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DIFFERENTIABLE STRUCTURES ON COMPLETE INTERSECTIONS—I†

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§1. INTRODUCTION AND SUMMARY OF RESULTS

A COMPLETE intersection $X_n(\boldsymbol{d}) \subset CP_{n+r}$ is the transversal intersection of hypersurfaces defined by polynomials whose degrees are given by the unordered r-tuple $\boldsymbol{d} = (d_1, \ldots, d_r)$. A different set of polynomials, with the same multidegree $\boldsymbol{d}$, defines a diffeomorphic variety; in fact n and $\boldsymbol{d}$ determine the isotopy class of the embedding $X_n \subset CP_{n+r}$. On the other hand, different multi-degrees may correspond to diffeomorphic varieties. The simplest examples occur in the case of algebraic curves: $X_1(1) = X_1(2) = S^2$ and $X_1(2, 2) = X_1(3) = S^1 \times S^1$. For these oriented 2-manifolds the diffeomorphism type is determined by the Euler characteristic, $e = d\left\{2 - \sum_{j=1}^{r}(d_j - 1)\right\}$ where d is the total degree, $d = \Pi d_j$.

A general problem is: what invariants, computed from n and $\boldsymbol{d}$, determine the diffeomorphism type, or the homotopy type, of X in higher dimensions. In this paper we apply the exact sequence of surgery theory to obtain some partial results. One consequence is that in any odd dimension there are arbitrarily large sets of multi-degrees for which the corresponding complete intersections are all diffeomorphic.

In §2 we study homotopy classification and show that, if n is odd and d has no divisors less than $(n + 3)/2$, the homotopy type of $X_n(\boldsymbol{d})$ is determined by n, d, and the Euler characteristic.

The surgery theory is presented in §3 with some related computations in KO-theory and stable cohomotopy in §4 and §5. The concluding result of §3 is that the number of diffeomorphism types of complete intersections of degree d, with d satisfying certain conditions and with given Pontryagin classes and Euler characteristics, is less than a bound depending only on n. In §6 a counting argument shows the existence of diffeomorphic complete intersections.

Some formulas for characteristic classes are given in §7. One consequence is a uniqueness result for the multidegree $\boldsymbol{d}$ in low codimension. If the dimension n and codimension r satisfy $2r \leq n$ and $n > 2$ then the degree and Pontryagin classes of a complete intersection determine the exact codimension and multidegree, see (7.1). (Thus codimension $r > n/2$ is necessary for the examples of §6.) Hartshorne has conjectured ([11], p. 1017) that a smooth subvariety in CP_{n+r} of codimension r satisfying $2r < n$ is a complete intersection. By (7.1) its total degree and Pontryagin classes would determine its multidegree. In §8 are two somewhat related uniqueness results: if X and Y are complete intersections, which are diffeomorphic, then they are ambiently isotopic. If X and Y are analytically equivalent (and if $n \geq 2$ and $c_1X \neq 0$), then X and Y are equivalent by a projective linear transformation and necessarily have the same multidegree. This gives, in (8.3), examples for which the moduli space of complex structures on the smooth manifold underlying a complete intersection is reducible.

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In §9 there is a survey of classification results applied to 3-folds and a number of explicit examples.

Some results of this paper were announced in [18] with an outline of the proof.

§2. HOMOLOGY PROJECTIVE SPACES

In this section we describe the homotopy classification of certain homology projective spaces which arise from complete intersections.

Definition: A d-twisted homology CP_n, $n = 2m+1$, is a simply connected, $2n$-dimensional CW-space M whose cohomology ring is isomorphic to

$$Z[x, y]/\{x^{m+1} = dy, y^2 = 0\}.$$

where the dimensions of x and y are 2 and $n+1$ respectively.

The usual CP_n is 1-twisted and conversely any 1-twisted homology CP_n has the homotopy type of CP_n (by Whitehead's theorem). The result we need is:

THEOREM 2.1. *If $n = 2m+1$ and if d has no divisors less than $m+2$, then any two d-twisted homology CP_n's are homotopy equivalent.*

In the case $n = 3$ this corresponds with known results on simply connected 6-manifolds ([29], §1.5) as we explain in detail in §9.

The connection with complete intersections is provided by the following.

PROPOSITION 2.2. (*i*) *There is a differentiable connected sum decomposition*

$$X_n(d) = M \# S^n \times S^n \# \cdots \# S^n \times S^n$$

where M is a smooth d-twisted homology CP_n unless the Kervaire invariant of $X_n(d)$ is defined and nonzero.

(*ii*) *Otherwise there is a topological connected sum*

$$X_n(d) = M \# K \# S^n \times S^n \# \cdots \# S^n \times S^n$$

where M is a d-twisted homology CP_n and K is the Kervaire manifold obtained by plumbing together at one point two copies of the tangent disk bundle to S^n.

(*iii*) *For d odd (and $n \neq 1$, 3 or 7) the first case holds if and only if $d = \pm 1$ mod 8. It holds always for $n = 1$, 3, or 7.*

The case d odd is given in [27] and the complete result in [5].

COROLLARY 2.3. *If $n = 2m+1$, and if d has no divisors less than $m+2$, then the homotopy type of $X_n(d)$ is determined by n, d, and the Euler characteristic or equivalently by its integral cohomology ring.*

An example in (9.2) below shows some condition on d is necessary.

Proof. For $n = 1$ the Euler characteristic suffices. For $n \geq 3$, the homotopy type of M is determined by n and d according to Theorem 2.1, d is odd so whether a K summand is present is determined by d mod 8, and the number of summands is determined by the Euler characteristic.

Twisted homology CP_n's also arise from certain free S^1-actions. Let M be a $2m$-connected $(4m+3)$-manifold with $H_{2m+1}(M) = Z/d$. If S^1 acts freely on M the Gysin sequence shows M/S^1 is a d-twisted homology CP_{2m+1}. If d has no divisors less than $m+2$, then such an action is unique up to equivariant homotopy. There is a natural example of this: let M be the intersection in C^{2m+2} of the unit sphere with the hypersurface defined by

$$z_0{}^d + z_1 z_2^{d-1} + z_2 z_3^{d-1} + \cdots + z_{2m} z_{2m+1}^{d-1} = 0.$$

The quotient M/S^1 is a d-twisted CP_{2m+1}, see [17]. Here M and the action are PL but not smooth in the structure induced from C^{2m+2}.

Proof of Theorem (2.1). Consider the map $CP_n \to K(Z/d,\ n+1)$ defined by the generator of $H^{n+1}(CP_n; Z/d)$ and let E be the total space of the induced fibration

$$K(Z/d, n) \to E \to CP_n.$$

We show that up to dimension $2n$ the integral cohomology of E is that of a d-twisted homology CP_n. Recall

$$H^j(K(Z/d, n); Z) = \begin{cases} Z & j = 0 \\ Z/d & j = n+1 \\ 0 & 1 \leq j \leq n \text{ or } n+2 \leq j \leq 2n+1 \end{cases}$$

since d has no divisors less than $m+1$.

Spectral sequence arguments show that

$$H^j(CP_n; Z) \to H^j(E; Z)$$

is an isomorphism for $j \leq n$;

$$0 \to H^j(CP_n; Z) \to H^j(E; Z) \to Z/d \to 0$$

is exact for j even, $n+1 \leq j \leq 2n$; and $H^j(E; Z) = 0$ for j odd, $1 \leq j \leq 2n+1$. Moreover, arguments using the spectral sequence with Z/p coefficients for p dividing d show that $H^j(E; Z/p)$ is cyclic for $0 \leq j \leq 2n$ so the sequence above does not split, that is $H^j(E; Z) = Z$ for j even, $0 \leq j \leq 2n$. Further if x generates $H^2(E; Z) = H^2(CP_n; Z)$, then $H^{n+1}(E; Z)$ is generated by an element y such that $x^{m+1} = dy$ and $x^j y$ generates $H^{2j+n+1}(E; Z)$ for $0 \leq j \leq m$.

Now let M be a d-twisted homology CP_n. The map $M \to CP_n$ classifying the generator in dimension 2 lifts to a map $M \to E$ whch induces an isomorphism in integral cohomology up through dimension $2n+1$. For any other such space the map $M' \to CP_n$ lifts first to E and then to M; hence it follows from the Whitehead theorem ([25], p. 405) that two d-twisted homology CP_n's are homotopy equivalent.

§3. DIFFERENTIABLE STRUCTURES

In this section we use the surgery exact sequence to obtain a bound on the number of smooth structures with given Pontryagin classes that a $2n$-dimensional twisted projective space M admits. Assume M has a given smooth structure; the surgery exact sequence ([4], II 4.10) is

$$0 \to hS(M) \to [M, G/O] \xrightarrow{\sigma} Z/2.$$

Following Brumfiel ([7], §4), to study $[M, G/O]$ we use the exact sequence

$$\widetilde{KO}^{-1}(M) \to \tilde{\pi}_s^0(M) \to [M, G/O] \to \widetilde{KO}^0(M) \to \tilde{J}(M)$$

induced from fibrations

$$SO \to SG \to G/O \to BSO \to BSG.$$

Here $\tilde{\pi}_s^0(M) = \lim[S^q \wedge M, S^q] = [M, SG]$ is reduced stable cohomotopy.

Since n is odd the surgery obstruction is a homomorphism, $\sigma: [M, G/O] \to Z/2$, ([22], p. 407). The following two lemmas will be proved in the next section.

LEMMA 3.1. *If M is a d-twisted homology CP_n and d is odd, then* $\widetilde{KO}^{-1}(M) = 0$.

LEMMA 3.2. *If d is odd the image of $[M, G/O]$ in $KO^0(M)$ is free.*

It follows that $\tilde{\pi}_s^0(M)$ is the torsion subgroup of $[M, G/O]$. If the homotopy equivalence $f: M_1 \to M$ represents an element of $hS(M)$, its image in $KO^0(M)$ is given by $f^{-1*}\tau M_1 - \tau M$. By (3.2) this image is determined by the Pontryagin classes of M_1, see ([3], Lemma 2.25). In fact, the Pontryagin character $ph = ch \circ c$, c is injective except for 2-torsion, and ch is injective since $H^*(M; Z)$ is torsion free ([2], p. 19).

We have proved the first part of the following:

THEOREM 3.3. *Let M be a d-twisted homology CP_n with d odd.*

(i) *The smooth structures on M with fixed Pontryagin classes, if any, are in* 1-1 *correspondence with the elements of* $\ker\{\sigma: \tilde{\pi}_s^0(M) \to Z/2\}$.

(ii) The order of $\tilde{\pi}_s^0(M)$ is bounded by the product of the orders of the even stable homotopy groups of spheres in dimensions $\leq 2n$.

Part (ii) follows immediately by using the spectral sequence converging to $\tilde{\pi}_s^{p+q}(M)$ with

$$E_2^{p,q} = H^p(M; \tilde{\pi}_s^q(\text{point})).$$

The coefficient groups are the reduced stable homotopy groups of spheres: 0, $Z/2$, $Z/2$, $Z/24, \ldots$ for $q = 0, -1, -2, -3, \ldots$ respectively.

Remark 3.4. If d is even (3.1) is false in some dimensions and the proof of (3.2) breaks down. Still one can show that the torsion subgroup of $[M, G/O]$ has order bounded independent of d and hence there is a bound on the number of smooth structures with fixed Pontryagin classes. In §5 a more precise bound will be obtained under additional assumptions on d.

It follows from (2.2) that for $d \equiv \pm 1 \bmod 8$ the diffeomorphism type of a complete intersection $X_n(d)$ is determined by the diffeomorphism type of M (the corresponding d-twisted homology CP_n) and the Euler characteristic. If also d has no divisors less than $m+2$ the homotopy type of M is determined. Also the connected sum decomposition shows there is a map $g: X_n(d) \to M$ collapsing the handles such that $g^*(\tau M \oplus \epsilon^1) = \tau X \oplus \epsilon^1$. In cohomology g^* is an isomorphism onto $H^{2*}(X;Z)$. Therefore the Pontryagin classes of M are determined by those of X. (One can also argue that M and X are cobordant and for these manifolds the Pontryagin numbers determine the classes.) Thus we have:

THEOREM 3.5. *The number of distinct diffeomorphism classes of complete intersections of dimension $2m+1$ with given Euler characteristic, Pontryagin classes, and total degree d such that $d \equiv \pm 1 \bmod 8$ and d has no divisors less than $m+2$ is bounded by $|\tilde{\pi}_s{}^0(M)|$ which in turn is less than a bound depending only on n.*

§4. *KO* THEORY

Consider first the Atiyah–Hirzebruch spectral sequence with $E_2^{p,q} = H^p(M;\ KO^q(x_0))$ converging to $E_\infty^{p,q} = KO_p^{p+q}(M)/KO_{p+1}^{p+q}(M)$ where the filtration on $KO^n(M)$ is induced by the skeletal filtration of M:

$$KO_p{}^n(M) = \ker\{KO^n(M) \to KO^n(M^{(p-1)})\}.$$

Since $H^*(M;Z)$ is free and concentrated in even dimensions, the spectral sequence shows $KO(M)$ has no odd torsion.

Replacing the map $f: M \to CP_n$ up to homotopy by an inclusion, the cohomology of the pair (CP_n, M) is given by

$$H^q(CP_n, M;Z) = \begin{cases} Z/d & q = n+2, n+4, \dots, 2n+1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

If we assume d is odd, all differentials in the spectral sequence for the pair are zero and we find $KO^0(CP_n,\ M) = 0$ and $KO^1(CP_n,\ M)$ is a finite group of odd order, ${}_d[(m+1)/2]$, when $n = 2m+1$. The exact sequence of the pair and the known results for CP_n([23], (3.9)) then show

$$\text{rank } \widetilde{KO}^0(M) = \text{rank } \widetilde{KO}^0(CP_n) = m \tag{4.2}$$

$$\text{torsion } KO^0(M) = \text{torsion } KO^0(CP_n) = \begin{cases} Z/2 & m \text{ even} \\ 0 & m \text{ odd.} \end{cases} \tag{4.3}$$

Let y generate the ring $KO^0(CP_n)$. Then $f^! Y^{m+1}$ generates the $Z/2$ summand when m is even, [23, (3.9)]. To prove (3.2) we must show that $f^! y^{m+1}$ maps to a nonzero element in $J(M)$.

If d is odd the projection $\pi: RP^{2n+1} \to CP_n$ can be lifted to E and hence to a map g such that

$$\begin{tikzcd} & M \arrow[d, "f"] \\ RP^{2n+1} \arrow[ur, "g"] \arrow[r, "\pi"] & CP_n \end{tikzcd} \quad \text{commutes.}$$

Then $g^! f^! y^{m-1} = \pi^! y^{m+1}$ which is nonzero in $KO(RP^{2n+1})$ and maps to a nonzero element of $J(RP^{2n+1})$, ([1], p. 169). By naturality $f^! y^{m+1}$ also maps to a nonzero element. This proves Lemma (3.2).

To prove (3.1) consider the spectral sequence converging to $K0^*(M)$. The term $E_2^{p,-p-1}$ is zero unless $p = 8k$ when it is $Z/2$. But $d_2: E_2^{8k-2,-8k} \to E_2^{8k,-8k-1}$ can be identified with $Sq^2 \circ \rho_2: H^{8k-2}(M;Z) \to H^{8k}(M;Z/2)$ which is onto for $k > 0$ and d odd. Thus $E_r^{p,-p-1} = 0$ for $r \geq 3$ and all $p > 0$.

§5. STABLE COHOMOTOPY

In this section we compare the surgery sequence for a d-twisted homology projective space M with the surgery sequence for CP_n. Since n is odd the surgery obstruction $\sigma: [M, G/0] \to Z/2$ is a homomorphism, ([22], p. 407).

LEMMA 5.1. *If d is odd the diagram*

$$\begin{tikzcd} {[M, G/O]} \arrow[dr, "\sigma"] & \\ & Z/2 \\ {[CP_n, G/O]} \arrow[uu, "f^*"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commutes.

Proof. We must show $\sigma(u \circ f) = \sigma(u)$ for $u \in [CP_n, G/O]$. Since d is odd f induces an isomorphism in cohomology with $Z/2$ coefficients. It follows from the naturality of the Steenrod square that if $v \in H^*(CP_n; Z/2)$ is the Wu class of CP_n then f^*v is the Wu class of M. The surgery obstruction can be computed by the formula ([7], (2.5)) $\sigma(u) = \langle v^2(M) \cdot U^*(\dot{K}), [M]_2 \rangle$ where $K = k_2 + k_6 + \cdots \in H^*(G/O; Z/2)$. A simple computation gives $\sigma(f^*u) = \sigma(u)$.

LEMMA 5.2. *If d has no divisors less than $n+3$, then $f^*: \tilde{\pi}_s^0(CP_n) \to \tilde{\pi}_s^0(M)$ is an isomorphism.*

Proof. Consider the spectral sequence for the pair with $E_2^{p,q} = H^p(CP_n, M; \tilde{\pi}_s^q(\text{point}))$. From the cohomology with Z coefficients given in (4.1) we deduce $E_2^{p,q} = 0$ for $p + q \geq 0$ since, by results of Serre ([25], 9.7.13), the coefficient groups are relatively prime to d for $-q < 2n+3$, while for $p \geq 2n+2$ the Z-cohomology is zero. Therefore $\pi_s^0(CP_n, M) = 0 = \pi_s^1(CP_n, M)$ and the lemma follows.

COROLLARY 5.3. *If d has no divisors less than $n+3$, then $\ker\{\sigma: [M, G/O] \to Z/2\}$ is isomorphic to $\ker\{\sigma: [CP_n, G/O] \to Z/2\}$.*

Some computations of the groups $\tilde{\pi}_s^0(CP_n)$ and of σ have been made by Brumfiel[6, 7]. For $n = 3$, $\tilde{\pi}_s^0(CP_3) = Z/2$ and σ is an isomorphism. For $n = 5$, $\tilde{\pi}_s^0(CP_5) = Z/2 \oplus Z/2 \oplus Z/3$ and $\sigma = 0$. In this case fixing a homotopy type M with d satisfying the hypothesis of (5.2), for each pair of Pontryagin classes admitting a smooth structure there are 12 distinct smoothings. In general for m even, $n = 2m+1$, σ is zero ([16], Proposition 3.8).

§6. COUNTING ARGUMENT

From (3.5) there is a bound, say N, on the number of distinct complete intersections in dimension $n = 2m+1$ with given total degree, d, Euler characteristic, e, and Pontryagin classes provided $d \equiv \pm 1 \bmod 8$ and d has no divisors less than $m+2$. The characteristic classes of $X_n(d)$ are given by polynomials in the symmetric functions of $d_1, \ldots, d_r$. Let $s_k = \sum_{j=1}^{r} d_j^k$. In the next section we will show

LEMMA 6.1. *The characteristic classes of $X_n(d)$ depend only on n, r, d, and $s_1, \ldots, s_n$.*

Our main result is a consequence of the following counting argument.

THEOREM 6.2. *Given integers n and N there are integers r and d, with d satisfying the conditions above, such that the number of distinct r-tuples $(d_1, \ldots, d_r)$ with product d and the same first n symmetric functions is greater than N.*

COROLLARY 6.3. *In any odd dimension and for any integer k, there are k distinct multidegrees for which the corresponding complete intersections are all diffeomorphic.*

The Corollary follows from (6.2) and (3.5).

The proof of (6.2) was suggested to us by A.O.L. Atkin. Consider $2r$ different primes $p_1, \ldots, p_{2r}$ and set $d = \Pi p_i$. Let A be the set of unordered r-tuples $(d_1, \ldots, d_r)$ such that $d = \Pi d_j$ and each d_j is the product of exactly two of the primes. Then $|A| = \frac{(2r)!}{2^r r!}$. Let $m = \max p_i$. Then $s_k < rm^{2k}$. Consider the map $A \to Z^n$ defined by $(d_1, \ldots, d_r) \mapsto (s_1, \ldots, s_n)$. The image lies in the box $B = \Pi_{k=1}^n [1, rm^{2k}]$ which contains $|B| = r^n m^{n(n+1)}$ elements. To prove the theorem we show we can choose the $2r$ primes less than m so that d satisfies the conditions and so that $|A| > N|B|$.

Now we are not free to take arbitrarily many primes less than m; the prime number theorem states that the number of such primes is asymptotically $m/\log m$. We meet the divisibility condition on d by excluding some small primes and the congruence condition by taking an even number of primes $\equiv \pm 3 \bmod 8$. These conditions can be met if we take $3r = m/\log m$ for m sufficiently large. Using Stirling's formula, $r! \sim \sqrt{2\pi} r^{r+1/2} e^{-r}$, the inequality becomes

$$2^{r+1/2} r^r e^{-r} > N r^n m^{n(n+1)},$$

or

$$\frac{2^r r^r e^{-r}}{r^n m^{n(n+1)}} > N/\sqrt{2}.$$

In fact as m tends to infinity with $3r = m/\log m$, the left hand side tends to infinity. (We leave this as a calculus exercise.)

§7. UNIQUENESS IN LOW CODIMENSION

The results of this section hold for both even and odd dimensional complete intersections. We give formulas for the characteristic classes of a complete intersection and show that the degree and Pontryagin classes uniquely determine the multidegree in low codimension.

We can identify a complete intersection $X = X_n(d) \subset CP_{n+r}$ with the same variety in a larger projective space: $X \subset CP_{n+r+s}$ where its multidegree is $(d_1, \ldots, d_r, 1, \ldots, 1)$.

THEOREM 7.1. (i) *Let $X_n(d) \subset CP_{n+r}$ be a complete intersection with $2r \leq n$, $n > 2$, and with each $d_j > 1$. Then any complete intersection with the same total degree and Pontryagin classes as $X_n(d)$ has the same multidegree where we identify d with $(d_1, \ldots, d_r, 1, \ldots, 1)$.* (ii) *In particular any complete intersection diffeomorphic to X_n (or even diffeomorphic to $X_n \# S^n \times S^n \# \cdots \# S^n \times S^n$), with $2r \leq n$ and $n > 2$, has the same multidegrees as X_n (so the parenthetical case cannot arise).*

The surfaces $X_2(4)$, $X_3(3, 2)$, and $X_2(2, 2, 2)$ are diffeomorphic since they are K-3 surfaces so the restriction $n > 2$ is necessary, see ([9], pp. 591–592).

To fix notation let H denote the line bundle on CP_{n+r} dual to the universal bundle. Then $c_1(H)$ is Poincaré dual to the homology class carried by CP_{n+r-1}. The embedding $i: X_n \hookrightarrow CP_{n+r}$ has normal bundle $\nu(i) = \gamma^{d_1} \oplus \cdots \oplus \gamma^{d_r}$ where $\gamma = i^*H$ and $\gamma^d = \gamma \oplus \cdots \oplus \gamma$ (d-times). The characteristic classes of $X_n(d)$ can be computed from the bundle equation $\tau X_n \oplus \nu(i) = i^*\tau CP_{n+r}$. Setting $x = i^*c_1(H)$ the Chern classes are given by

$$\{1 + c_1 + c_2 + \cdots\}\Pi(1 + d_j x) = (1 + x)^{n+r+1}$$

and the Pontryagin classes by

$$\{1 + p_1 + p_2 + \cdots\}\Pi(1 + d_j^2 x^2) = (1 + x^2)^{n+r+1}$$

The elementary symmetric functions σ_n are defined by $\sum_{n=0}^{\infty} \sigma_n x^n = \Pi_{j=1}^r(1 + d_j x)$ and the σ_n can be expressed in terms of the power sums $s_k = \sum_{j=1} d_j^k$ for $k \leq n$. Thus Lemma 6.1 is clear. At the end of this section we will give explicit formulas for the characteristic classes in terms of the s_k.

To prove 7.1 we need the following:

LEMMA 7.2. *Let $a_j = d_j^2$ and set $\mathbf{a} - \mathbf{1} = (a_1 - 1, \ldots, a_r - 1)$. Then $\sigma_k(\mathbf{a} - \mathbf{1})$ is given by a polynomial in $p_1, \ldots, p_k$ independent of r.*

Proof. Using $\dfrac{1 + ax^2}{1 + x^2} = 1 + \dfrac{(a-1)x^2}{1 + x^2}$ the Pontryagin class formula above can be rewritten as

$$\{1 + p_1 + p_2 + \cdots\}\Pi\left(1 + \frac{(a_j - 1)}{1 + x^2} x^2\right) = (1 + x^2)^{n+1}$$

and hence

$$\{1 + p_1 + p_2 + \cdots\}\sum_{k=0}^{\infty} \sigma_k(\mathbf{a} - \mathbf{1}) \frac{x^{2k}}{(1 + x^2)^k} = (1 + x^2)^{n+1}.$$

The lemma follows.

Proof of Theorem (7.1). Since $\sum_{k=0}^{\infty} \sigma_k(\mathbf{a} - \mathbf{1}) = \Pi_{j=1}^r(1 + a_j - 1) = d^2$ we have

$$d^2 - 1 - \sigma_1(\mathbf{a} - \mathbf{1}) - \cdots - \sigma_k(\mathbf{a} - \mathbf{1}) = \sigma_{k+1}(\mathbf{a} - \mathbf{1}) + \sigma_{k+2}(\mathbf{a} - \mathbf{1}) + \cdots$$

Since each $a_i \geq 1$, the right hand side is ≥ 0 for any $\mathbf{a}$ and is $= 0$ if and only if at most k of the a_i are >1. Let X_n be a complete intersection. For each $k \leq n/2$ the left hand side is determined by d and the Pontryagin classes of X_n. It follows that if $X_n(\mathbf{d})$ has the same total degree and Pontryagin classes as some complete intersection of codimension $r \leq n/2$, then X_n can have at most r of the $d_i > 1$. Further $1 - a_1, \ldots, 1 - a_r$ are the roots of the equation

$$t^r + \sigma_1(\mathbf{a} - \mathbf{1})t^{r-1} + \cdots + \sigma_r(\mathbf{a} - \mathbf{1}) = 0$$

and hence the Pontryagin classes of X_n determine the multidegree. For (ii) when $n > 2$, the homotopy type of X_n determines the total degree.

PROPOSITION 7.3. *Let $X_n \subset CP_{n+r}$ be a complete intersection of given codimension r with $n > 2$ and $2r \leq n+2$. Then the total degree and Pontryagin classes of X_n determine the multidegree.*

Proof. We need only extend the arguments above to cover the case $r = \left[\frac{n}{2}\right] + 1$. In that case $\sigma_k(\mathbf{a} - \mathbf{1})$ is determined by Pontryagin classes for $k \leq r-1$ and, since $\sigma_k(\mathbf{a} - \mathbf{1}) = 0$ for $k > r$,

$$\sigma_r(\mathbf{a} - \mathbf{1}) = d^2 - 1 - \sum_{k=1}^{r-1} \sigma_k(\mathbf{a} - \mathbf{1}).$$

Hence as above $(d_1, \ldots, d_r)$ is determined.

We conclude this section with some simple explicit formulas needed for the computation described in §9. The power sums s_k are much easier to calculate than the σ_k. We have

$$\Pi_{j=1}^r (1 + d_j x) = 1 + g_1(s_1)x + \frac{1}{2!} g_2(s_1, s_2)x^2 + \cdots$$

where the g_k are polynomials and $\sigma_k = \frac{1}{k!} g_k(s_1, \ldots, s_k)$. The first three are $g_1(s_1) = s_1$, $g_2(s_1, s_2) = s_1^2 - s_2$, and $g_3 = s_1^3 - 3s_1 s_2 + 2s_3$. They can be computed from the Newton formulae [13, p. 32]

$$s_k - g_1 s_{k-1} + \cdots + (-1)^k g_k k = 0, \quad k \geq 1.$$

Now let $t_k = \sum_{j=1}^{r'} e_j^k$. It is easy to see that

$$\left\{1 + g_1(s_1) + \frac{1}{2!} g_2(s_1, s_2) + \cdots\right\}\left\{1 + g_1(t_1) + \frac{1}{2!} g_2(t_1, t_2) + \cdots\right\}$$
$$= 1 + g_1(s_1 + t_1) + \frac{1}{2!} g_2(s_1 + t_1, s_2 + t_2) + \cdots.$$

This identity holds also if the s's and t's are regarded as independent transcendental variables. Therefore

$$\left\{1 + g_1(t_1 - s_1) + \frac{1}{2!} g_2(t_1 - s_1, t_2 - s_2) + \cdots\right\}\left\{1 + g_1(s_1) + \frac{1}{2!} g_2(s_1, s_2) + \cdots\right\}$$
$$= 1 + g_1(t_1) + \frac{1}{2!} g_2(t_1, t_2) + \cdots.$$

Hence $\frac{\Pi(1 + e_j x)}{\Pi(1 + d_j x)} = 1 + g_1(t_1 - s_1)x + \frac{1}{2!} g_2(t_1 - s_1, t_2 - s_2)x^2 + \cdots$. To apply this formula to compute the Chern classes of a complete intersection we take $e_1 = \cdots = e_{n+1+r} = 1$.

Then

$$c_1 = (n+1+r-s_1)x$$
$$c_2 = \frac{1}{2}\{(n+1+r-s_1)^2-(n+1+r-s_2)\}x^2$$
$$c_k = \frac{1}{k!}g_k(n+1+r-s_1,\ldots,n+1+r-s_k)x^k.$$

The Euler characteristic

$$e(X) = c_n(X)\cap[X] = d\frac{1}{n!}g_n(n+1+r-s_1,\ldots,n+1+r-s_n)$$

since $x^n \cap [X] = d$.

Similarly

$$p_1 = (n+1+r-s_2)x^2$$
$$p_2 = \frac{1}{2}\left\{(n+1+r-s_2)^2-(n+1+r-s_4)\right\}x^4, \text{ etc.}$$

§8. AMBIENT ISOTOPY AND PROJECTIVE LINEAR EQUIVALENCE

This section contains two results complementing the fact that complete intersections of the same multidegree are ambiently isotopic. First it turns out that for ambient isotopy of two complete intersections it is enough to assume they are diffeomorphic (and $n>2$). Second if the complete intersections are equivalent as analytic manifolds they are isotopic in a very strong sense: through projective linear transformations. The projective linear transformations of CP_{n+r} (maps induced by a linear map of C^{n+r+1}) form a connected group $\mathrm{PGL}(n+r+1, C)$. Furthermore such manifolds have the same multidegree so the diffeomorphic complete intersections of (6.3) are not analytically equivalent.

THEOREM 8.1. *If X_n and Y_n are complete intersections of dimension $n>2$ in CP_{n+r} and if X_n is diffeomorphic to Y_n, then there is a diffeomorphism of CP_{n+r} isotopic to the identity carrying X_n to Y_n.*

Proof. Case $2r \le n+2$. By (7.3) X_n and Y_n have the same multidegree so they are ambiently isotopic see ([16], §4). Case $2r \ge n+2$. If $i: X_n \to CP_{n+r}$ and $j: Y_n \to CP_{n+r}$ are the embeddings and $h: X_n \to Y_n$ is the diffeomorphism then i and $j\circ h$ are homotopic for $n>2$ since they induce the same map, a canonical isomorphism, on $H^2(\ ;Z)$ and this classifies maps to CP_{n+r}. The embeddings are then isotopic by Haefliger's theorem ([10], p. 47) whose hypotheses follow from the inequality $2r\ge n+2$.

For $n\le 2$ these maps may not be homotopic, the degree of X_n is an invariant of the homotopy class of the inclusion map but for $n=1$ or 2 the degree is not determined by X_n as a differentiable manifold. For example, the curves of degrees 1 and 2 in CP_2 are both diffeomorphic to S^2 but they represent different elements of $\pi_2(CP_2)$. Also they have different complements, see ([16], Theorem 1.2). In $n=2$ the K-3 surfaces $X_2(4)$, $X_2(3,2)$, and $X_2(2,2,2)$ are diffeomorphic but have degrees 4, 6 and 8 respectively ([9], pp. 591–592).

THEOREM 8.2. *If X_n and Y_n are complete intersections of dimension $n\ge 2$ in CP_{n+r}, with $c_1 \ne 0$ in case $n=2$, and if X_n and Y_n are equivalent as analytic manifolds, then* *there is a projective linear transformation of CP_{n+r} restricting to an equivalence of X_n with Y_n. Furthermore X_n and Y_n have the same multidegree.*

Proof. There is a correspondence between analytic maps of a complex manifold X to CP_N and analytic line bundles L over X with $N+1$ global sections which do not vanish simultaneously at any point of X, see ([8], v§2) for a description of this. Our proof extends to complete intersections the example on p. 177 of the book of Griffiths and Harris ([9], pp. 176–178) and removes a restriction on degree imposed there. We use their notation. The sections $s_0, \dots, s_N$ correspond to the map defined by $f(x) = [s_0(x), \dots, s_N(x)] \in CP_N$. The bundle $L = f^*H$, where $c_1(H)$ is Poincaré dual to a hyperplane. The sections $s_0, \dots, s_N$ span a subspace $E \subset H^0(X, \mathcal{O}(L))$ and other sections spanning the same space E correspond to maps which differ by a projective transformation of CP_N. In general $E = f^*H^0(CP_N, \mathcal{O}(H))$. If X_n is embedded as a complete intersection then $E = H^0(X, \mathcal{O}(L))$, see ([12], III ex. 5.5). Hence two embeddings will differ by a projective linear transformation if they pull back analytically equivalent line bundles. The line bundles form a group Pic $X = H^1(X, \mathcal{O}_X{}^*)$. For a complete intersection of dimension ≥ 2 the first term of the exact sequence

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X{}^*) \to H^2(X; Z)$$

vanishes ([9], p. 156) so a line bundle is uniquely determined by its first Chern class. Therefore homotopic embeddings correspond to equivalent line bundles. For $n > 2$ any two such maps are homotopic as seen in the proof of 8.1. For $n = 2$ we make use of the fact that $c_1(X)$ is a multiple of $c_1(L)$ in $H^2(X; Z)$ given by the formula at the end of §7. If f and $\bar{f}$ are analytic embeddings and if $c_1(X) \neq 0$ then $f^*c_1(H)$ and $\bar{f}^*c_1(H)$ are proportional. But both are indivisible cohomology classes ([19], Theorem (2.1)). Since both the corresponding line bundles admit global sections, the two Chern classes are both positive, so $f^*c_1(H) = \bar{f}^*c_1(H)$ and hence f and $\bar{f}$ are homotopic. This completes the proof of projective linear equivalence.

The coordinate ring of $X_n \subset CP_{n+r}$ is given in terms of the defining polynomials $P_1, \dots, P_r$ by $R = C[Z_0, \dots, Z_{n+r}]/\{P_1, \dots, P_r\}$. The ring R is graded by degree (since the P_j are homogeneous) and the associated Poincaré series $\sum_{k=0}^{\infty} \dim R_k t^k$ can be computed to be $(1-t)^{-n-r-1}\Pi_{j=1}^r(1-t^{d_j})$ where deg $P_j = d_j$, see ([21], p. 131). From this rational function the multidegree $\boldsymbol{d}$ can be recovered, up to the equivalence of Theorem 7.1.

Since there is a linear transformation carrying X_n onto Y_n their coordinate rings are isomorphic as graded rings. In fact, for a complete intersection the coordinate ring can be defined in terms of the line bundle L by $R_k = H^0(X_n, \mathcal{O}(L^{\otimes k}))$. Hence X_n and Y_n have the same multidegree.

An interesting consequence of this and a result of E. Sernesi was pointed out to us by B. Moishezon.

COROLLARY 8.3. *Let X_n be a smooth manifold which underlies a set of k complete intersections with different multidegrees. Then the moduli space of complex structures on X_n has at least k irreducible components provided $n > 2$ or $n = 2$ and X_n is not $K3$.*

Proof. If two complete intersections, X_n and Y_n, lie in the same irreducible component of the moduli space, they would be connected by a sequence of small deformations. Then by Sernesi [24], X_n would be analytically equivalent to a complete intersection of the same multidegree as Y_n. But by the theorem X_n and Y_n must then have the same multidegree. Hence by (6.3) we have

COROLLARY 8.4. *In any odd dimension n greater than* 1 *and for any integer k, there is a smooth manifold X_n for which the moduli space of complex structures on X_n has at least k irreducible components.*

An example in dimension two of moduli spaces with several components has been given by Horikawa [14].

§9. SOME EXAMPLES OF 3-FOLDS

The examples of diffeomorphic complete intersections with different multidegrees promised by the counting argument of §6 have very large codimension and degree. It does not seem likely a particular example would be found by that approach. Also the problem remains: given two complete intersections X_n and Y_n of the same degree both corresponding to smooth structures on a given d-twisted homology projective space M and with the same Pontryagin classes, how to determine if X_n and Y_n give the same element of $\pi_s^0(M)$.

For complete intersections of dimension 3 a sequence of papers by Wall [26], Jupp [15] and Žubr [28] gives a complete classification which we summarize.

(i) $X_3 = M \# a(S^3 \times S^3)$ where M is a smooth, d-twisted homology CP_3.

(ii) Two M's with the same degree d, and classes w_2 and p_1 are diffeomorphic.

(iii) These invariants satisfy

$$w_2 = 0 \text{ implies } p_1 \equiv 4d \bmod 24$$

$$w_2 \neq 0 \text{ implies } p_1 \equiv d \bmod 48 \text{ and } d \text{ is even.}$$

Here p_1 denotes $(p_1 \cup x) \cap [M]$; recall the map sending u to $(u \cup x) \cap [M]$ is an isomorphism of $H^4(M;Z)$ with Z.

(iv) Given d, w_2, and p_1 satisfying (iii) there is a smooth d-twisted homology CP_3 with the given invariants.

(v) The homotopy type of M is determined by d and w_2 unless $w_2 = 0$ and d is even, in which case there are two homotopy types determined by d and p_1 mod 48. (By (iii) and the work of Žubr [29] the homotopy type of M determines p_1 mod 48 unless d is odd, in which case only p_1 mod 24 is a homotopy invariant – as we saw in Theorem 2.1.)

(vi) For complete intersections the homotopy type of M is always determined by w_2 and d: if $w_2 = 0$ and d is even, $d \equiv 0 \bmod 8 \Rightarrow p_1 \equiv 4d \bmod 48$, $d \not\equiv 0 \bmod 8 \Rightarrow p_1 \equiv 4d + 24 \bmod 48$.
(This follows from the formula $p_1 = d(4 + r - s_2)$ of §7.) In general $w_2 = c_1 \bmod 2$ and for complete intersections $w_2 = p_1/d \bmod 2$.

Example 9.1. If two complete intersections in dimension 3 have the same degree d and classes p_1 and w_2, then one is diffeomorphic to the connected sum of the other with handles of the form $S^3 \times S^3$, e.g.

$$X_3(12, 10) = X_3(15, 4, 2) \# (13440)(S^3 \times S^3).$$

The invariants for this example and a second are:

d	p_1/d	e/d	c_1	d						
120	−238	−3494	−16	12	10					
120	−238	−3270	−14	15	4	2				
640	−113	−1353	−13	8	5	4	4			
640	−113	−1189	−11	10	2	2	2	2	2	2

By (7.1) it is impossible to have such an example in which one of the varieties is a hypersurface or, by (7.3), if both have codimension 2.

Example 9.2. Homotopy equivalence. The following invariants give two examples of pairs of varieties which are homotopy equivalent but not diffeomorphic; the two smoothings have distinct p_1's.

d	p_1/d	e/d	c_1	$\mathbf{d}$						
3780	−434	−11792	−28	15	14	3	3	2		
3780	−426	−11792	−28	18	7	6	5			
13500	−290	− 7012	−24	15	5	5	3	3	2	2
13500	−258	− 7012	−26	10	9	6	5	5		

Here the same simply-connected homotopy type has two distinct smoothings which underlie complex manifolds.

The examples

d	p_1/d	e/d	c_1	$\mathbf{d}$						
5184	−151	−2884	−19	6	6	6	6	4		
5184	−164	−2884	−18	9	8	3	3	2	2	2

show that the homotopy type is not determined by d and e alone (that is not by the integral cohomology ring.) They are distinguished by w_2.

Example 9.3. Diffeomorphic complete intersections. The following invariants describe two pairs of diffeomorphic varieties:

d	p_1/d	e/d	c_1	$\mathbf{d}$							
62720	−455	−17040	−35	16	10	7	7	2	2	2	
62720	−455	−17040	−35	14	14	5	4	4	4		
125440	−458	−17954	−36	16	10	7	7	2	2	2	2
125440	−458	−17954	−36	14	14	5	4	4	4	2	

In this case the first Chern class c_1 is also the same. This implies that the three symmetric functions $s_1 - r$, $s_2 - r$ and $s_3 - r$ are the same for the two multidegrees, hence appending the same additional term to each multidegree gives two new multidegrees with the same invariants. The second example comes from the first in this way by appending 2.

An example of diffeomorphic complete intersections with different first Chern classes would give an example of a disconnected moduli space independent of the results in §8. We believe such examples can be found.

These examples are easy to check but hard to happen upon. They were found by computer search. The main idea is that a partition of n, $n_1 + \cdots + n_r = n$, corresponds to a multidegree, $d_j = n_j + 1$, with first Chern class $c_1 = 4 - n$. The invariants d, $-p_1/d$, $-e/d$, c_1 were computed using a subroutine ([20], p. 69) to generate partitions of n. These were sorted into order of increasing d, $-p_1/d$, $-e/d$ and then read to find duplications. We thank Neil Rickert for instruction and guidance in this project.

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