$-forms

$$\frac{z_1^{j_1-1}....z_k^{j_k-1}P(x,w)\Omega}{\Pi(z_i^n - (x - a_i w)w^{n-1})} \ (1 \leq j_i) \quad \text{where} \quad \sum_1^k (j_i - 1) + degP + k + 2 = nk \tag{28}$$

(cf. [14, Theorem 2.9]). Here

$$\Omega = \sum_i (-1)^{i-1} z_i dz_1 \wedge ... \widehat{dz_i} .. \wedge dz_k \wedge dx \wedge dw + (-1)^{k+1}(x dz_1 \wedge ... \wedge dz_k \wedge dw - w dz_1 \wedge ... \wedge dz_k \wedge dx)$$

In the chart $x \neq 0$ such residue (of (28)) is given by:

$$\frac{z_1^{j_1-1}....z_k^{j_k-1}P(w)dw}{(z_1....z_k)^{n-1}} \tag{29}$$

Using (27), one can reduce powers of z_i i.e. we can assume:

$$1 \leq j_i \leq n-1 \tag{30}$$

and a basis of the eigenspace $H^0(\Omega^1_{X_n})_\chi$, with χ as in (25), can be obtained by selecting $P(w) = w^s$ where s must satisfy:

$$\sum_1^k (j_i - 1) + s + k + 2 \leq nk \tag{31}$$

The adjunction condition assuring that the residue of (28) will be regular on normalization of (27) is

$$-\sum_1^k (n - j_i)(n-1) + sn + n - 1 \geq 0 \tag{32}$$

To count the number of solutions of (31) and (32) i.e. $dimH^0(\Omega^1_{X_n})_\chi$ with χ given by (25), let $\bar{j}_i = n - j_i$. Then $1 \le \bar{j}_i \le n-1$ and (31),(32) have form $\sum_1^k(n-1-\bar{j}_i)+s+k+2 \le kn$, $-(\sum_1^k \bar{j}_i)(n-1)+sn+n > 0$. Hence:

$$s+2 \le \sum_1^k \bar{j}_i < \frac{(s+1)n}{n-1} = s+1+\frac{s+1}{n-1} \tag{33}$$

Notice that from (31) one has $s \le nk-k-2$ i.e. $\frac{s+1}{n-1} \le k - \frac{1}{n-1}$ and hence $\sum_1^k \bar{j}_i \le k+s$. In particular possible values of $\sum_1^k \bar{j}_i$ are $s+2, ...s+k$ and therefore for given $\bar{j}_i$, parameter s can take at most $k-1$ values $\sum \bar{j}_i - 2, ..., \sum \bar{j}_i - k$. In particular, multiplicities of the χ-eigenspaces do not exceed $k-1$.

Let $\sum \bar{j}_i = M$. Then from (33) one has $M-1-\frac{M}{n} < s \le M-2$ and hence the number of possible values of s is

$$M-2-\left[M-1-\frac{M}{n}\right] = -1-\left[-\frac{M}{n}\right] = \left[\frac{M}{n}\right]$$

as claimed in the Prop. 3.4. □

Remark 3.6. One can deduce the theorem 3.1 using Prop. 3.4 and the following:

Proposition 3.7. *([2], Prop. 6.5). For $x \in \mathbb{R}$ denote by $\langle x\rangle = x-[x]$ the fractional part of x. Assume that $gcd(i,n)=1$ and n does not divide either of $A_0,...,A_k$. Then for the curve (15) the dimension of the eigenspace corresponding to the eigenvalue $exp(\frac{2\pi\sqrt{-1}i}{n})$ of the automorphism of $H^{1,0}(X_{n,A_0,...,A_k},\mathbb{C})$ induced by the map $(x,y) \to (x, yexp(-\frac{2\pi\sqrt{-1}}{n}))$ equals to*

$$-\left\langle\frac{i\sum_0^k A_s}{n}\right\rangle + \sum_0^k\left\langle\frac{iA_s}{n}\right\rangle \tag{34}$$

Indeed, the equality of multiplicities (24) follows by comparison (26) with (34) since for $i=1$ (34) yields $-\frac{\sum A_s}{n} + \left[\frac{\sum A_s}{n}\right] + \sum\frac{A_s}{n} = \left[\frac{\sum A_s}{n}\right]$

Remark 3.8. Special case of Prop. 3.7 appears also in [26] (cf. lemma 6.1). The multiplicity of the latter corresponds to the case $j = n-i$ in Prop. 3.7.

4. Decomposition theorem for abelian covers of plane

The main result of this section relates the Albanese variety of ramified covers to the local Albanese varieties of ramification locus as follows.

Theorem 4.1. *Let $\mathcal{C}$ be a plane algebraic curve such that all irreducible components of its characteristic variety contain the identity of $\mathrm{Char}\pi_1(\mathbb{P}^2\setminus\mathcal{C})$. Assume that the Condition 2.2 is satisfied. Let $\pi_\Gamma : \pi_1(\mathbb{P}^2\setminus\mathcal{C}) \to \Gamma$ be a surjection onto a finite abelian group. Then the Albanese variety of the abelian cover $\bar{X}_\Gamma$ ramified over $\mathcal{C}$ and associated with π_Γ is isogenous to a product of isogeny components of local Albanese varieties of possibly non-reduced germs having as reduced singularity a singularity of $\mathcal{C}$.*

Proof. To each component of positive dimension of the characteristic variety corresponds an isogeny component of Albanese variety of $\bar{X}_\Gamma$ as follows.

Let $Char_j$ be an irreducible component of the characteristic variety $V_1(\mathbb{P}^2\setminus\mathcal{C})$ of $\mathcal{C}$ (cf. (2.1)) and let $\phi_j : \mathbb{P}^2\setminus\mathcal{C} \to \mathbb{P}^1\setminus D_j$ be the corresponding holomorphic map where D_j is a finite subset of $\mathbb{P}^1$. The cardinality of D_j is equal to $dim(Char_j)+1$ and $Char_j = \phi_j^*(Char\pi_1(\mathbb{P}^1 \setminus D_j))$. Denote by Γ_j the push-out of π_Γ. The map ϕ_j is dominant and yields the surjection $(\phi_j)_* : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \pi_1(\mathbb{P}^1 \setminus D_j)$ of the fundamental groups. With these notations we have the universal (for the groups filings the right left corner of) commutative diagram:

$$\begin{array}{ccc} \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) & \to & \pi_1(\mathbb{P}^1 \setminus D_j) \\ \downarrow & & \downarrow \\ \Gamma & \to & \Gamma_j \end{array} \tag{35}$$

A character of $H_1(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{Z})$, which is the image of a character of Γ for the map $\mathrm{Char}\Gamma \to \mathrm{Char}H_1(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{Z})$, can be obtained as a pullback of a character of $H_1(\mathbb{P}^1 \setminus D_j)$ if and only if it is a pullback of a character of Γ_j via maps in diagram (35). Let $\mathcal{D}_j \to \mathbb{P}^1$ the ramified cover with branching locus D_j, having Γ_j as its Galois group and let $\Phi_j : Alb(\bar{X}_\Gamma) \to Jac(\mathcal{D}_j)$ be the corresponding Albanese map. The Jacobian $Jac(\mathcal{D}_j)$ is an isogeny component of $Alb(\bar{X}_\Gamma)$. It depends only on $Char_j$ and Γ.

Next let $\chi_k, k = 1, .., N$ be the collection of characters of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ whose depth is greater than the depth of generic point on the component of characteristic variety to which it belongs. We shall call such characters *the jumping characters of* $\mathcal{C}$. It follows form our Condition 2.2 that jumping characters are exactly the intersection points of the components of characteristic variety.

We claim injectivity of the map of Albanese varieties induced by the holomorphic maps ϕ_j:

$$0 \to Alb(\bar{X}_\Gamma) \stackrel{\bigoplus \Phi_j}{\to} \bigoplus_j Jac(\mathcal{D}_j) \tag{36}$$

To see that $\mathrm{Ker} \bigoplus \Phi_j = 0$, consider the induced homomorphism

$$H_1(Alb(\bar{X}_\Gamma), \mathbb{C}) \to H_1(\bigoplus_j Jac(\mathcal{D}_j), \mathbb{C}). \tag{37}$$

The group Γ acts on both vector spaces and the homomorphism (37) is Γ-equivariant. For a character χ belonging to a single component of characteristic variety the depths is equal to the depth of the generic character in its component (cf. Condition 2.2) which in turn coincides with $H_1(\mathcal{D}_j, \mathbb{C})_\chi$. Therefore one has isomorphism $H_1(\bar{X}_\Gamma, \mathbb{C})_\chi \to H_1(\mathcal{D}_j, \mathbb{C})_\chi$. For a character $\chi = \chi_k$, i.e. for a character in the intersection of several components, again from Condition 2.2, one has injection: $H_1(\bar{X}_\Gamma, \mathbb{C})_\chi \to \oplus_{j, \chi \in \mathrm{Char}_j} H_1(\mathcal{D}_j, \mathbb{C})$. This implies (36).

To finish the proof of the Theorem 4.1 it suffices to show that each summand in the last term in (36) is isogenous to a product of components of local Albanese varieties of $\mathcal{C}$. Indeed Poincare complete reducibility theorem (cf. [5]) implies that the image of the middle map is isogenous to a direct sum of irreducible summands of the last term.

Denote by the same letter ϕ_j the extension of a regular map $\phi_j : \mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1 \setminus D_j$ to the map $\mathbb{P}^2 \setminus \mathcal{S}_j \to \mathbb{P}^1$ where $\mathcal{S}_j$ is the finite collection of indeterminacy points of the extension of ϕ_j to $\mathbb{P}^2$. Let $\mathcal{C}_d = \phi_j^{-1}(d), d \in D_j$. Then $\mathcal{C}$ contains the union of the closures $\bar{\mathcal{C}}_d$ of (which are possibly reducible and non reduced curves). Each $P \in \mathcal{S}_j$ belongs to at least $CardD_j$ irreducible components and, since $CardD_j > 1$, P is a singular point of $\mathcal{C}$. We claim the following:

Claim 4.2. Resolution $\tilde{\mathbb{P}}^2_{\mathcal{C},P} \to \mathbb{P}^2$ of the singularity at P contains exactly exceptional curve E_P such that the regular extension $\tilde{\phi}_j$ of ϕ_j to $\tilde{\mathbb{P}}^2_{\mathcal{C},P} \to \mathbb{P}^1$ induces a finite map $\tilde{\phi}_j : E_P \to \mathbb{P}^1$.

To see this, consider a sequence of blow ups $\tilde{\mathbb{P}}^2_{\mathcal{C},P,h}, h = 1, ..., N(\mathcal{C}, P)$ of the plane starting with the blow up of $\mathbb{P}^2$ at P and in which the last blow up produces the resolution of singularity of $\mathcal{C}$ at P. For each h, let $\phi_{j,h} : \tilde{\mathbb{P}}^2_{\mathcal{C},P,h} \to D_j$ be the extension of ϕ from $\mathbb{P}^2 \setminus \mathcal{C}$ to $\tilde{\mathbb{P}}^2_{\mathcal{C},P,h}$. For every base point Q of the map $\phi_{j,h}$ on $\tilde{\mathbb{P}}^2_{\mathcal{C},P,h}$ consider the pencil of tangent cones to fibers of the map $\phi_{j,h}$ The fixed (possibly reducible) component of the pencil of tangent cones $T_d, d \in \mathbb{P}^1$ to curves $\tilde{\phi}_j^{-1}(d)$ [4] either:

a) coincide with the tangent cone T_d to each curve $\phi_j^{-1}(d)$, or

b) there exist d such that the tangent cone T_d to $\phi_{j,h}^{-1}(d)$ at Q contains a line not belonging to the fixed component of the pencil of tangent cones.

Since on $\tilde{\mathbb{P}}^2_{\mathcal{C},P}$ (i.e. eventually after sufficiently many blow ups) no two fibers of ϕ intersect, in a sequence of blow ups desingularizing $\mathcal{C}$ at P, there is a point Q infinitesimally close to P at which the tangent cones satisfy b). At such point $Q \in \tilde{\mathbb{P}}^2_{\mathcal{C},j,h}$ any two distinct fibers of $\phi_{j,h}$ admit distinct tangents because otherwise, since we have one dimensional linear system, the common tangent to two fibers will belong to the fixed component. Let $E_P \subset \tilde{\mathbb{P}}^2_{\mathcal{C},j,h+1}$ be the exceptional curve of the blow up of $\tilde{\mathbb{P}}^2_{\mathcal{C},j,h}$. Exceptional curves preceding or following this one on the resolution tree (which up to this point did not have vertices with valency greater than 2!) belong to one of the fibers of ϕ_j. Restriction of $\phi_{j,h+1}$ onto E_P is the map claimed in (4.2).

Finally, the ramified Γ-covering of $\mathbb{P}^2$ lifted to $\mathbb{P}^2_{\mathcal{C},P}$ and restricted on the proper preimage of the curve E_P in $\tilde{\mathbb{P}}^2_{\mathcal{C},P}$ induces the map onto Γ_j-covering of $\mathbb{P}^1$ ramified at D_j. Hence the Jacobian of the latter covering is a component of the Jacobian of a covering of E_P. It follows from the Corollary 3.3 that Jacobian of this cover of E_P isogenous to product of Jacobians of cyclic covers. Each Jacobian of cyclic cover of exceptional curve, in turn, is a component of local Albanese variety of singularity with appropriately chosen multiplicities of components of $\mathcal{C}$ given the by data of the cyclic cover of E_P (cf. Theorem 2.8). $\square$

The following theorem 4.4 allows to describe the isogeny class of Albanese varieties of abelian covers in explicit examples considered in the next section. The Albanese variety of abelian cover with Galois group Γ will be obtained as a sum of isogeny components of Jacobians of abelian covers of the line associated with Γ and corresponding to the positive dimensional components of the characteristic variety of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$. To state the theorem we shall use the following partial order on the set of mentioned isogeny components.

Definition 4.3. *Let $\Psi_i : \mathcal{B} \to \mathcal{A}_i, i \in I$, be a collection of equivariant morphisms of abelian varieties endowed with the action of a finite abelian group Γ. An isotypical isogeny component of the collection $\mathcal{A}_i$ is an abelian variety of the form S^m where S is Γ-simple* [5]*. Define the partial order of the set of isotypical components of $\Pi_{i \in I} \mathcal{A}_i$*

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[4] i.e. union of lines which are tangent to a component of the curve $\phi_{j,h}^{-1}(d)$ for any d

[5] i.e. simple in the category of abelian varieties with Γ-action cf.[27]

as follows: $\mathcal{A} \geq \mathcal{A}'$ *if and only if each* $\mathcal{A}$ *and* $\mathcal{A}'$ *belongs to the image of one of* Ψ_i $(i \in I)$ *and* $\mathcal{A} = S^m, \mathcal{A}' = S^{m'}, m \geq m'$

Now we are ready to state the following description of the Albanese variety of abelian cover $\bar{X}_\Gamma$.

Theorem 4.4. *Let* $\mathcal{C}$ *be a plane curve as in Theorem 4.1 i.e. with fundamental group of the complement satisfying the Condition 2.2 and all components of characteristic variety containing the identity character. Let* $\pi_\Gamma : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \Gamma$ *be a surjection onto an abelian group and let* Γ_{Ξ_i} *be corresponding push-out group given by diagram (35). Let* $\bar{\mathbb{P}}_{\Gamma_{\Xi_i}}$ *denotes the ramified cover of* $\mathbb{P}^1$ *with covering group* Γ_{Ξ_i} *which is the compactification of the cover of the target map of* $\mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1 \setminus D_i$ *corresponding to the component* Ξ_i*.*

(1) For any i *there are* Γ*-equivariant morphisms*

$$Alb(\bar{X}_\Gamma) \to Jac(\bar{\mathbb{P}}_{\Gamma_{\Xi_i}}) \tag{38}$$

(2) Let $A_m, m \in M$ *be the set of maximal elements in the ordering of isotypical components of collection of morphisms in (1).*

Then there is an isogeny

$$Alb(\bar{X}_\Gamma) \to \oplus_{m \in M} A_m \tag{39}$$

Remark 4.5. The maps in (38) corresponding to different characters may coincide (this is always the case for example for conjugate characters). The theorem asserts that selection among jumping characters and component of characteristic varieties can be made so that maximal isotypical components in corresponding covers provide isotypical decomposition of $Alb(\bar{X}_\Gamma)$.

Proof. Morphisms $\bar{X}_\Gamma \to \bar{\mathbb{P}}^1_{\Gamma_{\Xi_i}}$ were constructed in the beginning of the proof of theorem 4.1.

Let $A_m, m \in M$ be collection of maximal isotypical components in the Albanese varieties which are targets of the maps (38). Composition of a map (38) with projection on the isogeny components $A_m, m \in M$ gives the map $Alb(\bar{X}_\Gamma) \to A_m$. Each isogeny component of $Alb(\bar{X}_\Gamma)$ is an isogeny component in one of varieties $\bar{\mathbb{P}}^1_{\Gamma_{\Xi_i}}$ and the dimension of Γ-eigenspace corresponding to any character coincides with the dimension of χ-eigenspace of the targets (38). Hence the map (39) has finite kernel.

Let χ be a character having non zero eigenspace on $H^1(A_m)$. Then by theorem 2.3 part (2), $dim H^1(A_m)_\chi = dim H^1(\mathcal{A})_\chi = dim H^1(\bar{X}_\Gamma)_\chi$ where $\mathcal{A}$ is one of the targets of the maps (38). Since $H^1(\bar{X}_\Gamma)$ is a direct sum of Γ-eigenspaces and the image of $H^1(\bar{X}_\Gamma)_\chi$ is non-trivial in exactly one summand in (39) one obtains the surjectivity in (39). $\square$

Remark 4.6. Multiplicities of isotypical components A_m are poorly understood in general as well as jumping characters (cf. [8] where the problem of bounding the multiplicities of the roots of Alexander polynomials of the complements to plane curves, which are in correspondence with the jumping characters, is discussed). Nevertheless in all known examples, the above theorem is sufficient to completely determine isogeny class of Albanese varieties of abelian covers.

5. Albanese varieties of abelian covers ramified over arrangements of lines.

In the case when ramification set is an arrangement of lines theorems 4.1 and 4.4 yield considerably simpler than in general case results. We shall start with:

Corollary 5.1. *Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{P}^2$ with double and triple points only which satisfies the assumptions*[6] *of Theorem 4.1. Let $X_n(\mathcal{A})$ be a compactification of the abelian cover of $\mathbb{P}^2 \setminus \mathcal{A}$ corresponding to the surjection $H_1(\mathbb{P}^2 \setminus \mathcal{A}, \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z})$.*

(1) Albanese variety of $X_n(\mathcal{A})$ is isogenous to a product of isogeny components of Jacobians of Fermat curves.

(2) $Alb(X_n(\mathcal{A}))$ is isogenous to a product and of Jacobians of Fermat curves if

(a) none of the characters in $\mathrm{Char}H^1(\mathbb{P}^2 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z}) \subset \mathrm{Char}H_1(\mathbb{P}^2 \setminus \mathcal{A}, \mathbb{Z})$ is a jumping character in the characteristic variety of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ and

(b) the pencils corresponding to positive dimensional components have no multiple fibers.

Proof. Each component of characteristic variety having a positive dimension corresponds to the map $\mathbb{P}^2 \setminus \mathcal{A} \to \mathbb{P}^1 \setminus D$ where $\mathrm{Card}D = 3$. Those induce maps of $Alb(X_n(\mathcal{A}))$ onto the Jacobians of abelian covers of $\mathbb{P}^1$ ramified along corresponding D. The Jacobian of such abelian cover of $\mathbb{P}^1$ is a component of the Jacobian of Fermat curve. (cf. Corollary 3.3 with $k = 2$). Hence the maximal isotypical isogeny components (cf. Theorem 4.4) are components of Jacobians of Fermat curves and therefore part (1) follows from theorem 4.4 i.e. $Alb(X_n(\mathcal{A}))$ is isogenous to a product of components of Fermat curves. Note that the Theorem 4.1 for arrangements of lines with double and triple points can be obtained follows from these arguments. Indeed, the isogeny components of Jacobians of Fermat curves are Jacobians of cyclic covers of $\mathbb{P}^1$ ramified at three points (cf. [15],[9]) and Jacobians of cyclic covers of $\mathbb{P}^1$ ramified at three points are local Albanese varieties of non-reduced singularities of the form $x^{a_1}(x-y)^{a_2}y^{a_3}$ (cf. Example 2.9).

If characteristic variety does not have jumping characters in subgroup $\mathrm{Char}H_1(\mathbb{P}^2 \setminus \mathcal{A}, \mathbb{Z}/n\mathbb{Z})$ of $\mathrm{Char}\pi_1(\mathbb{P}^1 \setminus \mathcal{A})$ then $Alb(X_n(\mathcal{A}))$ is just a product of Jacobians corresponding to positive dimensional components of characteristic variety (i.e. there are no "corrections" in A_m coming from Jacobians of covers corresponding to jumping characters). The assumption about absence of multiple fibers implies that map of $X_n(\mathcal{A})$ corresponding to each positive dimensional component of characteristic variety of $\mathcal{A}$ has as target the cover as in Remark 3.2 i.e. a Fermat curve. Hence $Alb(X_n(\mathcal{A}))$ is a product of Jacobians of Fermat curve and we obtain part of (2). □

Example 5.2. Consider Ceva arrangement $xyz(x-z)(y-z)(x-y) = 0$ and the universal $\mathbb{Z}_5$ cover (with the covering group which is the quotient of $\mathbb{Z}_5^6$ by the cyclic subgroup generated by $(1,1,1,1,1,1)$. Then the irregularity of the corresponding abelian cover is 30 (cf. [17], [24] section 3.3 ex.2). The characteristic variety consists of five 2-dimensional components $\Xi_i, i = 1, ..., 5$ (cf. [24]), each being the pull back of $H^1(\mathbb{P}^1 \setminus D, \mathbb{C}^*), \mathrm{Card}D = 3$ via either a linear projection from one of 4 triple points or via a pencil of quadrics three degenerate fiber of which form the 6 lines

[6] i.e. we consider only the cases when all irreducible components of characteristic variety contain the identity and also Condition 2.2 is satisfied.

of the arrangement. Each of these 5 pencils induces a map on the abelian cover of $\mathbb{P}^1$ branched at 3 points, which has as the Galois group the quotient of $\oplus_1^3\mu_5$ by the diagonally embedded group of roots of unity μ_5 of degree 5. This cover, i.e. $\bar{\mathbb{P}}_{\Xi_i}, i = 1, .., 5$, is the Fermat curve of degree 5. The Jacobian of degree-5 Fermat curve is isogenous to a product of Jacobians of three curves $C_i, i = 1, 2, 3$ of genus 2 each one being a cyclic cover of $\mathbb{P}^1$ ramified at three points. (cf. [9],[21]). Hence the Albanese variety of this abelian cover is isogenous to a product of 15 copies of the Jacobian of ramified at three points cover of $\mathbb{P}^1$ of degree 5. In this example there are no jumping characters (in particular in $\mathrm{Char}H_1(\mathbb{P}^2 \setminus \mathcal{A}, \mathbb{Z}/5\mathbb{Z})$) and the isogeny can be derived from Corollary 5.1

Example 5.3. Consider again Ceva arrangement and calculate the abelian component of (semi-abelian) Albanese variety (cf. section 2.3) of its Milnor fiber M given by $w^6 = \Pi l_i$. Notice that the characteristic polynomial of the monodromy is $(t-1)^5(t^2+t+1)$ (cf. [24]). The ζ_3-eigenspace of $H^1(M, \mathbb{C})$ can be identified with the contribution in sum (6) of the pullback of the character χ of $\mathbb{P}^1 \setminus D$ via the pencil of quadrics formed by lines of the arrangement. Here D is the triple of points corresponding to the reducible quadrics in the pencil and χ is the character taking the same value ω_3 on standard generators if $\pi_1(\mathbb{P}^1 \setminus D)$. This pencil can be lifted to the elliptic pencil on a compactification of M onto 3-fold cyclic cover of $\mathbb{P}^1$ ramified at D and corresponding to $\mathrm{Ker}\chi$. Moreover, above expression for the characteristic polynomial of the monodromy shows that the map induced by this pencil is isogeny i.e. the abelian (i.e. compact) component of the Albanese of M is the elliptic curve E_0. The semi-abelian variety with is the Albanese variety of M is an extension:

$$0 \to (\mathbb{C}^*)^5 \to Alb(M) \to E_0 \to 0 \tag{40}$$

Example 5.4. Consider abelian cover of $\mathbb{P}^2$ ramified along arrangement of lines dual to 9 inflection points of a smooth cubic with Galois group $\mathbb{Z}_n^9/\mathbb{Z}_n$. This arrangement has 9 lines and 12 triple points. An explicit equation is as follows:

$$(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0 \tag{41}$$

The characteristic variety consists of 12 components corresponding to 12 triple points and 4 additional two-dimensional components intersecting along cyclic subgroup of order 3. Characters at the intersection are jumping and have depth 2 (cf. [12],[28]) while depth of generic character in each positive dimensional component is 1. In coordinates of $\mathrm{Char}\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ corresponding to components of $\mathcal{A}$ described jumping characters have the form $(\omega, ..., \omega), \omega^3 = 1$.

In the case $n = 5$, in which according to Hirzebruch one obtains a quotient of the unit ball, the Albanese variety is isogenous to the product of 16 copies of Fermat curve of degree 5, as follows from Corollary 5.1 (2) or equivalently 48 copies of curves of Jacobians of curves of genus 2 with automorphism of order 10 or, what is the the same, the 2-dimensional variety of CM type corresponding to cyclotomic field $\mathbb{Q}(\zeta_5)$. For arbitrary n such that $\gcd(3, n) = 1$ one get several copies of Jacobians of Fermat curves of degree n corresponding to components of characteristic variety.

If n is divisible by 3, i.e. the jumping characters are present, then the condition 2.2 should be verified. To this end, we shall reinterpret the part of this condition dealing with the map between the cohomology of local systems. The cohomology of the local systems appearing in (2.2) can be identified with the eigenspaces of the (co)homology of abelian covers (cf.[24]). More precisely, the χ-eigenspace can be identified with the cohomology of the local system corresponding to the character χ. The eigenspace corresponding to the character belonging to 4 irreducible components of characteristic variety in turn can be interpreted as the dual space of $H^1(\mathbb{P}^2, \mathcal{I}_Z(3))$ where $Z \subset \mathbb{P}^2$ is the subscheme of triple points (cf. [24, (3.2.14),(3.2.15)] and corresponding remark). On the other hand, each of the above 4 components corresponds to a selection of a subset $Z_i \subset Z, CardZ_i = 9$, cf. [24, Section 3.3,Example 3] for description of these subsets, each of which is a complete intersection of two cubic curves. The cohomology of generic local system in such component is identified with the dual space of $H^1(\mathbb{P}^2, \mathcal{I}_{Z_i}(3))$. The condition 2.2 is interpreted as injectivity of the map

$$H^1(\mathbb{P}^2, \mathcal{I}_Z(3)) \to \bigoplus_{i=1}^{i=4} H^1(\mathbb{P}^2, \mathcal{I}_{Z_i}(3)) \tag{42}$$

induced by injections $\mathcal{I}_Z \to \mathcal{I}_{Z_i}, i = 1, ...4$. This injectivity is readily seen e.g. by interpreting terms in (42) using standard sequence: $0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_Z \to 0$ and similar sequences for Z_i.

Implication of verification of Condition 2.2 is that in this case the product of Jacobians of Fermat curves which are the Jacobians corresponding to positive dimensional components of characteristic variety must be factored by the product $E_0^{\kappa-\delta}$ where κ is the number of components containing a jumping character (taking value $exp(\frac{2\sqrt{-1}\pi}{3})$ or $exp(\frac{4\sqrt{-1}\pi}{3})$ on all 9 lines of arrangement) and δ is the depth of the jumping character [7].

In the case $n = 3$ the abelian cover with the covering group $\mathbb{Z}_3^9/\mathbb{Z}_3$ one obtains from theorem 4.4 or Corollary 4.4

$$Alb(\bar{\mathbb{P}}^2_{\mathbb{Z}_3^8}) = E_0^{16}/E_0^2 = E_0^{14} \tag{43}$$

Indeed, in this case $\kappa = 4, \delta = 2$.

In the case $3|n, n > 3$, the product of Jacobians corresponding to positive dimensional components has several copies of E_0 as isogeny components and $Alb(X_n)$ is the quotient of this product by $E_0^{\kappa-\delta} = E_0^2$.

Example 5.5. Consider Hesse arrangement $\mathcal{H}$ formed by 12 lines containing 9 inflection points of a smooth cubic. It was shown in [24] (cf. section 3, example 5) that the characteristic variety of the fundamental group of the complement to this arrangement consists of 10 three-dimensional components and 54 two-dimensional components none of which belongs to a three-dimensional component (intersection of components must be zero dimensional). As earlier, it is convenient to describe components in terms of corresponding pencils i.e. maps $\mathbb{P}^2 \setminus \mathcal{H} \to \mathbb{P}^1 \setminus h$ where h is a set of points of cardinality 4 or 3 so that the characters in each component formed by pullbacks via these maps. The pencils corresponding to components of dimension 3 are linear projections from each of 9 quadruple points and the additional pencil is the pencil of curves of degree 3 containing 4 cubic curves each being a union of a triple of lines in the arrangement $\mathcal{H}$. The 54 maps $\mathbb{P}^2 \setminus \mathcal{H} \to \mathbb{P}^1 \setminus h$

[7] cf.[28], Prop. 4.8. This effect of characters in the intersection of several components of characteristic varieties is erroneously omitted in the final formula in Example 3 in section 3.3 of [24].

($Card\ h = 3$) are restrictions of the maps corresponding to the pencil of quadrics union of which are 6-tuples of lines in $\mathcal{H}$ forming a Ceva arrangement.[8]

The pencil corresponding to 3-dimensional component of characteristic variety induces the map of abelian cover of the plane ramified along $\mathcal{H}$ with Galois group $(\mathbb{Z}_3)^{12}/\mathbb{Z}_3$ on the maximal abelian cover $\mathbb{Z}_3$ cover of $\mathbb{P}^1$ ramified at 4 points. In particular the Albanese variety in question maps onto the Jacobian J_{10} of curve of genus 10. Similarly each 2-dimensional component of characteristic variety induces map of Albanese of abelian cover of $\mathbb{P}^2$ onto maximal abelian 3-cover of $\mathbb{P}^1$ ramified at 3 points. The latter is Fermat curve of degree i.e. the elliptic curve with j-invariant zero.

We obtain that the Albanese variety of the cover considered by Hirzebruch (cf.[20]) is isogenous to

$$J_{10}^{10} \times E_0^{54} \tag{44}$$

Example 5.6. Variety of lines on a Fermat hypersurface Previous results imply immediately the following:

Theorem 5.7. *Let F_3 be variety if lines on Fermat cubic threefold:*

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \tag{45}$$

Then there is an isogeny:

$$Alb(F_3) = E_0^5 \tag{46}$$

This isogeny was observed recently [6]. Also, Roulleau cf. [29] obtained the isomorphism class of the Albanese variety of Fermat cubic threefold.

Proof. It follows from discussion in [32] that Fano surface F_3 is abelian cover of degree 3^4 of $\mathbb{P}^2$ ramified over Ceva arrangement. Hence the isogeny (46) follows as in example 5.2. □

6. Applications

6.1. **Mordell-Weil ranks of isotrivial families of abelian varieties.** Recall the following (cf. [26])

Proposition 6.1. *Let $\mathcal{A} \to \mathbb{P}^2$ be a regular model of an isotrivial abelian variety over $\mathbb{C}(x,y)$ with a smooth fiber A. Assume that there is a ramified abelian cover $X \to \mathbb{P}^2$ such that the pullback of $\mathcal{A}$ to X is trivial abelian variety over X. Let Γ be the Galois group of $\mathbb{C}(X)/\mathbb{C}(x,y)$. Then the trivialization of $\mathcal{A}$ over X yields the action of Γ on A and the Mordell-Weil rank of $\mathcal{A}$ is equal to $\dim_{\mathbb{Q}} Hom_{\Gamma}(Alb(X), \mathcal{A}) \otimes \mathbb{Q}$.*

[8]This was explained in [24]. Recall that in interpretation of inflection points of the cubic as points in affine plane over field $\mathbb{F}_3$, the twelve lines correspond to the full set of lines in this plane and 6 tuples are in one to one correspondence with quadruples of points in this finite plane no three of which are collinear. Counting first ordered quadruples of this type one sees that there are 9×8 choices for the first two points, 6 choices for the third point (it cannot be the third point on the line containing first two) and 3 choices for the forth). Therefore one get 54 unordered quadruples of points and hence 54 6-tuples of lines.

Let A be an abelian variety over $\mathbb{C}$. Given an abelian cover $X \to \mathbb{P}^2$ with covering group Γ and a homomorphism $\Gamma \to AutA$, an example of isotrivial abelian variety over $\mathbb{C}(x,y)$ as in Prop.6.1 can be obtained as a resolution of singularities of

$$\mathcal{A}_X = X \times A/\Gamma \tag{47}$$

where Γ acts on $X \times A$ diagonally: $(x,a) \to (\gamma \cdot x, \gamma \cdot a), \gamma \in \Gamma, x \in X, a \in A)$. The map $\mathcal{A}_X \to X/\Gamma = \mathbb{P}^2$ gives to $\mathcal{A}_X$ a structure of isotrivial abelian variety over $\mathbb{C}(x,y)$.

Calculations of Albanese varieties in examples of previous sections yield values of Mordell-Weil ranks of isotrivial abelian varieties in many examples as in Prop. 6.1.

Example 6.2. Let $J_{2,5}$ denote the Jacobian of a smooth projective model of genus 2 curve C given by equation: $y^5 = x^2(x-1)^2$ (i.e. one of the curves C_i in Example 5.2). Assume that the direct sum $\Gamma = \mathbb{Z}_5^5$ acts on C so that the generator of each summand acts as the multiplication by $\zeta, \zeta = exp(\frac{2\pi i}{5}) : (x,y) \to (x, \zeta y)$ (cf. 5.2). This induces the action of $\mathbb{Z}_5^5$ on $J_{2,5} = Jac(C)$. In example 5.2, we viewed Γ as the quotient of $\mathbb{Z}_5^6$ by $(1,1,1,1,1,1)$, so that each summand corresponds to monodromy about one of 6 lines in Ceva arrangement. Then an identification of $\mathbb{Z}_5^5$ and $\mathbb{Z}_5^6/\mathbb{Z}_5$ can be obtained by identifying the former group with the image in the latter of the subgroup of $\mathbb{Z}_5^6$ of elements $(a_1, a_2, a_3, a_4, a_5, -\sum_{i=1}^{i=5})a_i, a_i \in \mathbb{Z}_5$. In such presentation of Γ, the action of first 5 components of elements in $\mathbb{Z}_5^6$ on C is given by multiplication by ζ while action of the last component on C is trivial.

Consider isotrivial family $\mathcal{A}_X$ of abelian varieties over $\mathbb{P}^2$ given by (47) with the zero set of discriminant being the Ceva arrangement of lines which is the quotient of $X \times J_{2,5}$, where X is the abelian cover with the covering group $\mathbb{Z}_5^5$ considered in example 5.2. The action of Γ is the diagonal action of $\Gamma = \mathbb{Z}_5^5$ as in (47). The Albanese variety of the abelian cover X in example 5.2 is isogenous to $(J_{2,5})^{15}$ (cf. (5.2)) and hence the rank of the Mordell-Weil group of the quotient is equal to

$$\mathrm{rk} Hom_{\mathbb{Z}_5^5}(J_{2,5}^{15}, J_{2,5}) \otimes \mathbb{Q} \tag{48}$$

The characters of representation of $\Gamma = \mathbb{Z}_5^6/\mathbb{Z}_5$ on $H_1(J_{2,3}^{15})$ are the characters of representation of Γ on $H^{1,0}(X,\mathbb{C})$ i.e. the characters from the characteristic variety of Ceva arrangement. Clearly neither of two characters for described above action of Γ on $H^1(C,\mathbb{C})$, having the form $(a,a,a,a,a,1), a \in \mathbb{Z}_5$ in the basis of $\mathrm{Char}\Gamma$ dual to the one coming from direct sum presentation of $\mathbb{Z}_5^6$, belongs to the characteristic variety of Ceva arrangement. Hence the rank (48) is zero.

6.2. **Periodicity of Albanese varieties.**

Theorem 6.3. *Let $\mathcal{C}$ be a curve in $\mathbb{P}^2$ such that there exist a surjection $\pi : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \mathbb{Z}$* [9]*. Consider two sequences of cyclic covers composed of ramified and unramified covers corresponding to surjections $\pi_n : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$*

(1) The sequence of isogeny classes of Albanese varieties of a tower of cyclic branched covers with given ramification locus $\mathcal{C}$ corresponding to surjections π_n is periodic.[10]

[9] For any curve in $\mathcal{C}$ (including irreducible in which case $H_1(\mathbb{P}^1 \setminus \mathbb{C}, \mathbb{Z}) = \mathbb{Z}/(\deg\mathcal{C})\mathbb{Z})$ adding to $\mathcal{C}$ a generic line in $\mathbb{P}^2$ yields a curve admitting such surjection cf. [22].

[10] i.e. exist $N \in \mathbb{N}$ such that Albanese varieties of cyclic covers corresponding to $\pi_n, \pi_{n'}$ with $n \equiv n' \mod N$ are isogeneous.

(2) The sequence of isogeny classes of semi-abelian varieties which are Albanese varieties of unbranched covers a complement to a curve $\mathcal{C}$ corresponding to surjections π_n is periodic.

Proof. Let $\Delta_\pi(t)$ be the Alexander polynomial of $\mathcal{C}$ corresponding to the surjection π (cf. [22]). For each root ξ of $\Delta_\pi(t)$ let n_ξ be its order (recall that any root of Alexander polynomial of an algebraic curve is a root of unity). For each set Ξ of distinct roots of $\Delta_\pi(t)$ let $n_\Xi = lcm(n_\xi), \xi \in \Xi$ and let N be the least common multiple of integers n_Ξ. To each congruence class modulo N corresponds a subset Ξ (possibly empty) such that integers in this class are divisible by exactly one (or none) among the integers n_Ξ.

The rank of $H_n(X_n)$ depends only on the number of roots ξ such that $\xi^n = 1$ (cf. 2.3) i.e. on $n \mod N$. More precisely, let X_n (resp. $\bar{X}_n$) denotes unramified (resp. ramified) cover of $\mathbb{P}^2 \setminus \mathcal{C}$ (resp. $\mathbb{P}^2$). Then $H_1(X_n, \mathbb{C}) \to H_1(X_{n_N}, \mathbb{C})$ (resp. $H_1(\bar{X}_n, \mathbb{C}) \to H_1(\bar{X}_{n_N}, \mathbb{C})$) are isomorphisms for all n belonging to one of the congruence class modulo N. For n not belonging to any of these congruence classes, one has $H_1(X_n, \mathbb{C}) = H_1(\bar{X}_n, \mathbb{C}) = 0$. Moreover the map $H_1(X_n, \mathbb{Z}) \to H_1(X_{n_N}, \mathbb{Z})$ (resp. $H_1(\bar{X}_n, \mathbb{Z}) \to H_1(\bar{X}_{n_N}, \mathbb{Z})$) is injective (resp. has finite kernel and co-kernel). Hence the isogeny class of Albanese variety of X_n with n in one and only one congruence class as above is constant. Hence the claims (1) and (2) follow. □

Remark 6.4. This result can be compared with results on periodicity properties of Betti and Hodge numbers. For a curve $\mathcal{C}$ in $\mathbb{C}^2$ which a union of r components, let $h_a^{1,0}(n)$ (resp. $h_c^{1,0}(n)$) denote the sequence of the Hodge numbers of a smooth compactification $X_a(n)$ (resp. $X_c(n)$) of abelian (resp. cyclic) covers of the complement in the tower of abelian (resp. cyclic) cover of $\mathbb{P}^2$ corresponding to surjection $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) \to \mathbb{Z}^r \to (\mathbb{Z}/n\mathbb{Z})^r$ (resp. surjection onto $\mathbb{Z}_n$ which is the composition of the latter surjection with summing up of the coordinates in $\mathbb{Z}^r/n\mathbb{Z}^r$).

It follows from [18] that the sequence $h_a^{1,0}(n)$ is polynomial periodic (similarly, [22] implies that sequence $h_c^{1,0}(n)$ is periodic). Recall that $n \to a(n) \in \mathbb{N}$ is polynomial periodic if there are periodic functions $a_i(n)$ such that $a(n) = \sum a_i(n) n^i$.

Now let $K_{\mathcal{AB}}$ be the K-group of motives of abelian varieties over $\mathbb{C}$ up to isogeny. More precisely this is the K-group of the category $\mathcal{AB}$ defined as follows. The objects of $\mathcal{AB}$ are abelian varieties over $\mathbb{C}$ and $Hom_{\mathcal{AB}}(A, A')$ is the group of homomorphism between A, A' tensored with $\mathbb{Q}$. The K-group $K_{\mathcal{AB}}$ is an (infinitely generated) $\mathbb{Z}$-module with canonical surjection $dim : K_{\mathcal{AB}} \to \mathbb{Z}$ given by $A \mapsto dimA$. The theorem 6.3 implies that the sequence $Alb(X_c(n)) \in K_{\mathcal{AB}}$ is periodic. However isogeny components of Albanese varieties of *abelian* possibly non-cyclic covers span an infinitely generated subgroup of the K-group of isogeny classes of abelian varieties.

In particular, there are no periodic functions $a_i(n) \in K_{\mathcal{AB}}$ such that $Alb(X_a(n)) = a_i(n)n^i$ (though as was mentioned $dimAlb(X_a(n))$ is polynomially periodic). Details of this will be presented elsewhere.

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