HODGE GENERA OF ALGEBRAIC VARIETIES, II.

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Abstract. We study the behavior of Hodge-theoretic genera under morphisms of complex algebraic varieties. We prove that the additive χ_y-genus which arises in the motivic context satisfies the so-called "stratified multiplicative property", which shows how to compute the invariant of the source of a proper surjective morphism from its values on various varieties that arise from the singularities of the map. By considering morphisms to a curve, we obtain a Hodge-theoretic analogue of the Riemann-Hurwitz formula. We also study the contribution of monodromy to the χ_y-genus of a smooth projective family, and prove an Atiyah-Meyer type formula for twisted χ_y-genera. This formula measures the deviation from multiplicativity of the χ_y-genus, and expresses the correction terms as higher-genera associated to cohomology classes of the quotient of the total period domain by the action of the monodromy group. By making use of Saito's theory of mixed Hodge modules, we also obtain formulae of Atiyah-Meyer type for the corresponding Hirzebruch characteristic classes.

Contents

Date: December 14, 2009.

2000 *Mathematics Subject Classification.* Primary 57R20, 32S20, 14C30, 32S35, 32S50, 14D05, 14D06, 14D07, 55N33; Secondary 57R45, 32S60, 13D15, 16E20.

A. Libgober partially supported by an NSF grant. S. Cappell and J. Shaneson partially supported by grants from DARPA.

1. Introduction

In the mid 1950's, Chern, Hirzebruch and Serre [14] proved that if $F \to E \to B$ is a fiber bundle of closed, oriented, topological manifolds such that the fundamental group of B acts trivially on the cohomology of F, then the signatures of the spaces involved are related by a simple multiplicative relation: $\sigma(E) = \sigma(F)\sigma(B)$. A decade later, Atiyah [2], and respectively Kodaira [30], observed that without the assumption on the action of the fundamental group the multiplicativity relation fails. Moreover, Atiyah showed that the deviation from multiplicativity is controlled by the cohomology of the fundamental group of B.

The main goal of this paper is to describe in a systematic way multiplicativity properties of the Hirzebruch χ_y-genus (and the associated characteristic classes) for fibrations of algebraic manifolds. We extend the results of Chern, Hirzebruch and Serre in several different directions. First, we show that in the case when the Chern-Hirzebruch-Serre assumption of the triviality of the monodromy action is fulfilled, the χ_y-genus is multiplicative. Since by the Hodge index theorem, the signature of a Kähler manifold is one of the values of the χ_y-genus, this theorem can be viewed as an extension of the Chern-Hirzebruch-Serre result in the algebraic case. Secondly, we consider fibrations with non-trivial monodromy action, and prove a Hodge-theoretic analogue of the Atiyah signature formula. We also derive a formula for the χ_y-genus of E in which the correction from the multiplicativity of the χ_y-genus is measured via pullbacks under the period map associated with our fibration of certain cohomology classes of the quotient of the period domain by the action of the monodromy group. For some manifolds F serving as a fiber of the fibration in discussion, as well as in the case when one is interested in the value of polynomial χ_y yielding the signature, the quotient of the period domain is the classifying space of the monodromy group and our correction terms coincide with those of Atiyah. In fact, the Atiyah terms are the appropriate Novikov type higher-signatures, and our correction terms are extensions of these Novikov invariants to the algebraic category. We indicate that recently established birational properties of higher genera [6, 7, 35] are valid also for "the corrections to multiplicativity" that we introduce here. Analogous formulae for the associated Hirzebruch characteristic classes are also discussed.

We now present in more detail the content of each section and summarize our main results.

In Section 2, we study the behavior of χ_y-genera under maps of complex algebraic varieties. We first consider a morphism $f : E \to B$ of complex algebraic varieties with B smooth and connected, which is a fiber bundle in the (strong) complex topology, and show that under certain assumptions on monodromy the χ_y-genera are multiplicative (cf. Lemma 2.6 and Lemma 2.7). Such multiplicativity properties of genera were previously studied in certain special cases in connection with rigidity (e.g., see [26, 27, 33]). For instance, Hirzebruch's χ_y-genus is multiplicative in bundles of (stably) almost complex manifolds with structure group a compact connected Lie group (the latter condition implies trivial monodromy), and in fact it is uniquely characterized by this property. The proof of our multiplicativity result uses the fact that the Leray spectral sequences of the map f are spectral sequences in the category of mixed Hodge structures. The latter claim is a consequence of Saito's theory of mixed Hodge modules (e.g., see [37]), and is discussed in some detail in §2.3.

In Section 2.5, we consider proper morphisms that are allowed to have singularities, and extend the above multiplicativity property to this general stratified case. More precisely, we prove that, under the assumption of trivial monodromy along the strata of our map, the additive χ_y-genus that arises in the motivic context satisfies the so-called "stratified multiplicative property" (cf. Proposition 2.11 and Corollary 2.13). This property shows how to compute the invariant of the source of a proper surjective morphism from its values on various varieties that arise from the singularities of the map, thus yielding powerful topological constraints on the singularities of any algebraic map. It also provides a method of inductively computing these genera of varieties. A similar result was obtained by Cappell, Maxim and Shaneson, for the behavior of intersection homology Hodge-theoretic invariants, both genera and characteristic classes (see [10], and also [9]). Such formulae were first predicted by Cappell and Shaneson in the early 1990's, see the announcements [12, 43], following their earlier work on stratified multiplicative properties for signatures and associated topological characteristic classes defined using intersection homology, cf. [11] (see also [[8], §4], and [46] for a functorial interpretation of Cappell-Shaneson's L-classes).

In the special case of maps to a smooth curve, and under certain assumptions for the monodromy along the strata of special fibers, in §3.2 we obtain a Hodge-theoretic analogue of the Riemann-Hurwitz formula [29]. The proof uses Hodge-theoretic aspects of the nearby and vanishing cycles in the context of one-parameter degenerations of projective varieties.

The contribution of monodromy to χ_y-genera is studied in Section 4. This can be applied to compute the summands arising from singularities in the above formulae. For simplicity, we first consider a smooth proper map $f : E \to B$ of smooth projective varieties (thus a fibration in the strong topology), and compute $\chi_y(E)$ so that the (monodromy) action of $\pi_1(B)$ on the cohomology of the typical fiber is taken into account (see Theorem 4.1). The proof uses the Hirzebruch-Riemann-Roch theorem [26] and standard facts from the classical Hodge theory. Our formula (4.2) is a Hodge-theoretic analogue of Atiyah's formula for the signature of fiber bundles [2], and measures the deviation from multiplicativity of the χ_y-genus in the presence of monodromy. In Section 4.3, this deviation is expressed in terms of higher-genera associated to cohomology classes of the quotient of the total period domain by the action of the monodromy group. As a corollary of our formula (4.2), we point out that if the action of $\pi_1(B)$ on the cohomology of the typical fiber F preserves the Hodge filtration, then the χ_y-genus is still multiplicative, i.e., $\chi_y(E) = \chi_y(B) \cdot \chi_y(F)$. This assertion is false in the non-compact case (see Example 2.9 (2)). As a byproduct of the proof of Theorem 4.1, we obtain a Hodge-theoretic analogue of Meyer's formula for twisted signatures [31] (see Corollary 4.6). At the end of Section 4, we present several interesting extensions of these Hodge-theoretic Atiyah-Meyer formulae to more general situations, where we allow the spaces involved to be singular and/or non-compact.

In Section 5, we extend some of the above mentioned results on χ_y-genera to Atiyah-Meyer type formulae for the corresponding Hirzebruch characteristic classes. The proofs are much more involved, and use in an essential way the construction of Hirzebruch classes via Saito's theory of mixed Hodge modules (cf. [8]). At this end, we point out that many of our formulae for genera can be obtained from the corresponding formulae for characteristic classes. However, we chose to present the results for genera first, since they can be proven by using only standard facts of classical Hodge theory, thus becoming accessible to a wider audience.

It is conceivable that many of our results remain valid in a more general context (e.g., for compactifiable complex analytic varieties). However, for simplicity, we present here our Atiyah-Meyer type results in the algebraic setting.

We have tried to make this paper as self-contained as possible. For this reason, in §2.2 we provide necessary background on Saito's theory of mixed Hodge modules, and in §3.1 we recall Deligne's formalism of nearby and vanishing cycles. However, we assume reader's familiarity with certain aspects of Deligne's Hodge theory ([17, 34]).

In a future paper, we will consider extensions of our monodromy formulae to the singular setting, both for genera and characteristic classes. One possible such extension makes use of intersection homology [22, 23] and the BBDG decomposition theorem [5, 15]. This approach is motivated by the considerations in [10] (where the case of trivial monodromy was considered), and by an extension of the Atiyah-Meyer signature formula to the singular case, which is due to Banagl, Cappell and Shaneson [3].

Acknowledgements. *We are grateful to Jörg Schürmann for reading a first draft of this work, and for making many valuable comments and suggestions for improvement. We thank Mark Andrea de Cataldo, Alexandru Dimca and Nero Budur for many inspiring conversations on this subject, and Joseph Steenbrink and Shoji Yokura for commenting on a preliminary version of this work.*

2. HODGE GENERA AND SINGULARITIES OF MAPS

2.1. **Hodge genera. Definitions.** In this section, we define the Hodge-theoretic invariants of complex algebraic varieties, which will be studied in the sequel. We assume reader's familiarity with Deligne's theory of mixed Hodge structures [17].

For any complex algebraic variety Z, we define the χ_y^c-genus in terms of the Hodge-Deligne numbers of compactly supported cohomology of Z (cf. [16]). More precisely,

$$\chi_y^c(Z) = \sum_p \left(\sum_{i,q} (-1)^{i-p} h^{p,q}(H_c^i(Z;\mathbb{C})) \right) y^p = \sum_{i,p \geq 0} (-1)^{i-p} \dim_{\mathbb{C}} Gr_F^p H_c^i(Z,\mathbb{C}) \cdot y^p,$$

where $h^{p,q}(H_c^i(Z;\mathbb{C})) = \dim_{\mathbb{C}} Gr_F^p(Gr_{p+q}^W H_c^i(Z) \otimes \mathbb{C})$, with $F^\bullet$ and $W_\bullet$ the Hodge and respectively the weight filtration of Deligne's mixed Hodge structure on $H_c^i(Z)$. Similarly, we define the χ_y-genus of Z, $\chi_y(Z)$, by using the Hodge-Deligne numbers of $H^*(Z;\mathbb{C})$. Of course, for a complete variety Z we have that $\chi_y(Z) = \chi_y^c(Z)$. If Z is smooth and projective, then each cohomology group $H^i_c(Z;\mathbb{C}) = H^i(Z;\mathbb{C})$ has a pure Hodge structure of weight i, and the above formulae define Hirzebruch's χ_y-genus (cf. [26]). Note that for any complex variety Z, we have that $\chi^c_{-1}(Z) = \chi_{-1}(Z) = \chi(Z)$ is the usual Euler characteristic, where for the first equality we refer to [21], pp. 141-142. Similarly, χ_0 and χ_0^c are two possible extensions to singular varieties of the arithmetic genus.

The compactly supported χ_y-genus, χ_y^c, satisfies the so-called "scissor relations" for complex varieties, that is: $\chi_y^c(Z) = \chi_y^c(W) + \chi_y^c(Z \setminus W)$, for W a closed subvariety of Z. Therefore, χ_y^c can be defined on $K_0(Var_{\mathbb{C}})$, the Grothendieck group of varieties over $\mathbb{C}$ which arises in the motivic context.

More generally, we can define χ_y-genera on the Grothendieck group of mixed Hodge structures $K_0(mhs) = K_0(D^b mhs)$, where we denote by mhs the abelian category of (rational) mixed Hodge structures. Indeed, if $K \in mhs$, define

$$\chi_y([K]) := \sum_p \dim_{\mathbb{C}} Gr_F^p(K \otimes \mathbb{C}) \cdot (-y)^p, \tag{2.1}$$

where $[K]$ is the class of K in $K_0(mhs)$. This is well-defined on $K_0(mhs)$ since the functor Gr^p_F preserves exactness. For $K^\bullet$ a bounded complex of mixed Hodge structures, we define

$$[K^\bullet] := \sum_{i\in\mathbb{Z}} (-1)^i [K^i] \quad \in K_0(mhs)$$

and note that we have:

$$[K^\bullet] = \sum_{i\in\mathbb{Z}} (-1)^i [H^i(K^\bullet)].$$

In view of (2.1), we set

$$\chi_y([K^\bullet]) := \sum_{i\in\mathbb{Z}} (-1)^i \chi_y([K^i]). \tag{2.2}$$

In this language, we have that:

$$\chi_y^c(Z) = \chi_y([H_c^\bullet(Z;\mathbb{Q})])$$

and

$$\chi_y(Z) = \chi_y([H^\bullet(Z;\mathbb{Q})]),$$

where $H_c^\bullet(Z;\mathbb{Q})$ and $H^\bullet(Z;\mathbb{Q})$ are regarded as bounded complexes of mixed Hodge structures, with all differentials equal to zero.

2.2. **Basics of Saito's theory of mixed Hodge modules.**

Even though the theory of mixed Hodge modules is very involved, in this section we give a brief overview adapted to our needs, and we show some quick applications in the following sections. The standard references for the algebraic case are Saito's papers [38] and [37], §4, but see also the book [34], Chapter 14, for a brief survey.

We recall that for any complex algebraic variety Z, the derived category of bounded cohomologically constructible complexes of sheaves of $\mathbb{Q}$-vector spaces on Z is denoted by

$D^b_c(Z)$, and it contains as a full subcategory the category $\mathrm{Perv}_{\mathbb{Q}}(Z)$ of perverse $\mathbb{Q}$-complexes. The Verdier duality operator $\mathbb{D}_Z$ is an involution on $D^b_c(Z)$ preserving $\mathrm{Perv}_{\mathbb{Q}}(Z)$. Associated to a morphism $f : X \to Y$ of complex algebraic varieties, there are pairs of adjoint functors (f^*, Rf_*) and $(f^!, Rf_!)$ between the respective categories of cohomologically constructible complexes, which are interchanged by Verdier duality. For details, see the books [20, 40].

M. Saito associated to a complex algebraic variety Z an abelian category $MHM(Z)$, the category of *mixed Hodge modules* on Z, together with a faithful forgetful functor

$$rat : D^bMHM(Z) \to D^b_c(Z)$$

such that $rat(MHM(Z)) \subset \mathrm{Perv}_{\mathbb{Q}}(Z)$. For $M^\bullet \in D^bMHM(Z)$, $rat(M^\bullet)$ is called the underlying rational complex of $M^\bullet$.

We say that $M \in MHM(Z)$ is supported on S if and only if $rat(M)$ is supported on S. Saito showed that the category of mixed Hodge modules supported on a point coincides with the category of (graded) polarizable rational mixed Hodge structures. In this case, the functor rat associates to a mixed Hodge structure the underlying rational vector space.

Since $MHM(Z)$ is an abelian category, the cohomology groups of any complex $M^\bullet \in D^bMHM(Z)$ are mixed Hodge modules. The underlying rational complexes of the cohomology groups of a complex of mixed Hodge modules are the perverse cohomologies of the underlying rational complex, that is, $rat(H^j(M^\bullet)) = {}^p\mathcal{H}^j(rat(M^\bullet))$.

The Verdier duality functor $\mathbb{D}_Z$ lifts to $MHM(Z)$ as an involution, in the sense that it commutes with the forgetful functor: $rat \circ \mathbb{D}_Z = \mathbb{D}_Z \circ rat$.

For a morphism $f : X \to Y$ of complex algebraic varieties, there are induced functors $f_*, f_! : D^bMHM(X) \to D^bMHM(Y)$ and $f^*, f^! : D^bMHM(Y) \to D^bMHM(X)$, exchanged under the Verdier duality functor, and which lift the analogous functors on the level of constructible complexes. Moreover, if f is proper, then $f_! = f_*$.

Let us give a rough picture of what the objects in Saito's category of mixed Hodge modules look like. For Z *smooth*, $MHM(Z)$ is a full subcategory of the category of objects $((M, F), K, W)$ such that:

1. (M, F) is an algebraic holonomic filtered $\mathcal{D}$-module M on Z, with an increasing "Hodge" filtration F by coherent algebraic $\mathcal{O}_Z$-modules;
2. $K \in \mathrm{Perv}_{\mathbb{Q}}(Z)$ is the underlying rational sheaf complex, and there is a quasi-isomorphism $\alpha : DR(M) \simeq \mathbb{C} \otimes K$ in $\mathrm{Perv}_{\mathbb{C}}(Z)$, where DR is the de Rham functor shifted by the dimension of Z;
3. W is a pair of (weight) filtrations on M and K compatible with α.

For a singular variety Z, one works with local embeddings into manifolds and corresponding filtered $\mathcal{D}$-modules with support on Z. In this notation, the functor rat is defined by $rat((M, F), K, W) = K$.

A complex $M^\bullet \in D^bMHM(Z)$ is *mixed of weight* $\leq k$ *(resp.* $\geq k$*)* if $Gr^W_i H^j M^\bullet = 0$ for all $i > j + k$ (resp. $i < j + k$), and it is *pure of weight* k if $Gr^W_i H^j M^\bullet = 0$ for all $i \neq j + k$. If f is a map of algebraic varieties, then $f_!$ and f^* preserve weight $\leq k$, and f_* and $f^!$ preserve weight $\geq k$. If $M^\bullet \in D^bMHM(Z)$ is of weight $\leq k$ (resp. $\geq k$), then $H^j M^\bullet$ has weight $\leq j + k$ (resp. $\geq j + k$).

If $j : U \hookrightarrow Z$ is a Zariski-open subset in Z, then the intermediate extension $j_{!*}$ (cf. [5]) preserves the weights.

Following [37], there exists a unique object $\mathbb{Q}^H \in MHM(point)$ such that $rat(\mathbb{Q}^H) = \mathbb{Q}$ and $\mathbb{Q}^H$ is of type $(0,0)$. In fact, $\mathbb{Q}^H = ((\mathbb{C}, F), \mathbb{Q}, W)$, with $gr^F_i = 0 = gr^W_i$ for all $i \neq 0$, and $\alpha : \mathbb{C} \to \mathbb{C} \otimes \mathbb{Q}$ the obvious isomorphism. For a complex variety Z, define $\mathbb{Q}^H_Z := a^*_Z \mathbb{Q}^H \in D^b MHM(Z)$ with $a_Z : Z \to point$ the map to a point. If Z is *smooth* of dimension n, then $\mathbb{Q}_Z[n] \in Perv(\mathbb{Q}_X)$ and $\mathbb{Q}^H_Z[n] \in MHM(Z)$ is a single mixed Hodge module (in degree 0), explicitely described by $\mathbb{Q}^H_Z[n] = ((\mathcal{O}_Z, F), \mathbb{Q}_Z[n], W)$, where F and W are trivial filtrations so that $gr^F_i = 0 = gr^W_{i+n}$ for all $i \neq 0$. So if Z is smooth of dimension n, then $\mathbb{Q}^H_Z[n]$ is pure of weight n. By the stability of the intermediate extension functor, this shows that if Z is any algebraic variety and $j : U \hookrightarrow Z$ is the inclusion of a smooth Zariski-open subset, then the intersection cohomology module $IC^H_Z := j_{!*}(\mathbb{Q}^H_U[n])$ is pure of weight n.

More generally, if $\mathbb{V}$ is a polarized variation of Hodge structures of weight k with quasi-unipotent monodromy at infinity[1] defined on a Zariski-open subset U of Z, then $\mathbb{V}$ corresponds to a smooth mixed Hodge module $\mathbb{V}^H[n]$ on U (i.e., the associated rational complex is a local system) of weight $k+n$, whose underlying perverse sheaf is $\mathbb{V}[n]$ (see [36], Thm. 5.4.3, [38], §2, [34], Thm. 14.30). So the twisted (middle-perversity) intersection homology complex $IC^H_Z(\mathbb{V}) := j_{!*}(\mathbb{V}^H[n])$ is a mixed Hodge module, pure of weight $k+n$.

The following result is implicit in the work of Saito.

Proposition 2.1. *Let Z be a n-dimensional irreducible projective variety, and $\mathbb{V}$ a polarized variation of Hodge structures of weight k (with quasi-unipotent monodromy at infinity) defined on a Zariski-open dense subset of Z. Then the intersection cohomology group $IH^j(Z; \mathbb{V})$ carries a pure Hodge structure of weight $j+k$.*

Proof. Let $a_Z : Z \to point$ be the constant map to a point. Since a_Z is proper, a_{Z*} preserves the weights. The claim follows from the isomorphism

$$IH^j(Z; \mathbb{V}) \cong \mathbb{H}^{j-n}(Z; IC^H_Z(\mathbb{V})) \cong H^{j-n}(a_{Z*} IC^H_Z(\mathbb{V})),$$

by noting that $a_{Z*} IC^H_Z(\mathbb{V})$ is a pure complex of weight $n+k$, thus $H^{j-n}(a_{Z*} IC^H_Z(\mathbb{V}))$ is a pure Hodge module of weight $j+k$ supported over a point. Since over a point mixed Hodge modules are exactly the (graded) polarizable mixed Hodge structures, the latter is a Hodge structure of weight $j+k$.

□

The following corollary will be needed in the sequel:

Corollary 2.2. *Let Z be a smooth complex projective algebraic variety, and $\mathbb{V}$ a polarized variation of Hodge structures of weight k defined on Z. Then the cohomology group $H^j(Z; \mathbb{V})$ carries a pure Hodge structure of weight $j+k$.*

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[1] This condition is automatically satisfied if $\mathbb{V}$ is a variation of $\mathbb{Z}$-Hodge structures, by the monodromy theorem, [34], Thm. 11.8.

Remark 2.3. More generally, if the variety Z in Proposition 2.1 and Corollary 2.2 is not necessarily compact, and if $\mathbb{V}$ is a polarized variation of Hodge structures or, more generally, an admissible variation of mixed Hodge structures (for a definition, see [34], Def. 14.47 and the references therein) on (a Zariski-open dense subset of) Z, then the associated (intersection) cohomology groups carry natural mixed Hodge structures.

2.3. **Spectral sequences of mixed Hodge modules.** In this section, we justify the claim that certain Leray-type spectral sequences (e.g., the Leray spectral sequence of an algebraic morphism, or the hypercohomology spectral sequence) are in fact spectral sequences of mixed Hodge structures.

From the general theory of spectral sequences, since the category of mixed Hodge modules is abelian, the canonical filtration τ on $D^bMHM(Z)$ preserves complexes of mixed Hodge modules. Therefore, the second fundamental spectral sequence ([34], §A.3.4) for any (left exact) functor F sending mixed Hodge modules to mixed Hodge modules, that is, the spectral sequence

$$E_2^{p,q} = H^pF(H^q(M^\bullet)) \Longrightarrow H^{p+q}F(M^\bullet), \tag{2.3}$$

is a spectral sequences of mixed Hodge modules.

Note that the canonical t-structure τ on $D^bMHM(Z)$ corresponds to the perverse truncation ${}^p\tau$ on $D^b_c(Z)$. However, Saito ([37], Remark 4.6(2)) constructed another t-structure ${}'\tau$ on $D^bMHM(Z)$ that corresponds to the classical t-structure on $D^b_c(Z)$. By using the t-structure ${}'\tau$ in the construction of the second fundamental spectral sequence above, one can show that the classical Leray spectral sequences are in fact spectral sequences of mixed Hodge structures.

Example 2.4. *Leray spectral sequences.*
(1) Let Z be a complex algebraic variety. Then for $\mathcal{F}^\bullet$ a bounded complex of sheaves with constructible cohomology on Z, we have the spectral sequence with the E_2-term given by

$$E_2^{p,q} = H^p(Z;\mathcal{H}^q(\mathcal{F}^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(Z;\mathcal{F}^\bullet). \tag{2.4}$$

This spectral sequence is induced by the natural filtration on the complex $\mathcal{F}^\bullet$, and if $\mathcal{F}^\bullet$ underlies a complex of mixed Hodge modules, then the spectral sequence is compatible with mixed Hodge structures: this follows by using the t-structure ${}'\tau$ defined in [37], Remark 4.6(2), and the fundamental spectral sequence (2.3) for $F = \Gamma(Z,\cdot) = (a_Z)_*$, together with the fact that mixed Hodge modules over a point are (graded polarizable) mixed Hodge structures.

Similarly, by taking $F = \Gamma_c(Z,\cdot) = (a_Z)_!$ together with the t-structure ${}'\tau$ above, the compactly supported hypercohomology Leray spectral sequence

$$E_2^{p,q} = H^p_c(Z;\mathcal{H}^q(\mathcal{F}^\bullet)) \Longrightarrow \mathbb{H}^{p+q}_c(Z;\mathcal{F}^\bullet). \tag{2.5}$$

is a spectral sequence in the category of mixed Hodge structures, provided $\mathcal{F}^\bullet$ underlies a bounded complex of mixed Hodge modules.

(2) Let $f : E \to B$ be a morphism of complex algebraic varieties. The Leray spectral sequence for f

$$E_2^{p,q} = H^p(B, R^q f_*\mathbb{Q}_E) \Longrightarrow H^{p+q}(E) \tag{2.6}$$

is a spectral sequence of mixed Hodge structures: this follows by using $'\tau$ on $D^bMHM(B)$, and the example above applied to $\mathcal{F}^\bullet = Rf_*\mathbb{Q}_E$, that underlies $M^\bullet = f_*\mathbb{Q}_E^H \in D^bMHM(B)$. Similarly, there is a compactly supported version of the Leray spectral sequence for f in the category of mixed Hodge structures, namely

$$E_2^{p,q} = H_c^p(B, R^q f_!\mathbb{Q}_E) \Longrightarrow H_c^{p+q}(E). \tag{2.7}$$

By using the usual t-structure on $D^bMHM(B)$, hence the perverse t-structure on $D_c^b(B)$, we obtain that the perverse Leray spectral sequence for f,

$$E_2^{i,j} = \mathbb{H}^i(B, {}^p\mathcal{H}^j(f_*\mathcal{F}^\bullet)) \Longrightarrow \mathbb{H}^{i+j}(E; \mathcal{F}^\bullet) \tag{2.8}$$

is a spectral sequence in the category of mixed Hodge structures, provided $\mathcal{F}^\bullet$ underlies a bounded complex of mixed Hodge modules.

Remark 2.5. In the quasi-projective setting, a different proof of the fact that the spectral sequences (2.6) and (2.7) are spectral sequences of mixed Hodge structures was given by Arapura in [1].

2.4. **Multiplicativity properties of χ_y-genera.** In this section we use the above spectral sequences in studying multiplicative properties of the Hodge-theoretic genera. We first prove the following:

Lemma 2.6. *Let E, B, F be complex algebraic varieties with B smooth and connected, and let $p : E \to B$ be an algebraic morphism such that p is locally trivial in the strong (complex) topology of B, with fiber F. Assume that the local systems $R^j p_*\mathbb{C}_E$ are constant for each j (e.g., $\pi_1(B) = 0$). Then*

$$\chi_y(E) = \chi_y(B)\chi_y(F).$$

If moreover, all E, B and F are smooth, then

$$\chi_y^c(E) = \chi_y^c(B)\chi_y^c(F).$$

Proof. Consider the Leray spectral sequence of p, that is

$$E_2^{p,q} = H^p(B, R^q p_*\mathbb{C}_E) \Longrightarrow H^{p+q}(E)$$

In the category of algebraic varieties this is a spectral sequence of mixed Hodge structures (cf. §2.3).

Since the local systems $R^q p_*\mathbb{C}_E$ are constant, the corresponding variations of mixed Hodge structures are trivial as the natural morphism $H^0(B, R^q p_*\mathbb{C}_E) \to (R^q p_*\mathbb{C}_E)_y$ is a mixed Hodge structure isomorphism for any $y \in B$. Then we have $H^p(B, R^q p_*\mathbb{C}_E) = H^p(B) \otimes (R^q p_*\mathbb{C}_E)_y$ as mixed Hodge structures, and there are mixed Hodge structure isomorphisms

$$E_2^{p,q} = H^p(B) \otimes H^q(F) \tag{2.9}$$

Since all differentials in the Leray spectral sequence are mixed Hodge structure morphisms, thus strict with respect to the Hodge and weight filtrations, by [17], Lemme 1.1.11 we get a spectral sequence for the Hodge components of a given type (k, l):

$$E(k,l)_2^{p,q} := Gr_F^k Gr_{k+l}^W E_2^{p,q} \Longrightarrow Gr_F^k Gr_{k+l}^W H^{p+q}(E) \tag{2.10}$$

Now let $e^{k,l}$ be the Euler characteristic of Hodge-type (k, l), i.e., for a complex algebraic variety Z we define

$$e^{k,l}(Z) = \sum_i (-1)^i h^{k,l}(H^i(Z)).$$

By the invariance of Euler characteristics under spectral sequences, from (2.9) and (2.10) we obtain

$$\begin{aligned} e^{k,l}(E) &= \sum_i (-1)^i \dim\left(\oplus_{p+q=i} E(k,l)_2^{p,q}\right) \\ &= \sum_{r+t=k, s+u=l} e^{r,s}(B) e^{t,u}(F). \end{aligned}$$

The multiplicativity of χ_y follows by noting that for a variety Z,

$$\chi_y(Z) = \sum_{k,l} e^{k,l}(Z) \cdot (-y)^k.$$

The claim about the multiplicativity of the χ_y^c-genus follows by Poincaré Duality, by noting that if Z is a smooth complex algebraic variety of dimension n, then $\chi_y^c(Z) = (-y)^n \chi_{y^{-1}}(Z)$. Indeed, the Poincaré duality isomorphism takes classes of type (p, q) in $H_c^j(Z)$ to classes of type $(n-p, n-q)$ in $H^{2n-j}(Z)$, where $n = \dim Z$.

□

By noting that for any $b \in B$, $(R^j p_! \mathbb{C}_E)_b \cong H_c^j(F; \mathbb{C})$, a similar argument applied to the compactly supported Leray spectral sequence (2.7) of the map p, yields the following

Lemma 2.7. *Let E, B, F be complex algebraic varieties with B smooth and connected, and consider $p : E \to B$ an algebraic morphism such that p is locally trivial in the strong (complex) topology of B with fibre F. Assume that the local systems $R^j p_! \mathbb{C}_E$ are constant for each j (e.g., $\pi_1(B) = 0$). Then*

$$\chi_y^c(E) = \chi_y^c(B) \chi_y^c(F).$$

Remark 2.8. The same argument can be used to show that the results of the above lemmas hold for the Hodge-Deligne polynomials (or the E-functions) defined by $E(Z; u, v) = \sum_{k,l} e^{k,l}(Z) u^k v^l$ (and similarly for the E-functions $E_c(Z; u, v)$ defined by using the compactly supported cohomology).[2] In particular, these results hold for the weight polynomials $W(Z; t) := E(Z; t, t)$ and respectively $W_c(Z; t) := E_c(Z; t, t)$ considered in [19]. In fact, Lemma 2.6 is modeled after [19], Theorem 6.1.

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[2] Note that $\chi_y(Z) = E(Z; -y, 1)$, and similarly, $\chi_y^c(Z) = E_c(Z; -y, 1)$.

Example 2.9. (1) As an example, consider the case of the Hopf fibration defining $\mathbb{CP}^n$. Then $\chi_y(\mathbb{CP}^n) = \chi_y^c(\mathbb{CP}^n) = 1 + (-y) + \cdots + (-y)^n$, $\chi_y^c(\mathbb{C}^{n+1} \setminus \{0\}) = (-y)^{n+1} - 1$, $\chi_y^c(\mathbb{C}^*) = -y - 1$, and by Poincaré Duality, $\chi_y(\mathbb{C}^{n+1} \setminus \{0\}) = 1 - (-y)^{n+1}$, $\chi_y(\mathbb{C}^*) = 1 + y$. Thus the multiplicativity for both χ_y and χ_y^c holds.

(2) Let p be the Milnor fibration of a weighted homogeneous isolated hypersurface singularity at the origin in $\mathbb{C}^{n+1}$, that is, $F = \{p = 1\} \hookrightarrow E = \mathbb{C}^{n+1} \setminus \{p = 0\} \to B = \mathbb{C}^*$, for p a weighted homogeneous polynomial in $n + 1$ variables, with an isolated singular point at the origin. In this case, the monodromy is an algebraic morphism of finite order, so it preserves the filtrations in $H^*(F; \mathbb{C})$. In other words, each $R^j p_* \mathbb{Q}_E$ is a (non-constant) local system of mixed Hodge structures (note that this is not the case for general singularities). The mixed Hodge structure on $H^*(F; \mathbb{Q})$ is known by work of Steenbrink [44]. It turns out that even in this special case, the χ_y-genera are *not* multiplicative (but see Section §4 for the compact case). Here is a concrete example: let $p(x, y) = x^3 - y^2$ defining the cuspidal cubic in $\mathbb{C}^2$. Then, in the notations above and by [44], the (mixed) Hodge numbers of F are $h^{0,0}(H^0(F)) = 1$, $h^{1,0}(H^1(F)) = 1$, $h^{0,1}(H^1(F)) = 1$ and $h^{1,1}(H^1(F)) = 0$ (note that $H^2(F) = 0$ since F is affine of complex dimension 1). Therefore, we obtain that $\chi_y(F) = y$, so by Poincaré Duality it follows that $\chi_y^c(F) = (-y) \cdot \chi_{y^{-1}}(F) = -1$. It also follows easily that $\chi_y^c(E) = y^2 + y$, $\chi_y^c(B) = -y - 1$, and $\chi_y(E) = 1 + y$, $\chi_y(B) = 1 + y$.

Remark 2.10. The assumption of trivial monodromy is closely related to, but different from the situation of "algebraic piecewise trivial" maps coming up in the motivic context (e.g., see [8]). For example, the formula in Lemma 2.7 is true (without any assumption on monodromy) for a Zariski locally trivial fibration of possibly singular complex algebraic varieties (see [16], Corollary 1.9, or [8], Example 3.3). Note also that if all spaces involved are smooth, then a Zariski locally trivial fibration is a locally trivial fibration in the complex (strong) topology, and the monodromy action is trivial.

2.5. χ_y^c-genera and singularities of maps.

In this section, by analogy with the results of [9, 10], we discuss the behavior of the χ_y^c-genus under proper morphisms of algebraic varieties, and show that χ_y^c satisfies the *stratified multiplicative property* in the sense of [12]. The result will be further refined in §3, in the case of maps onto curves.

Let $f : X^n \to Y^m$ be a proper map of complex algebraic varieties of indicated dimensions. Such a map can be stratified with subvarieties as strata, i.e., there exist finite algebraic Whitney stratifications $\mathcal{X}$ of X and $\mathcal{S}$ of Y, such that for any component S of a stratum of Y, $f^{-1}(S)$ is a union of connected components of strata of $\mathcal{X}$, each of which is mapping submersively to S. This implies that $f_{|f^{-1}(S)} : f^{-1}(S) \to S$ is a locally trivial map of Whitney stratified spaces (see [24], §I.1.6). For simplicity, we will assume that f is smooth over the dense open stratum (e.g., the latter is connected). We denote by F the general fiber of f, and by F_S the fiber of f above the singular stratum $S \in \mathcal{S}$.

We can now prove the following formula, showing the deviation from multiplicativity of the χ_y^c-genus in the case of a stratified map (compare with results in [9, 10]):

Proposition 2.11. *Let $f : X \to Y$ be a proper surjective morphism of (possibly singular) complex algebraic varieties. Let $\mathcal{S}$ be the set of components of open strata of Y in a stratification of f, and assume $\pi_1(S) = 0$ for all $S \in \mathcal{S}$. For $S \in \mathcal{S}$, define $\hat{\chi_y^c}(\bar{S})$ inductively by the formula:*

$$\hat{\chi_y^c}(\bar{S}) = \chi_y^c(\bar{S}) - \sum_{W<S} \hat{\chi_y^c}(\bar{W}),$$

where the sum is over all $W \in \mathcal{S}$ with $W \subset \bar{S} \setminus S$. Then:

$$\chi_y^c(X) = \chi_y^c(Y)\chi_y^c(F) + \sum_{S\in\mathcal{S}, dimS<dimY} \hat{\chi_y^c}(\bar{S}) \cdot \left[\chi_y^c(F_S) - \chi_y^c(F)\right]. \tag{2.11}$$

Proof. Note that by the additivity of χ_y^c-genera, we have that $\hat{\chi_y^c}(\bar{S}) = \chi_y^c(S)$. Next, by additivity and multiplicativity for locally trivial fibrations with trivial monodromy (cf. Lemma 2.7), it is easy to see that:

$$\chi_y^c(X) = \chi_y^c(Y)\chi_y^c(F) + \sum_{S\in\mathcal{S}, \mathrm{dim}S<\mathrm{dim}Y} \chi_y^c(S) \cdot \left[\chi_y^c(F_S) - \chi_y^c(F)\right].$$

The result follows.

□

Example 2.12. *Smooth blow-up*
Let Y be a smooth subvariety of codimension $r+1$ in a smooth variety X. Let $\pi : \tilde{X} \to X$ be the blow-up of X along Y. Then π is an isomorphism over $X \setminus Y$ and a projective bundle (Zariski locally trivial) over Y, coresponding to the normal bundle of Y in X of rank $r+1$. The formula in Proposition 2.11 yields

$$\chi_y^c(\tilde{X}) = \chi_y^c(X) + \chi_y^c(Y) \cdot \left(-y + \cdots + (-y)^r\right). \tag{2.12}$$

Note that this formula also holds without any assumption on monodromy, by using instead Remark 2.10 and the fact that $\pi^{-1}(Y)$ is a Zariski locally trivial fibration over Y with fiber $\mathbb{CP}^r$ (see [16], §1.10).

In the notations of Proposition 2.11, we also obtain the following fact which was claimed in [10]:

Corollary 2.13. *Let $f : X \to Y$ be a proper surjective morphism of complex projective algebraic varieties, and assume $\pi_1(S) = 0$ for all $S \in \mathcal{S}$. Then with the obvious definition for $\hat{\chi}_y(\bar{S})$, the following holds:*

$$\chi_y(X) = \chi_y(Y)\chi_y(F) + \sum_{S\in\mathcal{S}, dimS<dimY} \hat{\chi}_y(\bar{S}) \cdot \left[\chi_y(F_S) - \chi_y(F)\right]. \tag{2.13}$$

Remark 2.14. More generally, by Remark 2.8 and additivity, both formulae (2.11) and (2.13) above are satisfied by the Hodge-Deligne polynomial $E_c(-; u, v)$ defined by means of compactly supported cohomology. In other words, the polynomial $E_c(-; u, v)$ satisfies the stratified multiplicative property.

2.6. χ_y**-genera and mixed Hodge modules.** In this section, by using Saito's theory of mixed Hodge modules we derive some easy additivity properties of χ_y-genera of complexes of mixed Hodge structures. We begin with a consequence of the fact that mixed Hodge modules over a point are just (graded-polarizable) mixed Hodge structures:

Lemma 2.15. *Let Z be a complex algebraic variety, and $a_Z : Z \to pt$ be the constant map to the point. For any bounded complex $M^\bullet$ of mixed Hodge modules on Z*

$$\mathbb{H}^p(Z; M^\bullet) := H^p((a_Z)_* M^\bullet) \quad \textit{and} \quad \mathbb{H}^p_c(Z; M^\bullet) := H^p((a_Z)_! M^\bullet)$$

are mixed Hodge structures. In particular, if M is a mixed Hodge module on Z whose underlying rational complex is the perverse sheaf $\mathcal{F}^\bullet$, then the hypercohomology group $\mathbb{H}^p(Z; \mathcal{F}^\bullet)$ (and resp. $\mathbb{H}^p_c(Z; \mathcal{F}^\bullet)$) is the rational vector space underlying $\mathbb{H}^p(Z; M)$ (and resp. $\mathbb{H}^p_c(Z; M)$), hence the former is a rational mixed Hodge structure.

As a corollary, we obtain some very useful facts for the global-to-local study of χ_y-genera. If Z is a complex algebraic variety, and $i : Y \hookrightarrow Z$ is a closed immersion, with $j : U \to Z$ the inclusion of the open complement, then there is a functorial distinguished triangle for $M^\bullet \in D^b MHM(Z)$, lifting the corresponding one from $D^b_c(Z)$ (cf [37], p. 321):

$$j_! j^* M^\bullet \to M^\bullet \to i_* i^* M^\bullet \xrightarrow{[1]} \tag{2.14}$$

In particular, by taking hypercohomology with compact supports in (2.14) and together with Lemma 2.15, we obtain that the following long exact sequence is a sequence in the category of mixed Hodge structures:

$$\cdots \to \mathbb{H}^p_c(U; j^* M^\bullet) \to \mathbb{H}^p_c(Z; M^\bullet) \to \mathbb{H}^p_c(Y; i^* M^\bullet) \to \cdots \tag{2.15}$$

Therefore,

$$\chi_y([\mathbb{H}^\bullet_c(Z; M^\bullet)]) = \chi_y([\mathbb{H}^\bullet_c(Y; M^\bullet)]) + \chi_y([\mathbb{H}^\bullet_c(U; M^\bullet)]). \tag{2.16}$$

As a corollary of (2.16), by induction on strata we obtain the following additivity property:

Corollary 2.16. *Let $\mathcal{S}$ be the set of components of strata of an algebraic Whitney stratification of the complex algebraic variety Z. Then for any $M^\bullet \in D^b MHM(Z)$,*

$$\chi_y([\mathbb{H}^\bullet_c(Z; M^\bullet)]) = \sum_{S \in \mathcal{S}} \chi_y([\mathbb{H}^\bullet_c(S; M^\bullet)]). \tag{2.17}$$

Remark 2.17. By taking $M^\bullet = \mathbb{Q}^H_Z$ in (2.17), we obtain the usual additivity of the χ^c_y-genus (recall that χ^c_y is defined on the Grothendieck group of complex varieties). This is a consequence of the fact that Deligne's and Saito's mixed Hodge structures on cohomology (with compact support) coincide. The latter assertion can be seen by construction if the variety can be embedded into a manifold, but in general it is a very deep result of Saito, see [39].

Each of the terms in the sum of the right-hand side of equation (2.17) can be further computed by means of the Leray spectral sequence for hypercohomology (cf. §2.3).

Let Z be an algebraic variety, $M^\bullet \in D^bMHM(Z)$, and fix an algebraic Whitney stratification with respect to which $\mathcal{F}^\bullet := rat(M^\bullet) \in D^b_c(Z)$ has constructible cohomology. Let $\mathcal{S}$ be the set of components of strata, and fix $S \in \mathcal{S}$. Then S is a non-singular complex algebraic variety. Moreover, each cohomology sheaf $\mathcal{H}^q(\mathcal{F}^\bullet)$ on S is a local system underlying a variation of mixed Hodge structures. We first prove the following extension of Lemmas 2.6 and 2.7 (in fact, the latter follow from this for some distinguished choices of $M^\bullet$):

Proposition 2.18. *Assume the local systems $\mathcal{H}^j(\mathcal{F}^\bullet)$ are constant on S for each $j \in \mathbb{Z}$, e.g. $\pi_1(S) = 0$. Then*

$$\chi_y([\mathbb{H}^\bullet(S; M^\bullet)]) = \chi_y(S) \cdot \chi_y([\mathcal{H}^\bullet(\mathcal{F}^\bullet)_x]) \tag{2.18}$$

and

$$\chi_y([\mathbb{H}^\bullet_c(S; M^\bullet)]) = \chi^c_y(S) \cdot \chi_y([\mathcal{H}^\bullet(\mathcal{F}^\bullet)_x]), \tag{2.19}$$

where $[\mathcal{H}^\bullet(\mathcal{F}^\bullet)_x] \in K_0(mhs)$ is the complex (with all differentials zero) of mixed Hodge structures corresponding to stalk cohomologies of $\mathcal{F}^\bullet$ at a point in S.

Proof. This is a consequence of the fact that the spectral sequences (2.4) and (2.5), applied to the variety S and to the complex $\mathcal{F}^\bullet|_S$, are spectral sequences of mixed Hodge structures.

By our assumption, the variations of mixed Hodge structures $\mathcal{H}^q(\mathcal{F}^\bullet)$ on S are trivial, hence by (2.4) and as in the proof of Lemma 2.6 there are mixed Hodge structure isomorphisms

$$E_2^{p,q} = H^p(S) \otimes V^q, \tag{2.20}$$

where $V^q := \mathcal{H}^q(\mathcal{F}^\bullet)_x \cong H^0(S; \mathcal{H}^q(\mathcal{F}^\bullet))$, for any $x \in S$.

Now let $e^{k,l}$ be the Euler-characteristic of Hodge-type (k, l), i.e., for a bounded complex $K^\bullet$ of mixed Hodge structures

$$e^{k,l}([K^\bullet]) = \sum_i (-1)^i h^{k,l}([K^i]).$$

Since all differentials in (2.4) are mixed Hodge structure morphisms, an argument similar to that in Lemma 2.6, yields that

$$e^{k,l}([\mathbb{H}^\bullet(S; M^\bullet))]) = \sum_{r+t=k, s+u=l} e^{r,s}([H^\bullet(S)]) \cdot e^{t,u}([V^\bullet]).$$

Formula (2.18) follows by noting that

$$\chi_y([K^\bullet]) = \sum_{k,l} e^{k,l}([K^\bullet]) \cdot (-y)^k.$$

Formula (2.19) follows similarly, by working instead with the spectral sequence (2.5). □

Remark 2.19. It follows from the discussion in §2, that for any $x \in S$, we have:

$$\chi_y([\mathcal{H}^\bullet(\mathcal{F}^\bullet)_x]) = \chi_y([i_x^*\mathcal{F}^\bullet]),$$

where $i_x : \{x\} \hookrightarrow S$ is the inclusion of the point.

Altogether, Corollary 2.16 and Proposition 2.18 yield the following global-to-local formula:

Theorem 2.20. *Let $\mathcal{S}$ be the set of components of strata of an algebraic Whitney stratification of the complex algebraic variety Z. Assume that for $M^\bullet \in D^bMHM(Z)$ with underlying complex $\mathcal{F}^\bullet \in D^b_c(Z)$, the local systems $\mathcal{H}^j(\mathcal{F}^\bullet)$ are constant on each pure stratum $S \in \mathcal{S}$ for each $j \in \mathbb{Z}$, e.g. $\pi_1(S) = 0$ for all $S \in \mathcal{S}$. Then*

$$\chi_y([\mathbb{H}^\bullet_c(Z; M^\bullet)]) = \sum_{S \in \mathcal{S}} \chi^c_y(S) \cdot \chi_y([\mathcal{H}^\bullet(\mathcal{F}^\bullet)_{x_S}]), \tag{2.21}$$

for some points $x_S \in S$.

3. A Hodge-theoretic analogue of the Riemann-Hurwitz formula

In this section, we apply the above formalism and properties of χ_y-genera to the study of Hodge-type invariants for a projective morphism onto a curve. We first recall the definition of *nearby and vanishing cycle functors*.

3.1. **Vanishing and nearby cycles.** Let X be a (separated and reduced) complex analytic space of dimension $n+1$, and $f : X \to \Delta$ be a holomorphic map onto the unit disc in $\mathbb{C}$, which is smooth over the punctured disc Δ^*. The total space X is homotopy equivalent to $X_0 = f^{-1}(0)$ by a fiber-preserving retraction $r : X \to X_0$. So the inclusion $i_t : X_t = f^{-1}(t) \hookrightarrow X$ followed by this retraction yields the specialization map $r_t : X_t \to X_0$.

Let $i_0 : X_0 \hookrightarrow X$ be the inclusion map, and define the canonical fiber X_∞ by

$$X_\infty := X \times_{\Delta^*} \hbar,$$

where $\hbar$ is the complex upper-half plane (that is, the universal cover of the punctured disc via the map $z \mapsto \exp(2\pi i z)$). Then X_∞ is homotopy equivalent to any smooth fiber X_t of f. Let $k : X_\infty \to X$ be the induced map.

Definition 3.1. The *nearby cycle complex* is defined by

$$\psi_f(\mathbb{Q}_X) := i_0^* Rk_* k^* \mathbb{Q}_X \in D^b_c(X_0).$$

By using a resolution of singularities, it can be shown that $\psi_f(\mathbb{Q}_X) = Rr_{t*}\mathbb{Q}_{X_t}$, for $t \neq 0$ small enough. The *vanishing cycle complex* $\phi_f(\mathbb{Q}_X) \in D^b_c(X_0)$ is the cone on the comparison morphism $\mathbb{Q}_{X_0} = i_0^*\mathbb{Q}_X \to \psi_f(\mathbb{Q}_X)$, that is, there exists a canonical morphism *can* : $\psi_f(\mathbb{Q}_X) \to \phi_f(\mathbb{Q}_X)$ such that

$$i_0^*\mathbb{Q}_X \to \psi_f(\mathbb{Q}_X) \overset{can}{\to} \phi_f(\mathbb{Q}_X) \overset{[1]}{\to} \tag{3.1}$$

is a distinguished triangle in $D^b_c(X_0)$.

In fact, by replacing $\mathbb{Q}_X$ by any complex in $D^b_c(X)$, we obtain in this way functors $\psi_f, \phi_f : D^b_c(X) \to D^b_c(X_0)$. It is well-known that if X is locally a complete intersection (e.g., X is smooth), then $\psi_f \mathbb{Q}_X[n]$ and $\phi_f \mathbb{Q}_X[n]$ are perverse complexes. This is just a particular case of the fact that the shifted functors ${}^p\psi_f := \psi_f[-1]$ and ${}^p\phi_f := \phi_f[-1]$ take perverse sheaves on X into perverse sheaves on the central fiber X_0 (cf. [40], Thm. 6.0.2).

Remark 3.2. The above construction of the vanishing and nearby cycles comes up in the following global context (for details, see [20], §4.2). Let X be a complex algebraic (resp. analytic) variety, and $f : X \to \mathbb{C}$ a non-constant regular (resp. analytic) function. Then for any $t \in \mathbb{C}$, one can construct functors

$$\mathcal{F}^\bullet \in D^b_c(X) \mapsto \psi_{f-t}(\mathcal{F}^\bullet), \phi_{f-t}(\mathcal{F}^\bullet) \in D^b_c(X_t)$$

where $X_t = f^{-1}(t)$ is assumed to be a non-empty hypersurface, by simply repeating the above considerations for the function $f - t$ restricted to a tube $T(X_t) := f^{-1}(\Delta)$ around the fiber X_t (here Δ is a small disc centered at t). At least in the algebraic context, the tube (which is an analytic space) can be chosen so that $f : T(X_t) \setminus X_t \to \Delta^*$ is a topologically locally trivial fibration, where $\Delta^* := \Delta \setminus \{t\}$. In the analytic context, for this to be true one needs to assume, for example, that f is proper on the tube $T(X_t)$.

Of particular importance is the fact that the nearby and vanishing functors can be defined at the level of Saito's mixed Hodge modules [36, 37]. More precisely, if f is a non-constant regular (resp. holomorphic) function on the complex algebraic (resp. analytic) space X, then one has functors $\psi_f, \phi_f : MHM(X) \to MHM(X_0)$, compatible with the corresponding perverse cohomological functors on the underlying perverse sheaves by the functor

$$rat : MHM(X) \to \mathrm{Perv}_{\mathbb{Q}}(X)$$

which assigns to a mixed Hodge module the underlying $\mathbb{Q}$-perverse sheaf. In other words, $rat \circ \psi_f = {}^p\psi_f \circ rat$, and similarly for ϕ_f. As a consequence of this fact, if X is smooth, for each $x \in X_0$ we have canonical mixed Hodge structures on the groups

$$H^j(M_x;\mathbb{Q}) = \mathcal{H}^j(\psi_f \mathbb{Q}_X)_x, \quad \tilde{H}^j(M_x;\mathbb{Q}) = \mathcal{H}^j(\phi_f \mathbb{Q}_X)_x, \tag{3.2}$$

where M_x denotes the Milnor fiber of f at x.

By the identification in (3.2), we note that $\mathrm{Supp}(\phi_f \mathbb{Q}_X) \subset \mathrm{Sing}(X_0)$ (see [20], Example 4.2.6 and Prop. 4.2.8). Then we have the following vanishing result (e.g., see [40], Example 6.0.13), whose proof we include here for completeness of the exposition:

Lemma 3.3. *Assume X is smooth and $f : X \to \Delta$ is proper. If $s := dim_{\mathbb{C}} Sing(X_0)$, then*

$$\mathbb{H}^j(X_0, \phi_f \mathbb{Q}_X) = 0, \quad \text{for } j \notin [n-s, n+s].$$

Proof. Since X is a smooth complex manifold of dimension $n+1$, it follows that $\mathbb{Q}_X[n+1]$ is perverse on X. Therefore ${}^p\phi_f \mathbb{Q}_X[n+1] = \phi_f \mathbb{Q}_X[n] \in \mathrm{Perv}(X_0)$. If $\Sigma := \mathrm{Sing}(X_0)$, then $\mathrm{Supp}(\phi_f \mathbb{Q}_X) \subset \Sigma$ yields that $\phi_f \mathbb{Q}_X[n]|_\Sigma \in \mathrm{Perv}(\Sigma)$, see [20], Cor. 5.2.5. The proof follows since (noting that X_0 and Σ are compact)

$$\mathbb{H}^j(X_0, \phi_f \mathbb{Q}_X) \cong \mathbb{H}^{j-n}(X_0, \phi_f \mathbb{Q}_X[n]) \cong \mathbb{H}^{j-n}(\Sigma, \phi_f \mathbb{Q}_X[n]|_\Sigma) = 0 \ \text{ for } j - n \notin [-s, s],$$

where the vanishing follows from [20], Proposition 5.2.20.

□

3.2. **A Riemann-Hurwitz formula for χ_y-genera.** Let $f : X \to C$ be a surjective projective algebraic morphisms from a smooth $(n+1)$-dimensional complex algebraic variety onto a smooth algebraic curve. Let $\Sigma(f) \subset C$ be the critical locus of f. Then f is a submersion over $C^* := C \setminus \Sigma(f)$, hence locally differentiably trivial (by Ehresmann's fibration theorem). For a point $c \in C$ we let X_c denote the fiber $f^{-1}(c)$.

We want to relate the χ_y^c-genera of X and respectively C via the singularities of f, and to obtain a stronger version of Prop. 2.11 in our setting. The outcome is a Hodge-theoretic version of a formula of Iversen, or of the Riemann-Hurwitz formula for the Euler characteristic (e.g., see [20], Cor. 6.2.5, Remark 6.2.6, or [29], (III, 32)), see Theorem 3.4 and Example 3.6 below. The proof uses the additivity of the χ_y^c-genus, together with the study of genera of singular fibers of f by means of vanishing cycles at a critical value.

Theorem 3.4. *Let $f : X \to C$ be a projective algebraic morphism from a smooth $(n+1)$-dimensional complex algebraic variety onto a non-singular algebraic curve C. Let $\Sigma(f) \subset C$ be the set of critical values of f, and set $C^* = C \setminus \Sigma(f)$. If the action of $\pi_1(C^*)$ on the cohomology of the generic fibers X_t of f is trivial (e.g., $\pi_1(C^*) = 0$), then*

$$\chi_y^c(X) = \chi_y^c(C) \cdot \chi_y^c(X_t) - \sum_{c \in \Sigma(f)} \chi_y([\mathbb{H}^\bullet(X_c; \phi_{f-c}\mathbb{Q}_X)]) \tag{3.3}$$

Proof. Under our assumptions, the fibers of f are complex projective varieties, and fibers over points in C^* are smooth. By additivity, we can write:

$$\chi_y^c(X) = \chi_y^c(X^*) + \sum_{c \in \Sigma(f)} \chi_y(X_c),$$

where $X^* := f^{-1}(C^*)$. Then by Lemma 2.6, we have that $\chi_y^c(X^*) = \chi_y^c(C^*)\chi_y(X_t)$, where X_t is the smooth (generic) fiber of f.

Now let $c \in \Sigma(f)$ be a critical value of f and restrict the morphism to a tube $T(X_c) := f^{-1}(\Delta_c)$ around the singular fiber X_c, where Δ_c denotes a small disc in C centered at c. By our assumptions, $f : T(X_c) \to \Delta_c$ is a proper holomorphic function, smooth over Δ_c^*, with fibers complex projective varieties, that is, a one-parameter degeneration of complex projective varieties. Then there is a long exact sequence of mixed Hodge structures (e.g., see [32], Thm. 1.1):

$$\cdots \to H^j(X_c; \mathbb{Q}) \to \mathbb{H}^j(X_c; \psi_{f-c}\mathbb{Q}_X) \to \mathbb{H}^j(X_c; \phi_{f-c}\mathbb{Q}_X) \to \cdots, \tag{3.4}$$

where $\mathbb{H}^j(X_c; \psi_{f-c}\mathbb{Q}_X)$ carries the *limit mixed Hodge structure* defined on the cohomology of the canonical fiber (usually denoted X_∞) of the one-parameter degeneration $f : T(X_c) \to \Delta_c$ (e.g., see [34], §11.2). However, a consequence of the definition of the limit mixed Hodge structure is that (cf. [34], Cor. 11.25)

$$\dim_{\mathbb{C}} F^p H^j(X_\infty; \mathbb{C}) = \dim_{\mathbb{C}} F^p H^j(X_t; \mathbb{C}),$$

where X_t is the generic fiber of the family (and of f). Therefore,

$$\chi_y(X_\infty) := \chi_y([\mathbb{H}^\bullet(X_c; \psi_{f-c}\mathbb{Q}_X)]) = \chi_y(X_t).$$

With this observation, from (3.4) we obtain that for a critical value c of f the following holds:

$$\begin{aligned} \chi_y(X_c) &= \chi_y([\mathbb{H}^\bullet(X_c; \psi_{f-c}\mathbb{Q}_X)]) - \chi_y([\mathbb{H}^\bullet(X_c; \phi_{f-c}\mathbb{Q}_X)]) \\ &= \chi_y(X_t) - \chi_y([\mathbb{H}^\bullet(X_c; \phi_{f-c}\mathbb{Q}_X)]). \end{aligned}$$

By additivity and Lemma 2.6, this yields (3.3).

□

Remark 3.5. The key point in the proof of the above theorem was to observe that in a one-parameter degeneration of complex projective manifolds the χ_y-genus of the canonical fiber coincides with the χ_y-genus of the generic fiber of the family. Note that this fact is not true for the corresponding E-polynomials, since, while the Hodge structure on the cohomology of the generic fiber is pure, the limit mixed Hodge structure on the cohomology of the canonical fiber carries the monodromy weight filtration.

Example 3.6. If X is smooth and f has only isolated singularities, then

$$\chi_y^c(X) = \chi_y^c(C)\chi_y^c(X_t) + (-1)^{n+1} \sum_{x \in \mathrm{Sing}(f)} \chi_y([\tilde{H}^n(M_x; \mathbb{Q})]), \tag{3.5}$$

where M_x is the Milnor fiber of f at x. Indeed, if f has only isolated singular points, then each critical fiber X_c has only isolated singularities and the corresponding vanishing cycles $\phi_{f-c}\mathbb{Q}_X$ are supported only at these points. Then (3.5) follows from the identification in (3.2) and the vanishing of Lemma 3.3.

Remark 3.7. In the special case of the Euler characteristic $\chi = \chi_{-1}$, the formula in Theorem 3.4 and in the example above holds for any proper analytic morphism onto a curve, without any assumption on the monodromy (see [20], Cor. 6.2.5). This follows from the multiplicativity of the Euler characteristic χ under fibrations, the additivity of compactly supported Euler characteristic χ_c, and from the fact that $\chi = \chi_c$ (cf. [21], pp. 141-142).

We will extend the formula of Example 3.6 to the case of general singularities. By (3.3), it suffices to restrict f over a small disc Δ_c centered at a critical value $c \in \Sigma(f)$ and to study the polynomial $\chi_y([\mathbb{H}^\bullet(X_c; \phi_{f-c}\mathbb{Q}_X)])$. Recall that the fibers of f are complex projective algebraic varieties, which are smooth over points in $\Delta_c \setminus \{c\}$.

Fix an algebraic Whitney stratification of X_c with respect to which $\phi_{f-c}\mathbb{Q}_X$ is constructible. For each $q \in \mathbb{Z}$ and each pure stratum $S \subset \mathrm{Sing}(X_c)$, $\mathcal{H}^q(\phi_{f-c}\mathbb{Q}_X)$ is a local coefficient system on S with stalk $\tilde{H}^q(M_S; \mathbb{Q})$, where M_S is the local Milnor fibre at a point in S. Then, according to Theorem 2.20, for $M^\bullet = \phi_f\mathbb{Q}_X[n]$ and by assuming trivial monodromy along all strata $S \subset \mathrm{Sing}(X_c)$, we obtain

$$\chi_y([\mathbb{H}^\bullet(X_c; \phi_{f-c}\mathbb{Q}_X)]) = \sum_{S \subset \mathrm{Sing}(X_c)} \chi_y^c(S) \cdot \chi_y([\tilde{H}^\bullet(M_S; \mathbb{Q})]),$$

where M_S is the local Milnor fibre of f at a point in S.

All these facts yield the following general result:

Corollary 3.8. *Let $f : X \to C$ be a projective algebraic morphism from a smooth $(n+1)$-dimensional complex algebraic variety onto a non-singular algebraic curve C. Let $\Sigma(f) \subset C$ be the set of critical values of f, and set $C^* = C \setminus \Sigma(f)$. Assume each special fiber X_c has an algebraic stratification with respect to which the corresponding vanishing cycle complex is constructible, and moreover the monodromy along each pure stratum is trivial. If the action of $\pi_1(C^*)$ on the cohomology of the generic fibers X_t of f is trivial, then*

$$\chi_y^c(X) = \chi_y^c(C) \cdot \chi_y^c(X_t) - \sum_{c \in \Sigma(f)} \sum_{S \subset Sing(X_c)} \chi_y^c(S) \cdot \chi_y([\tilde{H}^\bullet(M_S;\mathbb{Q})]) \tag{3.6}$$

4. THE PRESENCE OF MONODROMY. ATIYAH-MEYER FORMULAE FOR THE χ_y-GENUS.

In this section we prove Hodge-theoretic analogues (in the category of complex algebraic varieties) of the Atiyah formula for the signature of a fibre bundles in the presence of monodromy [2], and of Meyer's twisted signature formula [31]. We first state and prove our formulae in the context of smooth projective varieties, then point out several interesting extensions to more general situations.

4.1. **Hirzebruch classes of complex projective manifolds and the Hirzebruch-Riemann-Roch theorem.** Recall that if X is a smooth complex projective variety, its Hirzebruch class $\widetilde{T}_y^*(T_X)$ corresponds to the (un-normalized) power series

$$\widetilde{Q}_y(\alpha) := \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}} \in \mathbb{Q}[y][[\alpha]], \quad \widetilde{Q}_y(0) = 1+y.$$

In fact,

$$\widetilde{T}_y^*(T_X) := td^*(X) \cup ch^*(\lambda_y(T_X^*)),$$

where $td^*(X)$ is the total Todd class of X, ch^* is the Chern character, and $\lambda_y(T_X^*) := \sum_p \Lambda^p T_X^* \cdot y^p$ is the total λ-class of (the cotangent bundle of) X. Hirzebruch's class appears in the *generalized Hirzebruch-Riemann-Roch theorem* (cf. [26], §21.3, but for the version needed here see also [47, 8]), which asserts that if E is a holomorphic vector bundle on X then the χ_y-characteristic of E, which is defined by

$$\chi_y(X,E) := \sum_{p \geq 0} \chi(X, E \otimes \Lambda^p T_X^*) \cdot y^p = \sum_{p \geq 0} \left(\sum_{i \geq 0} (-1)^i \mathrm{dim} H^i(X, \Omega(E) \otimes \Lambda^p T_X^*) \right) \cdot y^p,$$

with T_X^* the cotangent bundle of X and $\Omega(E)$ the coherent sheaf of germs of sections of E[3], can in fact be expressed in terms of the Chern classes of E and the tangent bundle of X,

---

[3] For X smooth and projective, $\chi_y(X, \mathcal{O}_X)$ agrees with the Hodge-theoretic χ_y-genus defined in the first part of this paper. Indeed, by Deligne's theory, one has the following equality for the Hodge numbers: $h^{p,q} = \mathrm{dim}_{\mathbb{C}} H^q(X, \Lambda^p T_X^*)$.

or more precisely

$$\chi_y(X,E)=\int_X \left(ch^*(E)\cup \widetilde{T}_y^*(T_X)\right)\cap [X]. \tag{4.1}$$

In particular, if $E=\mathcal{O}_X$ we have that

$$\chi_y(X)=\int_X \widetilde{T}_y^*(T_X)\cap [X].$$

Also note that the value $y=0$ in (4.1) yields the classical Hirzebruch-Riemann-Roch theorem (in short, HRR) for the holomorphic Euler characteristic of E, that is (cf. [26]),

$$\chi(X,E)=\int_X (ch^*(E)\cup td^*(X))\cap [X].$$

4.2. **χ_y-genera of smooth projective families.** Let $f:E\to B$ be a smooth proper map of smooth complex projective varieties. By Ehresmann's theorem, f is a differentiable fibration. The cohomology groups $H^k(E_b)$ of fibers fit into a local system $R^kf_*\mathbb{Q}_E$. By a result of Griffiths, this local system underlies a weight k (geometric) variation of Hodge structures on B such that the Hodge structure at $b\in B$ is just the Hodge structure we have on $H^k(E_b)$. Let

$$\mathcal{H}_k:=R^kf_*\mathbb{Q}_E\otimes_{\mathbb{Q}}\mathcal{O}_B.$$

This is a holomorphic bundle with a flat connection $\bigtriangledown:\mathcal{H}_k\to\mathcal{H}_k\otimes_{\mathcal{O}_B}\Omega^1_B$, and it admits a finite decreasing filtration $\{\mathcal{F}^p_k\}_p$ by holomorphic sub-bundles satisfying Griffiths' transversality condition $\bigtriangledown(\mathcal{F}^p_k)\subset\mathcal{F}^{p-1}_k\otimes\Omega^1_B$. Set $\mathcal{H}^{p,k-p}:=Gr^p_{\mathcal{F}}\mathcal{H}_k$. This is a holomorphic bundle, not necessarily flat, of rank $h^{p,k-p}_b:=\dim H^{p,k-p}(E_b;\mathbb{C})$. The numbers $h^{p,k-p}_b$ for $b\in B$, remain constant in the family since they depend upper-semicontinuously on b.

One of the main results of this section is the following:

Theorem 4.1. *Let $f:E\to B$ be a smooth proper map of smooth complex projective varieties. Then the Hirzebruch χ_y-genus of E can be computed by the following formula:*

$$\chi_y(E)=\int_B \left(ch^*\left(\chi_y(f)\right)\cup \widetilde{T}_y^*(T_B)\right)\cap [B], \tag{4.2}$$

where $\chi_y(f):=\sum_{p,q\geq 0}(-1)^q\mathcal{H}^{p,q}\cdot y^p\in K(B)[y]$ is the K-theory χ_y-genus of f.

Before proving the theorem, we make few remarks and state some immediate consequences.

Remark 4.2. (1) Formula (4.2) shows the deviation from multiplicativity of the χ_y-genus of fiber bundles in the presence of monodromy. The right-hand side of (4.2) is a sum of polynomials, one of the summands being $\chi_y(B)\cdot\chi_y(F)$. Indeed, the zero-dimensional piece of $ch^*(\chi_y(f))$ is $\chi_y(F)$.

(2) Formula (4.2) is the Hodge-theoretic analogue of Atiyah's signature formula (cf. [2], (4.3)) in the complex algebraic setting. Indeed, if $y = 1$, then by [47], Remark 3, and in the notation of [2] §4,

$$\widetilde{T}_1^*(T_B) = \prod_{i=1}^{dim B} \frac{\alpha_i}{\tanh(\frac{1}{2}\alpha_i)} =: \widetilde{\mathcal{L}}(B),$$

where α_i are the Chern numbers of the tangent bundle of B. Moreover, it is known that $\chi_1(E) = \sigma(E)$ is the usual signature (cf. [26]), and in a similar fashion one can show that $(-1)^q \mathcal{H}^{p,q}$ is the K-theory signature $\mathrm{Sign}(f)$ from [2]. In other words, the value at $y = 1$ of (4.2) yields

$$\sigma(E) = \int_B \left(ch^*(\mathrm{Sign}(f)) \cup \widetilde{\mathcal{L}}(B) \right) \cap [B].$$

(3) In [2], Atiyah pointed out that the non-multiplicativity examples for the signature of holomorphic fibrations $Z \to C$ having $\dim_{\mathbb{C}} Z = 2$, $\dim_{\mathbb{C}} C = 1$, with non-trivial monodromy action on the cohomology of the fiber G, also show the non-multiplicativity of the Todd genus. Examples of non-multiplicativity in higher dimensions can be obtained as follows. Let $D \to C$ be an arbitrary holomorphic fiber bundle with fiber F and having a trivial monodromy. Then the Todd (and hence χ_y) genus is non-multiplicative for the fibration $Z \times_C D \to D$, since $Z \times_C D$ also fibers over Z with fiber F and trivial monodromy, and

$$Td(Z \times_C D) = Td(F)Td(Z) \neq Td(F)Td(C)Td(G) = Td(D)Td(G) \tag{4.3}$$

More such examples can be obtained via standard constructions, e.g. fiber or direct products of Atiyah examples, or higher dimensional examples as above.

(4) Theorem 4.1 can be extended so that we allow E and F to be singular. We can also discard the compactness assumption on the base B, but in this case we need to allow contributions "at infinity" in our formula; see the discussion at the end of this section for precise formulations of these general results.

An immediate corollary of Theorem 4.1 is the following:

Corollary 4.3. *Under the assumptions of the above theorem, if moreover $R^k f_* \mathbb{Q}_E$ is a local system of Hodge structures for each k, i.e. the monodromy action of $\pi_1(B)$ on the cohomology of the fiber preserves the Hodge filtration, then*

$$\chi_y(E) = \chi_y(B) \cdot \chi_y(F), \tag{4.4}$$

where F is the typical fiber of the family.

Proof. Indeed, if $R^k f_* \mathbb{Q}_E$ is a local system of Hodge structures, the Griffiths transversality condition for the flat connection $\bigtriangledown : \mathcal{H}_k \to \mathcal{H}_k \otimes_{\mathcal{O}_B} \Omega^1_B$ reduces to $\bigtriangledown(\mathcal{F}^p_k) \subset \mathcal{F}^p_k \otimes \Omega^1_B$. It follows that all bundles $\mathcal{F}^p_k$, whence the bundles $Gr^p_{\mathcal{F}} \mathcal{H}_k = \mathcal{H}^{p,k-p}$, are flat. Since the rational Chern classes in positive degrees of flat bundles are trivial (cf. [28]), we obtain

$$ch^* \left(\sum_{p,q} (-1)^q \mathcal{H}^{p,q} y^p \right) = \sum_{p,q} (-1)^q \mathrm{rank}(\mathcal{H}^{p,q}) y^p = \chi_y(F).$$

The result follows.

□

Remark 4.4. (1) The above corollary is false in the non-compact case, e.g., for the Milnor fibration of the cuspidal cupid that was already considered in §2.4 (in general, the monodromy of a weighted homogeneous hypersurface singularity is an algebraic morphism, thus induces a morphism of mixed Hodge structures in cohomology).

(2) An example of fibration with the action as in Corollary 4.3 can be given as follows. Let G be a finite group of biholomorphic maps acting freely on the smooth projective varieties E and F, so that the action on the cohomology of F is non-trivial. If we let G act diagonally on $E \times F$, then the fibration:

$$(E \times F)/G \to E/G \tag{4.5}$$

has F as its fiber, the monodromy action coincides with the action of G on the cohomology of F, and it preserves the Hodge filtration on $H^*(F)$ since the monodromy transformations are biholomorphic.

Remark 4.5. *Higher χ_y-genera.* If X is a smooth projective variety, $\pi := \pi_1(X)$, and $\alpha \in H^*(B\pi;\mathbb{Q})$, we define higher χ_y-genera of X by the formula

$$\chi_y^{[\alpha]}(X) := \int_X \left(u^*(\alpha) \cup \widetilde{T}_y^*(T_X)\right) \cap [X], \tag{4.6}$$

where $u : X \to B\pi$ is the classifying map of the universal cover of X.

Under the assumptions of Corollary 4.3, formula (4.2) can be rephrased in terms of higher χ_y-genera as follows. From the classifying space description of each of the bundles $\mathcal{H}^{p,q}$, it is clear that $ch^*(\mathcal{H}^{p,q})$ is induced from an universal characteristic class $ch^*(H^{p,q}) \in H^*(BGL(r;\mathbb{C});\mathbb{Q})$, where $r = \text{rank}\mathcal{H}^{p,q}$. Moreover, the assumption that the action of $\pi_1(B)$ preserves the Hodge filtration, hence the (p,q)-type, yields that the classifying map $B \to BGL(r;\mathbb{C})$ for $\mathcal{H}^{p,q}$ factors (up to homotopy) as $B \overset{u}{\to} B\pi \overset{v}{\to} BGL(r;\mathbb{C})$, where u is the classifying map of the universal cover of B, and v is induced by the monodromy action. It follows that $ch^*(\mathcal{H}^{p,q}) = u^*(v^*ch^*(H^{p,q}))$. Set $\alpha_\pi^{p,q} := v^*ch^*(H^{p,q})$. Then the Chern character of the K-theory χ_y-genus of f can be written as

$$ch^*(\chi_y(f)) = u^*\left(\sum_{p,q}(-1)^q \alpha_\pi^{p,q} \cdot y^p\right)$$

Then in the notation of (4.6) and under the assumptions of Corollary 4.3, formula (4.2) asserts that $\chi_y(E)$ can be written as a Hodge polynomial in higher genera of B, namely

$$\chi_y(E) = \sum_{p,q}(-1)^q \chi_y^{[\alpha_\pi^{p,q}]}(B) \cdot y^p. \tag{4.7}$$

However, since we assumed that each $\mathcal{H}^{p,q}$ is a flat bundle, we have that $\chi_y^{[\alpha_\pi^{p,q}]}(B) = h^{p,q}(F) \cdot \chi_y(B)$, and the equation (4.7) yields the multiplicativity of Corollary 4.3.

We now return to the proof of Theorem 4.1.

Proof. Recall from §2.3 that the Leray spectral sequence of the map f, that is,

$$E_2^{p,q} = H^p(B, R^q f_* \mathbb{Q}_E) \Longrightarrow H^{p+q}(E), \tag{4.8}$$

is a spectral sequence of mixed Hodge structures. In fact all mixed Hodge structures involved in the case of a smooth projective family are pure (and by a result of Deligne, the Leray spetral sequence of f degenerates at E_2). Indeed, the local systems $R^q f_* \mathbb{Q}_E$ underlie geometric variations of pure Hodge structures (thus admissible in the sense of Steenbrink-Zucker-Kashiwara; see [34], Theorem 14.49 and the references therein), and it is known (e.g., from Saito's work [37], see also [34], Theorem 14.50) that the cohomology of a smooth projective variety with coefficients in such a variation admits a pure Hodge structure.

By definition, we have

$$\chi_y(E) = \sum_{i,p} (-1)^i \mathrm{dim} Gr_F^p H^i(E; \mathbb{C}) \cdot (-y)^p = \sum_p \chi^p(E) \cdot (-y)^p,$$

where we let $\chi^p(E) := \sum_i (-1)^i \mathrm{dim} Gr_F^p H^i(E; \mathbb{C})$ be the Euler characteristic associated to the exact functor Gr_F^p. Since the differentials in the Leray spectral sequence of f are morphisms of (mixed) Hodge structures, thus strict with respect to the Hodge filtrations, it follows that

$$\begin{aligned} \chi^p(E) &= \sum_{k,l} (-1)^{k+l} \mathrm{dim} Gr_F^p H^k(B, R^l f_* \mathbb{C}_E) \\ &= \sum_l (-1)^l \left(\sum_k (-1)^k \mathrm{dim} Gr_F^p H^k(B, R^l f_* \mathbb{C}_E) \right) \\ &= \sum_l (-1)^l \chi^p(B, R^l f_* \mathbb{Q}_E). \end{aligned}$$

Therefore,

$$\begin{aligned} \chi_y(E) &= \sum_p \left(\sum_l (-1)^l \chi^p(B, R^l f_* \mathbb{Q}_E) \right) \cdot (-y)^p \\ &= \sum_l (-1)^l \left(\sum_p \chi^p(B, R^l f_* \mathbb{Q}_E) \cdot (-y)^p \right) \\ &= \sum_l (-1)^l \chi_y(B, R^l f_* \mathbb{Q}_E). \end{aligned}$$

So, we reduced the problem to the following setting, which is a Hodge-theoretic analogue of the situation considered by Meyer [31]: B is a smooth projective variety, $\mathbb{V}^l := R^l f_* \mathbb{Q}_E$ is a geometric variation of pure Hodge structures of weight l on B (in fact, for what follows, one may replace $\mathbb{V}^l$ by a weight l polarized variation of Hodge structures on B, or more generally, by an admissible variation of mixed Hodge structures), and we consider the χ_y-genus of B twisted by $\mathbb{V}^l$, that is, $\chi_y(B, \mathbb{V}^l)$, which encodes the Hodge numbers of the (mixed, if $\mathbb{V}^l$ is replaced by an admissible variation) Hodge structures on the cohomology groups $H^k(B;\mathbb{V}^l)$, $k \in \mathbb{Z}_{\geq 0}$.

If we let, as before, $\mathcal{H}_l := \mathbb{V}^l \otimes_{\mathbb{Q}} \mathcal{O}_B$ be the flat bundle associated to $\mathbb{V}^l$, then we have an isomorphism

$$H^k(B;\mathbb{V}^l \otimes \mathbb{C}) \cong \mathbb{H}^k(B;\Omega^\bullet_B \otimes_{\mathcal{O}_B} \mathcal{H}_l),$$

and the Hodge filtration on $H^k(B;\mathbb{V}^l \otimes \mathbb{C})$ is induced by the filtration $F^\bullet$ on the de Rham complex that is defined by Griffiths' transversality

$$F^p(\Omega^\bullet_B \otimes_{\mathcal{O}_B} \mathcal{H}_l) := \left[\mathcal{F}^p_l \mathcal{H}_l \overset{\bigtriangledown}{\to} \Omega^1_B \otimes \mathcal{F}^{p-1}_l \mathcal{H}_l \overset{\bigtriangledown}{\to} \cdots \overset{\bigtriangledown}{\to} \Omega^i_B \otimes \mathcal{F}^{p-i}_l \mathcal{H}_l \overset{\bigtriangledown}{\to} \cdots \right]$$

The associated graded is the complex

$$Gr^p_F(\Omega^\bullet_B \otimes_{\mathcal{O}_B} \mathcal{H}_l) = \left(\Omega^\bullet_B \otimes_{\mathcal{O}_B} Gr^{p-\bullet}_{\mathcal{F}} \mathcal{H}_l,\ Gr_F \bigtriangledown\right)$$

with the induced differential.

Then

$$\begin{aligned}
\chi^p(B,\mathbb{V}^l) &= \sum_k (-1)^k \dim\, Gr^p_F H^k(B,\mathbb{V}^l \otimes \mathbb{C}) \\
&= \sum_k (-1)^k \dim\, Gr^p_F \mathbb{H}^k(B;\Omega^\bullet_B \otimes_{\mathcal{O}_B} \mathcal{H}_l) \\
&\overset{(*)}{=} \sum_k (-1)^k \dim\, \mathbb{H}^k(B; Gr^p_F(\Omega^\bullet_B \otimes_{\mathcal{O}_B} \mathcal{H}_l)) \\
&= \chi(B, \Omega^\bullet_B \otimes_{\mathcal{O}_B} Gr^{p-\bullet}_{\mathcal{F}} \mathcal{H}_l),
\end{aligned}$$

where $(*)$ follows from [[34], Theorem 3.18 (iv)] and the fact proved by M. Saito that $(\mathbb{V}^l, \Omega^\bullet_B \otimes_{\mathcal{O}_B} \mathcal{H}_l)$ is a cohomological (mixed, if $\mathbb{V}^l$ is replaced by an admissible variation) Hodge complex in the sense of Deligne (recall B is smooth and compact; see also [34], Theorem 10.9 for the case of a pure polarized variation).

The last term in the above equality can be computed by using the invariance of the Euler characteristic under spectral sequences. In general, if $\mathcal{K}^\bullet$ is a complex of sheaves on a topological space B, then there is the following spectral sequence calculating its hypercohomology (e.g., see [20], §2.1):

$$E_1^{i,j} = H^j(B,\mathcal{K}^i) \Longrightarrow \mathbb{H}^{i+j}(B;\mathcal{K}^\bullet).$$

Assuming $\chi(B,\mathcal{K}^\bullet)$ is defined, it can be computed by

$$\chi(B,\mathcal{K}^\bullet) = \sum_{i,j} (-1)^{i+j} \dim\, H^j(B,\mathcal{K}^i) = \sum_i (-1)^i \chi(B;\mathcal{K}^i)$$

Therefore the twisted χ_y-genus $\chi_y(B,\mathbb{V}^l)$ can be computed as follows (where we neglect the cup product symbol or replace it by "$\cdot$" where there is no danger of confusion):

$$\chi_y(B,\mathbb{V}^l) \quad = \quad \sum_p \chi^p(B,\mathbb{V}^l) \cdot (-y)^p$$

$$\begin{aligned}
&= \sum_{p} \chi(B, \Omega_B^{\bullet} \otimes_{\mathcal{O}_B} Gr_{\mathcal{F}}^{p-\bullet}\mathcal{H}_l) \cdot (-y)^p \\
&= \sum_{i,p} (-1)^i \chi(B, \Omega_B^i \otimes Gr_{\mathcal{F}}^{p-i}\mathcal{H}_l) \cdot (-y)^p \\
&\overset{(HRR)}{=} \sum_{i,p} (-1)^i \left(\int_B ch^*(Gr_{\mathcal{F}}^{p-i}\mathcal{H}_l) ch^*(\Omega_B^i) td^*(B) \cap [B] \right) \cdot (-y)^p \\
&= \int_B \sum_{i,p} \left(ch^*(Gr_{\mathcal{F}}^{p-i}\mathcal{H}_l) \cdot (-y)^{p-i} \right) \cdot \left(ch^*(\Omega_B^i) td^*(B) \cdot y^i \right) \cap [B] \\
&= \int_B \left(\sum_s ch^*(Gr_{\mathcal{F}}^{s}\mathcal{H}_l) \cdot (-y)^s \right) \cdot \left(td^*(B) \sum_i ch^*(\Omega_B^i) \cdot y^i \right) \cap [B] \\
&= \int_B \left(\sum_s ch^*(Gr_{\mathcal{F}}^{s}\mathcal{H}_l) \cdot (-y)^s \right) \cdot td^*(B) ch^*\left(\lambda_y(T_B^*) \right) \cap [B] \\
&= \int_B \left(\sum_s ch^*(Gr_{\mathcal{F}}^{s}\mathcal{H}_l) \cdot (-y)^s \right) \cdot \widetilde{T}_y^*(T_B) \cap [B].
\end{aligned}$$

Coming back to the computation of $\chi_y(E)$, we obtain that

$$\begin{aligned}
\chi_y(E) &= \sum_l (-1)^l \chi_y(B, R^l f_* \mathbb{Q}_E) \\
&= \sum_l (-1)^l \int_B \left(\sum_s ch^*(Gr_{\mathcal{F}}^{s}\mathcal{H}_l) \cdot (-y)^s \right) \cdot \widetilde{T}_y^*(T_B) \cap [B] \\
&= \int_B \left(\sum_{l,s} (-1)^l ch^*(Gr_{\mathcal{F}}^{s}\mathcal{H}_l) \cdot (-y)^s \right) \cdot \widetilde{T}_y^*(T_B) \cap [B] \\
&= \int_B \left(\sum_{l,s} (-1)^l ch^*(\mathcal{H}^{s,l-s}) \cdot (-y)^s \right) \cdot \widetilde{T}_y^*(T_B) \cap [B] \\
&= \int_B \left(\sum_{p,q} (-1)^q ch^*(\mathcal{H}^{p,q}) \cdot y^p \right) \cdot \widetilde{T}_y^*(T_B) \cap [B].
\end{aligned}$$

□

As an important corollary of the proof of Theorem 4.1 we obtain the following Hodge-theoretic analogue of Meyer's signature formula [31]:

Corollary 4.6. *Let Z be a smooth projective variety and $\mathbb{V}$ be a geometric, or polarized (or more generally, an admissible) variation of (mixed) Hodge structures on Z, with associated* *flat bundle with "Hodge" filtration* $(\mathcal{V}, \mathcal{F}^\bullet)$. *Then the twisted* χ_y*-genus* $\chi_y(Z, \mathbb{V})$ *can be computed by the formula*

$$\chi_y(Z, \mathbb{V}) = \int_Z \left(ch^*(Hc_y(\mathcal{V})) \cup \widetilde{T}^*_y(T_Z) \right) \cap [Z], \tag{4.9}$$

where $Hc_y(\mathcal{V})$ *is the* K*-theory Hodge polynomial characteristic of* $\mathcal{V}$ *defined by*

$$Hc_y(\mathcal{V}) = \sum_p Gr^p_{\mathcal{F}} \mathcal{V} \cdot (-y)^p.$$

It is now easy to see, with minor changes in the proof of Theorem 4.1, that we have in fact the following general result:

Theorem 4.7. *Let* $f : E \to B$ *be a quasi-projective surjective morphism of complex algebraic varieties, with* B *smooth and projective. Assume that the sheaves* $R^s f_* \mathbb{Z}_E$, $s \in \mathbb{Z}$ *are locally constant on* B. *Then the* χ_y*-genus of* E *can be computed by the following formula:*

$$\chi_y(E) = \int_B \left(ch^*\left(\chi_y(f)\right) \cup \widetilde{T}^*_y(T_B) \right) \cap [B], \tag{4.10}$$

where $\chi_y(f) := \sum_{i,p} (-1)^i Gr^p_{\mathcal{F}} \mathcal{H}_i \cdot (-y)^p \in K(B)[y]$ *is the* K*-theory* χ_y*-genus of* f.

Proof. Indeed, by our assumptions of f, the local systems $\mathcal{L}_s := R^s f_* \mathbb{Q}_E$, $s \in \mathbb{Z}$, underlie geometric, hence admissible variations of mixed Hodge structures (see [34], Thm. 14.49). Therefore, formula (4.9) applies to each of these variations. The rest follows from the Leray spectral sequence of the map f (cf. (2.6)), by noting as in the proof of Thm. 4.1, that $\chi_y(E) = \sum_s (-1)^s \chi_y(B; \mathcal{L}_s)$.

□

Remark 4.8. As stated in [31], Meyer's formula for the signature $\sigma(Z; \mathcal{L})$ of a Poincaré local system $\mathcal{L}$ (that is, a local system equipped with a nondegenerate bilinear pairing $\mathcal{L} \otimes \mathcal{L} \to \mathbb{R}_Z$) on a closed oriented smooth manifold Z of even dimension involves a twisted Chern character and the total L-polynomial of Z (as opposed to Atiyah's formula [2], where an un-normalized version of the L-polynomial is used). More precisely ([31]),

$$\sigma(Z; \mathcal{L}) = \int_Z \left(\widetilde{ch}^*([\mathcal{L}]_K) \cup L(Z) \right) \cap [Z],$$

where $[\mathcal{L}]_K$ is the K-theory signature of $\mathcal{L}$, $L(Z)$ is the total Hirzebruch L-polynomial of Z, and $\widetilde{ch}^* := ch^* \circ \psi^2$ is a modified Chern character obtained by composition with the second Adams operation. Similarly, following [27], p.61–62 (see also [41], §6), we can reformulate our Hodge-theoretic Atiyah-Meyer formulae in terms of the normalized Hirzebruch classes $T^*_y(T_Z)$ corresponding to the power series

$$Q_y(\alpha) := \widetilde{Q}_y(\alpha(1+y)) \cdot (1+y)^{-1} = \frac{\alpha(1+y)}{1 - e^{-\alpha(1+y)}} - \alpha y \in \mathbb{Q}[y][[\alpha]],$$

by using instead a modified Chern character, $ch^*_{(1+y)}$, whose value on a complex vector bundle ξ is

$$ch^*_{(1+y)}(\xi) = \sum_{j=1}^{rk\xi} e^{\beta_j(1+y)},$$

for β_j the Chern roots of ξ. (In this notation, Meyer's modified Chern character is simply $ch^*_{(2)}$.) For example, in the notations of Corollary 4.6, formula (4.9) is equivalent to

$$\chi_y(Z, \mathbb{V}) = \int_Z \left(ch^*_{(1+y)}(Hc_y(\mathcal{V})) \cup T^*_y(T_Z) \right) \cap [Z]. \tag{4.11}$$

A similar formula can be obtained for $\chi_y(U; \mathbb{V})$, the twisted χ_y polynomial associated to the canonical mixed Hodge structure on $H^*(U; \mathbb{V})$, for U any smooth (not necessarily compact) complex variety and $\mathbb{V}$ an admissible variation of mixed Hodge structure on U. (The existence of such mixed Hodge structures follows for example from Saito's theory, see also [34], Thm. 14.50 and the references therein.) In this case, in order to obtain a cohomological mixed Hodge complex whose Hodge filtration induces the Hodge filtration on $H^*(U; \mathbb{V} \otimes \mathbb{C})$, we need to use the twisted logarithmic de Rham complex associated to the Deligne extension of $\mathbb{V}$ on a good compactification of U. More precisely, let $(\mathcal{V}, \bigtriangledown)$ be the corresponding vector bundle on U with its flat connection and Hodge filtration. Then we can choose a smooth compactification $j : U \to Z$ such that $D = Z \setminus U$ is a divisor with normal crossings, and for each half-open interval of length one there is a unique extension of $(\mathcal{V}, \bigtriangledown)$ to a vector bundle $(\bar{\mathcal{V}}^I, \bar{\bigtriangledown}^I)$ with a logarithmic connection on Z such that the eigenvalues of the residues lie in I (cf. [18]). If we set $\bar{\mathcal{V}} := \bar{\mathcal{V}}^{[0,1)}$, then the twisted logarithmic de Rham complex $\Omega^\bullet_Z(\log D) \otimes \bar{\mathcal{V}}$ is quasi-isomorphic (on Z) to $Rj_*\mathbb{V} \otimes \mathbb{C}$, and the filtration $\mathcal{F}^\bullet$ on $\mathcal{V}$ extends to a filtration $\bar{\mathcal{F}}^\bullet \subset \bar{\mathcal{V}}$ since the variation of mixed Hodge structures was assumed to be admissible. As before, by Griffiths' transversality, we can filter the logarithmic twisted de Rham complex, and Saito proved that this becomes part of a cohomological mixed Hodge complex that calculates $H^*(U; \mathbb{V})$. By repeating the arguments in the proof of Theorem 4.1, we obtain the following formula, analogous to (4.9), involving contributions at infinity (i.e., forms on Z, with logarithmic poles along D):

$$\chi_y(U; \mathbb{V}) = \int_Z \left(ch^*(Hc_y(\bar{\mathcal{V}})) \cup ch^*(\sum_i \Omega^i_Z(\log D) \cdot y^i) \cup td^*(Z) \right) \cap [Z]. \tag{4.12}$$

This explains why under the assumptions of Corollary 4.3, the multiplicativity of the χ_y-genus fails in the non-compact case (cf. Remark 4.4).

In view of formula (4.12), we can obtain an even more general Atiyah type result for an algebraic map f as in Theorem 4.7 by dropping the compactness assumption on its target B. We leave the details as an exercise for the interested reader.

4.3. **Higher χ_y-genera and period domains.** In this section, we define higher χ_y-genera of variations of Hodge structures which correspond to cohomology classes of the quotients of Griffths period domains (cf. [25]). These higher genera are analogous to the previously considered higher genera corresponding to the cohomology classes of the fundamental group (cf. Remark 4.5), and for some types of variations of Hodge structures coincide with the latter. These classes allow to obtain a formula for the χ_y-genus of a fibration in terms of characteristic classes of the base, which yields the multiplicativity in a variety of cases including the case of trivial monodromy group.

Let $(B, \mathbb{V})$ be a pair where B is a Kähler manifold and $\mathbb{V}$ is a (integer) polarized variation of pure Hodge structures of weight k. If V is the stalk of $\mathbb{V}$ at a point in B, let $\epsilon = \pm 1$ be the type of the bilinear form Q on V (i.e. $Q(x,y) = \epsilon Q(y,x)$), and η_V be the partition $\mathrm{dim} V = \sum_{p+q=k} h^{p,q}$ where $h^{p,q} = \mathrm{dim} H^{p,q}$. Let D_{η_V} be the classifying space of pure Hodge structures of type (ϵ, η_V). This space is a subset in the flag manifold (consisting of flags in V satisfying the Riemann bilinear relations) and in particular it is the base of the universal flag bundle $\mathcal{F}_{\eta_V} := \cdots \subset \mathcal{F}^p \subset \cdots$ $(\mathrm{rank} \mathcal{F}^p = \sum_{i \le p} \mathrm{dim} H^{i,k-i})$ having the flags as its fiber, and also for the bundles $\mathcal{H}^{p,k-p}$ which are the quotients $\mathcal{F}^p / \mathcal{F}^{p+1}$.

Let $\bar{\Gamma}$ be the monodromy group corresponding to $\mathbb{V}$. This is a subgroup in the subgroup of $GL(\mathrm{dim} V, \mathbf{Z})$ consisting of transformations preserving Q. The group $\bar{\Gamma}$ acts on D_{ϵ,η_V} and some subgroup $\Gamma \subseteq \bar{\Gamma}$ of finite index acts freely on D_{ϵ,η_V}. The pair $(B, \mathbb{V})$ defines the period map:

$$\pi : B \to D_{\epsilon,\eta_V}/\Gamma.$$

The action of the group Γ on V and D_{ϵ,η_V} induces the action on the total space of $\mathcal{F}_{\eta_V}$ so that the projection $\mathcal{F}_{\eta_V} \to D_{\epsilon,\eta_V}$ is Γ-equivariant. The latter map induces the locally trivial fibration: $\mathcal{F}_{\eta_V}/\Gamma \to D_{\epsilon,\eta_V}/\Gamma$ and, moreover, for any $h^{p,q} \in \eta_V$ the bundle $\mathcal{H}^{p,q}$ over D_{ϵ,η_V} descends to a bundle over the quotient $D_{\epsilon,\eta_V}/\Gamma$.

Definition 4.9. *Let $\alpha \in H^*(D_{\epsilon,\eta_V}/\Gamma)$. The higher genus $\chi_y^{[\alpha]}$ is given by:*

$$\chi_y^{[\alpha]} = \pi^*(\alpha) \cup \widetilde{T}_y(B)[B].$$

Among variations of Hodge structures one can single out those for which, in the case $\epsilon = -1$ there are at most two non-vanishing Hodge numbers, and for $\epsilon = +1$ if $p \neq q$ then all $h^{p,q} = 0$ except for at most two of them for which one has $h^{p,q} = 1$. In this case, the period domain is simply-connected since it is the Siegel upper-half plane for $\epsilon = -1$ and $SO(2, h^{p,p})/U(1) \times SO(h^{p,p})$, i.e. the quotient by the maximal compact subgroup, for $\epsilon = +1$ (cf. [13], p.145). In this case the period map factors as $B \to B\pi_1(B) \to D_{\epsilon,\eta_V}/\Gamma = B(\Gamma)$ (the latter is the classifying space of Γ), and the $\chi_y^{[\alpha]}$ coincides with the higher-χ_y genus considered in [7] (see also Remark 4.5). We shall refer to such variations as topological variations of Hodge structures.

Next we shall consider a geometric variation of Hodge structures arising from a smooth proper map of smooth projective varieties $f : E \to B$ (or of compact Kähler manifolds). The Kähler class ω_E of E induces the Kähler class ω_F on each fiber F, which is left invariant by the monodromy of this fibration. In particular the Hodge form on $H^k(F)$ (given by

$Q(\alpha,\beta)=(\omega_F^{dimF-2k}\cup\alpha\cup\beta)[F])$ is a monodromy invariant. The primitive cohomology of F yields a polarized variation of pure Hodge structures, and the fundamental group $\pi_1(B)$ acts via the monodromy representation on $\oplus_k H^k_{prim}(F)$ and also on $\prod_k D_{(-1)^k,\eta_k}$, where η_k is the partition of $\dim H^k(F)_{prim}=\sum_{p+q=k}\dim H^{p,q}(F)\cap H^k(F)_{prim}$. Let $\bar{\Gamma}$ be the quotient on $\pi_1(B)$ by the kernel of this action, and Γ be a subgroup of finite index acting freely.

Theorem 4.10. *Let $f: E\to B$ be a smooth proper map of smooth projective varieties, and*

$$\pi: B\to(\prod_k D_{(-1)^k,\eta_k})/\Gamma$$

be the total period map. Let π_k be the projection of the target of π on the k-th component, and let $\alpha_\Gamma^{p,q}=\pi_k^*(ch^*(\mathcal{H}^{p,q}))$ be the pull back to the quotient of the total period map of the Chern character of the bundle $\mathcal{H}^{p,q}$. Then $\chi_y(E)$ is given by the formula (compare with (4.7))

$$\chi_y(E)=\sum_{p,q}(-1)^q\chi_y^{[\alpha_\Gamma^{p,q}]}(B)\cdot y^p. \tag{4.13}$$

The proof follows from the formula (4.2) similarly to the way (4.7) was derived from (4.2).

Remark 4.11. If $\Gamma=1$ (i.e., the monodromy group $\bar{\Gamma}$ is trivial or finite) we obtain multiplicativity. More generally, if the $H^{p,q}$ are monodromy invariant then the period map is homotopic to the map to a point and again one has multiplicativity.

Remark 4.12. Fibrations for which the fibers are curves or K3 surfaces induce topological variations of Hodge structures and hence the χ_y-genus of the total space can be expressed in terms of Novikov's type χ_y-genus as in Remark 4.5. On the other hand, for fibrations with fibers of higher dimensions one needs the generalization of the higher χ_y-genus except for very special cases.

Remark 4.13. The generalization of the higher χ_y-genus given in definition (4.9) has the strong birational invariance property of higher genera corresponding to twisting by the cohomology classes of the fundamental group. More precisely, we have the K-equivalence relation among pairs $(B_1,\mathbb{V}_1)$ and $(B_2,\mathbb{V}_2)$ generated by the elementary K-equivalence $f_1: X\to B_1, f_2: X\to B_2$ such that $f_1^*(\mathbb{V}_1)=f_2^*(\mathbb{V}_2)$. The monodromy groups Γ and total period domains D of $\mathbb{V}_1$ and $\mathbb{V}_2$ are the same, and for any α in $H^*(D/\Gamma)$ the $\chi_y^{[\alpha]}$-genera of $(B_i,\mathbb{V}_i)$ coincide. This follows from the push-forward formulas in the same way as in [7]. We shall discuss characterization of such generalized twisted χ_y invariants elsewhere.

5. Atiyah-Meyer type characteristic class formulae.

In this section, we present characteristic class versions of the above Atiyah-Meyer formulae for the χ_y-genus. The proofs of these characteristic class formulae are much more involved, and make use of Saito's theory of mixed Hodge modules and the construction of the motivic Hirzebruch classes (cf. [8]), which we recall here. For full details on this construction, the reader is advised to consult [8, 10].

Let Z be a complex algebraic variety. Then for any $p \in \mathbb{Z}$ one has a functor of triangulated categories

$$gr_p^F DR : D^b MHM(Z) \to D^b_{coh}(Z)$$

commuting with proper push-down, where $D^b_{coh}(Z)$ is the bounded derived category of sheaves of $\mathcal{O}_Z$-modules with coherent cohomology sheaves. If $\mathbb{Q}^H_Z \in D^b MHM(Z)$ denotes the constant Hodge module on Z, and if Z is smooth and pure dimensional then $gr^F_{-p} DR(\mathbb{Q}^H_Z) \simeq \Omega^p_Z[-p] \in D^b_{coh}(Z)$. The transformations $gr_p^F DR(M)$ are functors of triangulated categories, so they induce functors on the level of Grothendieck groups. Thus, if $G_0(Z) \simeq K_0(D^b_{coh}(Z))$ denotes the Grothendieck group of coherent sheaves on Z, we obtain the following group homomorphism commuting with proper push-down:

$$gr^F_{-*} DR : K_0(MHM(Z)) \to G_0(Z) \otimes \mathbb{Z}[y, y^{-1}], \tag{5.1}$$

$$[M] \mapsto \sum_p \left(\sum_i (-1)^i [\mathcal{H}^i (gr^F_{-p} DR(M))] \right) \cdot (-y)^p.$$

We can now make the following definitions (see [8, 10])

Definition 5.1. The transformation $\widetilde{MHT_y}$ is defined as the composition of transformations [4]:

$$\widetilde{MHT_y} := td_* \circ gr^F_{-*} DR : K_0(MHM(Z)) \to H^{BM}_{2*}(Z) \otimes \mathbb{Q}[y, y^{-1}], \tag{5.2}$$

where td_* is the Baum-Fulton-MacPherson Todd class transformation [4], which is linearly extended over $\mathbb{Z}[y, y^{-1}]$. Note that $\widetilde{MHT_y}$ commutes with proper push-forward.

Remark 5.2. Let $K^0(Z)$ be the Grothendieck group of complex algebraic vector bundles on Z. If Z an algebraic manifold, the canonical map $K^0(Z) \to G_0(Z)$ induced by taking the sheaf of sections is an isomorphism, and the Todd class transformation of the classical Grothendieck-Riemann-Roch theorem is explicitly described by $td_*(\cdot) = ch^*(\cdot) td^*(T_Z) \cap [Z]$.

Definition 5.3. The *Hirzebruch class* of an n-dimensional complex algebraic variety Z is defined by the formula

$$\widetilde{T}_{y*}(Z) := \widetilde{MHT_y}([\mathbb{Q}^H_Z]). \tag{5.3}$$

Similarly, if Z is an n-dimensional complex algebraic manifold, and $\mathbb{V}$ a polarized variation of Hodge structures on Z, we define *twisted Hirzebruch characteristic classes*

$$\widetilde{T}_{y*}(Z; \mathbb{V}) = \widetilde{MHT_y}([\mathbb{V}^H]), \tag{5.4}$$

where $\mathbb{V}^H[n] = ((\mathcal{V}, \bigtriangledown), \mathcal{F}_{-\bullet}, \mathbb{V}[n])$ is the smooth mixed Hodge module on Z corresponding to $\mathbb{V}$, with $\mathcal{F}_{-\bullet} := \mathcal{F}^\bullet$ (e.g., see [34], Thm. 14.30 and the references therein).

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[4] The special case of the transformation $\widetilde{MHT_y}$ at $y = 1$ was previously used by Totaro [45] for finding numerical invariants of singular varieties, more precisely Chern numbers that are invariant under small resolutions.

By [[8], Lemma 3.1 and Theorem 3.1], the following normalization holds: if Z is smooth and pure dimensional, then $\widetilde{T}_{y*}(Z) = \widetilde{T}^*_y(T_Z) \cap [Z]$, thus $\widetilde{T}_{y*}(Z)$ is an extension to the singular setting of (the Poincaré dual of) the un-normalized Hirzebruch class.

We can now prove the following Meyer-type formula for the twisted Hirzebruch characteristic classes

Theorem 5.4. *Let Z be a complex algebraic manifold of pure dimension n, and $\mathbb{V}$ a polarized variation of Hodge structures on Z with associated flat bundle with "Hodge" filtration $(\mathcal{V}, \mathcal{F}^\bullet)$. Then*

$$\widetilde{T}_{y*}(Z;\mathbb{V}) = \left(ch^*(Hc_y(\mathcal{V})) \cup \widetilde{T}_y^{\,*}(T_Z)\right) \cap [Z] = ch^*(Hc_y(\mathcal{V})) \cap \widetilde{T}_{y*}(Z), \tag{5.5}$$

where $Hc_y(\mathcal{V})$ is the K-theory Hodge polynomial characteristic of $\mathcal{V}$.

Proof. Let $\mathbb{V}^H[n] = ((\mathcal{V}, \bigtriangledown), \mathcal{F}_{-\bullet}, \mathbb{V}[n])$ be the smooth mixed Hodge module on Z corresponding to $\mathbb{V}$, with $\mathcal{F}_{-\bullet} := \mathcal{F}^\bullet$ the increasing filtration on the $\mathcal{D}$-module $\mathcal{V}$. It follows from Saito's work that there is a filtered quasi-isomorphism between $(DR(\mathbb{V}^H), F_{-\bullet})$ and the usual filtered de Rham complex $(\Omega^\bullet_Z(\mathcal{V}), F^\bullet)$ with the filtration induced by Griffiths' transversality, that is:

$$F^p\Omega^\bullet_Z(\mathcal{V}) : \left[\mathcal{F}^p \overset{\bigtriangledown}{\to} \Omega^1_Z \otimes \mathcal{F}^{p-1} \overset{\bigtriangledown}{\to} \cdots \overset{\bigtriangledown}{\to} \Omega^i_Z \otimes \mathcal{F}^{p-i} \overset{\bigtriangledown}{\to} \cdots\right].$$

Therefore,

$$\begin{aligned}
\widetilde{T}_{y*}(Z;\mathbb{V}) &= td_*\left(\sum_p \left(\sum_i (-1)^i[\mathcal{H}^i(gr^F_{-p}DR(\mathbb{V}^H))]\right) \cdot (-y)^p\right)\\
&= td_*\left(\sum_p \left(\sum_i (-1)^i[\mathcal{H}^i(gr^p_F\Omega^\bullet_Z(\mathcal{V}))]\right) \cdot (-y)^p\right)\\
&= td_*\left(\sum_p \left(\sum_i (-1)^i[\Omega^i_Z \otimes Gr^{p-i}_{\mathcal{F}}\mathcal{V}]\right) \cdot (-y)^p\right)\\
&= \sum_p \sum_i (-1)^i\ td_*([\Omega^i_Z \otimes Gr^{p-i}_{\mathcal{F}}\mathcal{V}]) \cdot (-y)^p\\
&\overset{(GRR)}{=} \sum_p \sum_i (-1)^i\ ch^*(\Omega^i_Z \otimes Gr^{p-i}_{\mathcal{F}}\mathcal{V}) \cup td^*(Z) \cap [Z] \cdot (-y)^p\\
&= \sum_p \sum_i \left(ch^*(Gr^{p-i}_{\mathcal{F}}\mathcal{V}) \cdot (-y)^{p-i}\right) \cup \left(ch^*(\Omega^i_Z) \cdot y^i\right) \cup td^*(Z) \cap [Z]\\
&= \left(\sum_q ch^*(Gr^q_{\mathcal{F}}\mathcal{V}) \cdot (-y)^q\right) \cup \left(\sum_i ch^*(\Omega^i_Z) \cdot y^i\right) \cup td^*(Z) \cap [Z]\\
&= ch^*(Hc_y(\mathcal{V})) \cup ch^*(\lambda_y(T^*_Z)) \cup td^*(Z) \cap [Z]\\
&= ch^*(Hc_y(\mathcal{V})) \cup \tilde{T}_y^{\,*}(T_Z) \cap [Z],
\end{aligned}$$

where (GRR) means an application of the classical Grothendieck-Riemann-Roch theorem. $\square$

Remark 5.5. More generally, Saito [37] showed that an admissible variation of mixed Hodge structures on a smooth variety Z (e.g., a geometric variation of mixed Hodge structures), with underlying local system $\mathcal{L}$, gives rise to a smooth mixed Hodge module on Z, call it $\mathcal{L}^H[\mathrm{dim} Z]$, with $\mathcal{L}[\mathrm{dim} Z]$ as its underlying perverse sheaf. Then one can define as in (5.4) twisted Hirzebruch characteristic classes associated to such admissible variations. Note that Theorem 5.4 remains true in this greater generality, i.e., we can let $\mathbb{V}$ be a geometric variation of (mixed) Hodge structures, or more generally, an admissible variation of mixed Hodge structures. Then if Z is complete, Corollary 4.6 can be obtained from Theorem 5.4 by pushing forward to a point via the constant map $Z \to pt$.

Jörg Schürmann [42] communicated to us that the following Atiyah-type result can be obtained as a direct application of the Verdier-Riemann-Roch formula for a smooth proper morphism (cf. [8], Cor. 3.1(3)), if one makes the identification $\mathcal{H}^{p,q} \simeq R^q f_*(\Lambda^p T_f^*)$, with T_f^* the dual of the tangent bundle T_f to the fibers of f (cf. [34], Proposition 10.28). However, the proof we give here is based only on the definition of the Hirzebruch classes and on Theorem 5.4 in the context of geometric variations of Hodge structures.

Theorem 5.6. *Let $f : E \to B$ be a smooth projective morphism between complex algebraic manifolds. Then the following holds:*

$$f_*\widetilde{T}_{y_*}(E) = ch^*\left(\chi_y(f)\right) \cap \widetilde{T}_{y_*}(B), \tag{5.6}$$

where $\chi_y(f) = \sum_{p,q\geq 0}(-1)^q \mathcal{H}^{p,q} \cdot y^p$ is the K-theory χ_y-genus of f.

Proof. Since f is proper and the transformation $\widetilde{MHT_y}$ commutes with proper pushdowns, we first obtain the following:

$$f_*\widetilde{T}_{y_*}(E) = f_*(\widetilde{MHT_y}([\mathbb{Q}_E^H])) = \widetilde{MHT_y}([f_*\mathbb{Q}_E^H]). \tag{5.7}$$

Now let $\tau_{\leq}$ be the natural truncation on $D^bMHM(B)$ with associated cohomology $H^\bullet$. Then for any complex $M^\bullet \in D^bMHM(B)$ we have the identification (e.g., see [20], p. 95-96; [40], Lemma 3.3.1)

$$[M^\bullet] = \sum_{i\in\mathbb{Z}}(-1)^i[H^i(M^\bullet)] \in K_0(D^bMHM(B)) \cong K_0(MHM(B)). \tag{5.8}$$

In particular, if for any $k \in \mathbb{Z}$ we regard $H^{i+k}(M^\bullet)[-k]$ as a complex concentrated in degree k, then

$$\left[H^{i+k}(M^\bullet)[-k]\right] = (-1)^k[H^{i+k}(M^\bullet)] \in K_0(MHM(B)). \tag{5.9}$$

Therefore, if we let $M^\bullet = f_*\mathbb{Q}_E^H$, we obtain that

(5.10)
$$f_*\widetilde{T}_{y_*}(E) = \sum_{i\in\mathbb{Z}}(-1)^i\widetilde{MHT_y}([H^i(f_*\mathbb{Q}_E^H)]) = \sum_{i\in\mathbb{Z}}(-1)^i\widetilde{MHT_y}(\left[H^{i+\mathrm{dim}B}(f_*\mathbb{Q}_E^H)[-\mathrm{dim}B]\right]).$$

Note that $H^i(f_*\mathbb{Q}^H_E) \in MHM(B)$ is the smooth mixed Hodge module on B whose underlying rational complex is (recall that B is smooth)

$$\mathrm{rat}(H^i(f_*\mathbb{Q}^H_E)) = {}^p\mathcal{H}^i(Rf_*\mathbb{Q}_E) = (R^{i-\dim B}f_*\mathbb{Q}_E)[\dim B], \tag{5.11}$$

where for the second equality we refer to [34], Example 13.20. In this case, each of the local systems $\mathcal{L}_s := R^s f_*\mathbb{Q}_E$ underlies a geometric variation of Hodge structures.

Altogether, (5.10) becomes

$$f_*\widetilde{T}_{y*}(E) = \sum_{i\in\mathbb{Z}}(-1)^i\widetilde{T}_{y*}(B;\mathcal{L}^H_i), \tag{5.12}$$

where, by analogy with Definition 5.3, $\mathcal{L}^H_i[\dim B] := H^{i+\dim B}(f_*\mathbb{Q}^H_E)$ is the smooth mixed Hodge module whose underlying perverse sheaf is $\mathcal{L}_i[\dim B]$.

Our formula (5.6) follows now from Theorem 5.4 and Remark 5.5.

□

Remark 5.7. If B in the above theorem is also complete, then by pushing (5.6) down to a point via the constant map $B \to pt$, we get back the result of Theorem 4.1.

An immediate corollary of Theorem 5.6 is the following extension of [[10], Cor. 3.12], whose proof imitates that of Corollary 4.3.

Corollary 5.8. *Under the assumptions of the above theorem, if moreover $R^k f_*\mathbb{Q}_E$ is a local system of Hodge structures for each k, i.e. the monodromy action of $\pi_1(B)$ on the cohomology of the fiber preserves the Hodge filtration, then*

$$f_*\widetilde{T}_{y*}(E) = \chi_y(F)\widetilde{T}_{y*}(B), \tag{5.13}$$

where F is the fiber of f.

Remark 5.9. Note that in the proof of Theorem 5.6 we only used the fact that f is proper (so that the characteristic class transformation $\widetilde{MHT_y}$ commutes with f_*), and that the local systems $\mathcal{L}_s := R^s f_*\mathbb{Q}_E$, $s \in \mathbb{Z}$, underly admissible variation of mixed Hodge structures on the smooth variety B. Therefore Theorem 5.6 admits the following generalization, where we allow the generic fiber and the domain of the map to be singular:

Theorem 5.10. *Let $f : E \to B$ be a projective surjective morphism of complex algebraic varieties, with B smooth, such that the sheaves $R^s f_*\mathbb{Z}_E$, $s \in \mathbb{Z}$ are locally constant on B. Then*

$$f_*\widetilde{T}_{y*}(E) = ch^*(\chi_y(f)) \cup \widetilde{T}_{y*}(B), \tag{5.14}$$

where $\chi_y(f) := \sum_{i,p}(-1)^i Gr^p_{\mathcal{F}}\mathcal{H}_i \cdot (-y)^p \in K(B)[y]$ is the K-theory χ_y-genus of f.

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