ON THE TOPOLOGY OF SOME
EVEN-DIMENSIONAL ALGEBRAIC HYPERSURFACES

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1. INTRODUCTION

The topological structure of odd-dimensional hypersurfaces was investigated recently by several authors [5], [9], [10]. These results are the generalization of the classical handle-decomposition of Riemann surfaces and can be summarized as follows.

Theorem. Let $V_n$ be a non-singular algebraic hypersurface of an odd dimension $n$ and of degree $d$ in an $(n + 1)$-dimensional complex projective space. Let $T$ denote the PL-manifold which is obtained by gluing a cone to the spherical boundary of the plumbing of two copies of the tangent bundles of the sphere $S^n$. Then there are two cases

(i) If $d \equiv \pm 3 \pmod{8}$ or $n = 1, 3, 7$, then

$$V_n = k(S^n \times S^n) \# M_n$$

(ii) If $d \equiv \pm 3 \pmod{8}$ and $n \neq 1, 3, 7$, then

$$V_n = (k - 1)(S^n \times S^n) \# T \# M_n$$

where $k = (1/2)rkH_n(V_n, \mathbb{Z})$ and $M_n$ is a manifold (generally piecewise-linear in case (ii)) which has the same homology as the projective space $CP^n$.

Note also that $M_n$ can either be realized as a rational singular projective hypersurface [5] or as a gluing of two copies of some $D^{n+1}$ disk bundle over $CP^{n-1/2}$ (see Remark 5 below).

The topology of even-dimensional hypersurfaces is considered in [4], where it is proved that it is possible to split off the maximal number of the handles $S^n \times S^n$. The manifold which remains after removing these handles is in general decomposable into a connected sum of more simple manifolds.

In this paper we describe the decomposition of some even-dimensional algebraic hypersurfaces into the connected sum of indecomposable almost differentiable manifolds (PL-manifolds with the differentiable structure on the complement of a point).

Let $F_{2n}(E)$ be the $2n$-dimensional manifold which is the plumbing of the tangent bundles of the sphere $S^n$ according to the graph $E$ [3]. The boundary of $F_{2n}(E)$ is homeomorphic to a sphere and we denote by $F_{2n}(E)$ the closed PL-manifold which is obtained by adding the cone over the sphere $S^{2n-1}$ to the boundary of $F_{2n}(E)$. Let $V_n$ be hypersurface of dimension $n$ and degree $d$ in $CP^{n+1}$. Our main result is the following:

Theorem. Suppose that $n \geq 2$, $n \equiv 2 \pmod{4}$ and $d$ is even. Then

$$V_n = a(S^n \times S^n) \# bF_{2n}(E) \# M_n$$

where $\approx$ stands for a PL-homeomorphism. $a$ =
The manifold $M_c^d$ can be obtained by gluing two copies of a certain $D^d$-bundle over $\mathbb{CP}^n$ by means of a PL-homeomorphism of the boundary.

In conclusion we formulate some results concerning the decomposition of other classes of even-dimensional hypersurfaces; their proof is based on the same ideas used in the proof of the main theorem.

§ 2. PROOF OF THE MAIN THEOREM

Recall that all non-singular hypersurfaces of a given degree and dimension are diffeomorphic. So it is enough to describe the decomposition for some special model of a hypersurface $V_n^d$. Let us consider the hypersurface $V_n^d(c)$ which is the projective completion of the affine hypersurface defined by the equation

$$P_n(Z_1, \ldots, Z_n) = Z_1^d + Z_2^{d-1} + Z_3^{d-1} + \cdots + Z_n^{d-1} + Z_{n+1} = c.$$  

The properties of this hypersurface are considered in [5]. For $c$ sufficiently close to zero we denote by $F_n^d$ the intersection of $V_n^d(c)$ with the ball $B_\varepsilon$ of a small radius $\varepsilon$ centered at the point with coordinates $Z_1 = Z_2 = \cdots = Z_n = 0$. Let

$$G_n^d = V_n^d(c) - F_n^d.$$  

**Proposition [5]**

(i). The hypersurface $V_n^d(c)$ is a non-singular projective variety for $c \neq 0$ and has a single isolated singularity for $c = 0$ at the point $Z_1 = \cdots = Z_{n+1} = 0$.

(ii). $F_n^d$ is an $(n-1)$-connected parallelizable $2n$-manifold with boundary and its $n$-dimensional Betti number $b_n$ is given by

$$b_n = \frac{1}{d} \left( (d-1)^{n+2} - 1 \right).$$  

(iii). The characteristic polynomial $\Delta_n(t)$ of the monodromy of the isolated singularity of $V_n^d(0)$ can be computed by the recursive equation

$$\Delta_{n+1}(t) = \Delta_n^{-1}(t) \frac{t^{d(d-1)^{n+1} - 1}}{t^{d(d-1)^n} - 1}.$$  

**Sketch of the proof.** (i) can be verified by direct computation. (ii) is a consequence of the fact that the polynomial $P_n(Z_1, \ldots, Z_{n+1})$ is weighted homogeneous of the weight

$$\left( d, d-1, \ldots, \frac{d(d-1)^i}{(d-1)^i + (-1)^{i+1}}, \ldots, \frac{d(d-1)^n}{(d-1)^n + (-1)^{n+1}} \right).$$  

(iii) can be checked by using the Milnor–Orlik[8] algorithm for the computation of a characteristic polynomial of the weighted homogeneous singularities[5].

From now on we consider the case $n$ is even and $n \geq 2$. We denote by $H_\ast(V_n^d, \mathbb{Z})$ the group of vanishing cycles, i.e. $\text{Ker}(H_\ast(V_n^d, \mathbb{Z}) \rightarrow H_\ast(\mathbb{CP}^{n+1}, \mathbb{Z}))$. This group can be also described as the image of the Hurewicz homomorphism $\pi_\ast(V_n^d) \rightarrow H_\ast(V_n^d)$ or as the orthogonal complement to the homology class $h$ of the intersection of $V_n^d$ and $\mathbb{CP}^{n+1}[4]$. Let us denote by $l$ the projective space defined by $Z_1 = c'Z_0, Z_2 = Z_4 = \cdots$.
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. . .= \sum_{i=1}^{n} a_i = 0$ belonging to $V_n^d(c)$. Note that the intersection of $l$ with the small ball $B$, centered at the point with coordinates $Z_1 = Z_2 = \cdots = Z_n = Z_{n+1} = 0$ is a disk. Denote this disk by $l_1$ and $l = l_1$ denote by $l_2$. By abuse of notation we let $l_1$ (resp. $l_2$) also denote the relative homology class in $H_n(F_n^d, \partial F_n^d)$ (resp. in $H_n(G_n^d, \partial G_n^d)$) defined by $l_1$ (resp. $l_2$).

**Lemma 1.** (a) The homomorphism

$$i_F: H_n(F_n^d, \mathbb{Z}) \to H_n(V_n^d, \mathbb{Z})$$

is an imbedding and its image is the subgroup of vanishing cycles, $H_n(V_n^d, \mathbb{Z})_0$.

(b) The group $H_n(G_n^d, \mathbb{Z})$ is generated by $h$ and the group $H_n(G_n^d, \partial G_n^d, \mathbb{Z})$ by $l_2$. In $H_n(G_n^d, \partial G_n^d, \mathbb{Z})$ holds the relation $dl_2 = h$.

(c) $\partial F_n^d$ is a $\mathbb{Q}$-homology sphere with $H_{n-1}(\partial F_n^d, \mathbb{Z}) = \mathbb{Z}/d$.

**Proof.** (a) From (iii) of the proposition we conclude that $\Lambda_n(1) = d$ for $n$ even. Hence $H_{n-1}(\partial F_n^d)$ is finite of order $d$ [6]. By Poincare duality $H_n(\partial F_n^d) = 0$. We now consider the following Mayer–Vietoris sequence

$$0 \to H_n(F_n^d) \oplus H_n(G_n^d) \to H_n(V_n^d) \to H_{n-1}(\partial F_n^d) \to 0. \quad (1)$$

It follows that $i_F$ is an imbedding. The image of $i_F$ belongs to the subgroup $H_n(V_n^d, \mathbb{Z})_0$ because the Hurewicz homomorphism is onto for the $(n-1)$-connected $2n$-manifolds. The rank of $H_n(V_n^d, \mathbb{Z})$ is equal to $(1/d)(d-1)^{n-2} + (d-1)] + 1$ [4], [9]. Hence

$$rkH_n(V_n^d, \mathbb{Z})_0 = \frac{1}{d} [(d-1)^{n-2} + (d-1)] = rkH_n(F_n^d, \mathbb{Z})$$

by (ii) of the proposition. It follows that $i_F(H_n(F_n^d))$ has finite index in $H_n(V_n^d, \mathbb{Z})_0$.

From the diagram

$$\begin{array}{c}
H_n(F_n^d) \xrightarrow{i_F} H_n(V_n^d, \mathbb{Z})_0 \\
\downarrow \quad \quad \quad \downarrow \\
H_n(\partial F_n^d, \mathbb{Z})^* \leftarrow H_n(V_n^d, \mathbb{Z})_0^*
\end{array}$$

we obtain that this index equals $1$ because

$$[H_n(F_n^d): H_n(F_n^d)^*] = [H_n(V_n^d)_0: H_n(V_n^d)^*_0] = d$$

(wher for an abelian group $E$ we denote by $E^*$ the dual group $\text{Hom}(E, \mathbb{Z})$).

(b) It follows from the sequence (1) that $H_n(G_n^d, \mathbb{Z})$ is a free cyclic group. Because the number of elements in $H_{n-1}(\partial G_n^d) = H_{n-1}(\partial F_n^d)$ is $d$ we obtain that the square of the generator of $H_n(G_n^d, \mathbb{Z})$ is equal to $d$, i.e. $h$ is a generator of $H_n(G_n^d, \mathbb{Z})$. The other assertions of (b) follows from this.

(c) The manifold $\partial F_n^d$ is $(n-2)$-connected [6] and because $H_{n-1}(\partial F_n^d, \mathbb{Z})$ is a torsion group (see (a)) we obtain the first part of the assertion. The second one follows from (b).

**Lemma 2.** If $d$ is even and $n \equiv 2 \pmod{4}$, then there exists an element $v \in H_n(F_n^d, \partial F_n^d, \mathbb{Z})$ such that $v^2 = -1/d$. 
Proof. For any $2n$-dimensional manifold $X$ we denote by $S(H_n(X))$ the intersection form on $H_n(X, \mathbb{Q})$. Since $\partial F^d$ is a rational homology sphere we have ([3] p. 48)

$$(- S(H_n(F^d))) \oplus (S(H_n(G^d))) = S(H_n(V^d)).$$

Moreover it is computed in [4] that $l_i^2 = (1/d)[1 - (1 - d)^{2i+1}]$. Since $l_i = (1/d)h$ and $h^i = d$, we have $l_i^2 = 1/d$. Hence $l_i^2 = (1/d)[1 - (1 - d)^{2i+1}] - (1/d)$. According to [4], for $d > 2$ there exist elements $e_i$ and $e_i$ in $H_n(V^d, \mathbb{Z})$ and hence in $H_n(F^d, \mathbb{Z})$ such that $e_i^2 = 0$ and $e_i e_j = 1$. Let $a = (l_i, e_i)$ and $b = (l_i, e_i)$. Then the element

$$v = l_i + \left( b(a - 1) - \frac{1}{2d} [1 - (1 - d)^{2i+1}] \right) e_i + (1 - a) e_i$$

has square $-1/d$. Note that under the assumption of the lemma the coefficient of $e_i$ is an integer.

**Lemma 3.** Let $E$ be a free abelian group endowed with a non-singular integer symmetric bilinear form $\beta$. Let $E'/E = \mathbb{Z}/d$. Then $\beta$ induces on $E'$ a bilinear form with values in $\mathbb{Z}_{(d)}$-subgroup of $\mathbb{Q}$, which consists of the fractions with the divisors of $d$ as denominator. If there also exists an element $v \in E'$ such that $v^3 = \pm 1/d$ then there exists an orthogonal decomposition $E' = A \oplus \{v\}$ where $A$ is an inner product space and $\{v\}$ is a subgroup generated by $v$.

**Proof.** This is a special case of Theorem 3.2 from [7].

Now we are ready to conclude the proof of the main theorem. Note that the class $v$ which was built during the proof of Lemma 2 can be represented by an immersed disk whose boundary is the sphere $\partial l_i$. According to A. Haefliger’s theorem for manifolds with boundary ([1] Theorem 4.1) we may suppose that $v$ is realizable by an embedded disk with the same boundary.

Let $T$ denote the tubular neighbourhood of this disk. Let us consider the manifolds $M^d_n = G^d_n \cup T$ and $N^d_n = F^d_n - T$. The abelian group $H_n(N^d_n, \mathbb{Z})$ endowed with the intersection form is isomorphic to $A$ from Lemma 3, which built for $E = H_n(F^d_n, \mathbb{Z})$. It follows that the intersection form on $H_n(N^d_n, \mathbb{Z})$ is unimodular, even and indefinite. Because $N^d_n$ is parallelizable (iii) of proposition) it follows that $N^d_n = a(S^n \times S^n) \cong bF^d(E_8)$. ( $\cong$ denotes the boundary connected sum.)

Now we consider $M^d_n$ which is $M^d_n$ with added cone to the boundary. Let $P_1$ denote the union of $l_2$ with the disk in $F^d_n$ which represents $V$. Obviously $P_1$ is diffeomorphic to the projective space $CP^n_2$. It follows from (2) that the self-intersection index of $P_1$ is equal to zero. Let $P_2$ be obtained by a slight translation of $P_1$ in such a fashion that it does not intersect $P_1$. Let $T_1$ and $T_2$ be non-intersecting tubular neighbourhoods of $P_1$ and $P_2$ respectively. We prove that $M^d_n - T_1 - T_2$ is an $h$-cobordism between $\partial T_1$ and $\partial T_2$. Indeed from the exact sequence of closed subspace

$$\rightarrow H_i(P_1) \rightarrow H_{i-1}(M - P_1) \rightarrow H_{i-1}(M) \rightarrow H_{i-1}(P_1) \rightarrow$$

it follows that $M^d_n - T_1$ has the same homotopy type as $P_2$. Using the exact sequence of the pair we obtain that

$$H_i(M^d_n - T_1, T_2) = H_i(M^d_n - T_1, T_2) = 0.$$
Hence the assertion of the theorem about $M_n^d$ follows from the theorem about $h$-cobordism for PL-manifolds.

§3. SOME CONCLUDING REMARKS

Remark 1. The PL-homeomorphism which is mentioned in the main theorem is not in general smooth, because $M_n^d$ does not generally admit a differential structure. For example for $M_6^d$ we obtain the following values of the Pontrjagin classes $p_1 = -8h^2$; $p_2 = 156h^4$ ($h$ denotes the standard generator of $H^2(M_n^d, \mathbb{Z}) = H^2(V_n^d, \mathbb{Z})$). Furthermore, the signature of $M_n^d$ is zero; thus if we assume that $M_n^d$ is a differentiable manifold we obtain $[2]

\begin{equation}
0 = \text{sign}(M_n^d) = \frac{1}{945} (62p_3 - 13p_2p_1 + 2p_1^3)
\end{equation}

and therefore $p_1$ cannot be an integer cohomology class.

Remark 2. The hypersurfaces $V_n^d$ for which $n = 2 \pmod{4}$ and $d$ is even are exactly the hypersurfaces whose intersection forms on $H_*(V_n^d, \mathbb{Z})$ have even type (see [4]).

Remark 3. If $n \equiv 0 \pmod{4}$ or $d$ is odd then a decomposition $V_n^d = N_n^d \neq M_n^d$ where $N_n^d$ is $(n-1)$-connected and $rkH_* (N_n^d) = rkH_* (V_n^d) - 2$ exists in the following cases. Either $d \equiv 0 \pmod{8}$ (and then sign $M_n^d = 0$), or $d \equiv 2 \pmod{8}$ and all prime divisors of $d$ has form $4l + 1$ (and then sign $M_n^d = 2$). Indeed, the existence of such a decomposition will imply that $H_* (F_n^d, \mathbb{Z})$ has a subgroup $A$ of corank 1 on which the intersection form is unimodular. It would then follow that in $H_* (F_n^d, \partial F_n^d, \mathbb{Z})$ there exists an element $v$ for which $v^2 = \pm 1/d$, but that is possible just in the cases listed above.

Note that non-existence of such a decomposition in the cases $d \neq 0, 2 \pmod{8}$ and $n \equiv 0 \pmod{4}$ or $d$ is odd and $n \equiv 2 \pmod{4}$ follows from the fact that the intersection form on $H_* (V_n^d, \mathbb{Z})$ has even type together with the following lemma.

Lemma 4. If $n \equiv 2 \pmod{4}$ and $d$ is even then $\text{sign} V_n^d \equiv 0 \pmod{8}$, otherwise $\text{sign} V_n^d = d \pmod{8}$.

Proof. As above let $l$ denote the homology class of $\mathbb{C}P^{n+2}$ in $V_n^d$ and $h$ denote the class of the intersection of $V_n^d$ with $\mathbb{C}P^{n+2}$. Each element $c \in H_* (V_n^d, \mathbb{Z})$ can be represented as $c = c_0 + \kappa l$ where $c_0 \in H_* (V_n^d, \mathbb{Z})$ and $\kappa = (c, h)$. It follows then that $(h, c) \equiv (c, c) \pmod{2}$ if $n \equiv 0 \pmod{4}$ or $n \equiv 2 \pmod{4}$ and $d$ is odd. Hence from the van der Blij lemma (see [7]) it follows that $d - h^2 = \text{sign} V_n^d \pmod{8}$.

If $n \equiv 2 \pmod{4}$ and $d$ is even then the intersection form on $H_* (V_n^d, \mathbb{Z})$ has even type and the congruence $\text{sign} V_n^d \equiv 0 \pmod{8}$ follows from [7].

Remark 4. Nevertheless if $n = 4$ (resp. $n = 8$) we have the following decomposition

$$V_n^d \neq P = N_n^d \neq M_n^d$$

where $P$ denotes the projective plane over the quaternions (resp. over the Caley numbers), $N_n^d$ is an $(n-1)$-connected manifold and $M_n^d$ is a gluing of two copies of a $D^n$-disk bundle over $\mathbb{C}P^{n+2}$, and $rkH_* (V_n^d \neq P) = rkH_* (N_n^d) + 2$. 

Remark 5. Let $n$ be odd and $V_n = M_n \neq N_n$ where $\text{rk}H_*(N_n) = \text{rk}H_*(V_n)$ and $N_n$ is an $(n - 1)$-connected manifold [5], [9], [10]. Then $M_n$ also can be represented as a gluing of two copies of a $D^{n-1}$-bundle over $\mathbb{CP}^{n-1/2}$ by means of some homeomorphism of the boundary.

Indeed, let $P_1$ denote $\mathbb{CP}^{n-1/2}$ embedded in $M_n$ such that the induced map $H_i(\mathbb{CP}^{n-1/2}, \mathbb{Z}) \to H_i(M_n, \mathbb{Z})$ is an isomorphism for $i \leq n - 1$. Let $P_2$ be obtained by translation of $P_1$ in such a fashion that it does not intersect $P_1$. Let $T_1$ and $T_2$ be non-intersecting tubular neighbourhoods of $P_1$ and $P_2$ respectively. Then by a computation similar to the above of homology groups it can be shown that $M_n - T_1 - T_2$ is an $h$-cobordism and therefore $M_n$ is equivalent to a gluing of $T_1$ and $T_2$ by means of some homeomorphism of the boundaries.

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