On the fundamental group of the complement to a discriminant variety

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1. Introduction.

Let $i:V \to \mathbb{P}^n$ be a closed embedding of a smooth complex algebraic variety into the projective space, $\check{V} \subset \check{\mathbb{P}}^n$ the dual variety of $i(V)$. Its points parametrize hyperplanes which are tangent to $i(V)$, or equivalently, singular hyperplane sections of $i(V)$. In this paper we discuss the group $\pi_1(\check{\mathbb{P}}^n-\check{V})$ and compute it in some special cases.

If $L \subset \mathbb{P}^n$ is a general 2-plane, then by the Zariski-Lefschetz type theorem ([Z_3],[LH]) $\pi_1(\check{\mathbb{P}}^n-\check{V}) \simeq \pi_1(L-L\cap\check{V})$. The intersection $L\cap\check{V}$ is either empty or a plane irreducible curve with nodes and cusps as singularities. Its degree, the number of nodes and cusps can be computed by generalized Plücker formulas (see n°2). It should be said that the group $\pi_1(\mathbb{P}^2-C)$ for a nodal-cuspidal plane curve C is known only in a few cases. We discuss in n°3 the previously known examples of Zariski ([Z_1],[Z_2]) of such groups. The presence of cusps is an essential obstacle, since, as it had been recently proven by Fulton-Deligne (see [D]), $\pi_1(\mathbb{P}^2-C)$ is always abelian if C has only nodes.

In the above mentioned examples of Zariski $V = \mathbb{P}^1$

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There is a close relation between the braid groups of Riemann surfaces and the homotopy groups of the diffeomorphisms groups of Riemann surfaces (see [B]). In section 5 we speculate on a possible generalization of this relation in the case of an arbitrary embedding $i:V \to \mathbb{P}^n$.

2. The discriminant variety of a linear system.

Let V be a nonsingular projective algebraic variety over complex numbers, $\underline{L}$ an invertible sheaf on V, E a linear subspace of $H^0(V,\underline{L})$, $\mathbb{P}(E)$ the corresponding linear system of divisors on V. Define the discriminant variety $\mathrm{Disc}(E,\underline{L})$ of the linear system $\mathbb{P}(E)$ as the subset of points $x \in \mathbb{P}(E)$ such that the corresponding divisor D_x is not smooth (every positive divisor is considered as a closed subscheme of V). This set is always closed in the Zariski topology of the projective space $\mathbb{P}(E)$ and hence has a unique structure of a reduced algebraic subvariety of $\mathbb{P}(E)$.

The most interesting case in which we will be involved is the case where $\underline{L}$ is a very ample sheaf and $E = H^0(V,\underline{L})$. In this case the complete linear system $\mathbb{P}(E)$ defines a closed embedding $i:V \to \mathbb{P}(E^*)$. The discriminant variety in this case, denoted simply by $\mathrm{Disc}(\underline{L})$, coincides with the dual variety $\check{i}(V)$ of $i(V)$. The latter is defined as the set of all points x in the dual projective space $\mathbb{P}(E^*)^{\vee} = \mathbb{P}(E)$ such that the corresponding hyperplane H_x is tangent to $i(V)$ somewhere. An equivalent definition of $\check{i}(V)$ can be given also as follows (see [KL] p. 335). Let $Z \subset \mathbb{P}(E^*) \times \mathbb{P}(E)$ be the canonical incidence correspondence between points and hyperplanes, $p_1:Z \to \mathbb{P}(E^*)$, $p_2:Z \to \mathbb{P}(E)$ the projections, and $i:\mathbb{P}^1 \to \mathbb{P}^n$ is the Veronese embedding v_n. In this case $\check{\mathbb{P}}^n - v_n(\mathbb{P}^1)^\vee$ is canonically isomorphic to the space $S^n(\mathbb{P}^1)$ of all unordered n-tuples of distinct points on $\mathbb{P}^1$. The fundamental group of this space is known as the n-th braid group of the Riemann sphere. It has been recomputed by many authors who apparently were not aware of Zariski's papers (see [B]). A general plane section of $v_n(\mathbb{P}^1)^\vee$ is a plane curve of degree $2(n-1)$ with $3(n-2)$ cusps and $2(n-2)(n-3)$ nodes. For $n = 3$ (a cuspidal quartic) the group $\pi_1(\mathbb{P}^2-C)$ was computed algebraically by S. Abhyankar ([A]).

Fixing a point $x_0 \in \check{\mathbb{P}}^n - \check{V}$ the fundamental group $\pi_1(\check{\mathbb{P}}^n-\check{V};x_0)$ has a natural representation ρ in the diffeotopy group of the hyperplane section H_{x_0} of $i(V)$ corresponding to the point x_0, that is, the group of diffeomorphisms of H_{x_0} modulo isotopy. The image of this representation can be called the universal monodromy group. It has many interesting homomorphisms into the automorphism group of different objects functorially associated to H_{x_0}, e.g cohomology groups. The images of these homomorphisms were studied in many situations (see, for example, [HA]). The computation of $\pi_1(\check{\mathbb{P}}^n-i(V))^\vee$ will be achieved if we know the universal monodromy group, the kernel of ρ, and the extension of the former group by the latter.

In section 4 we carry out the above program for the cases of the Veronese embedding $v_3:\mathbb{P}^2 \to \mathbb{P}^9$ and the Segre embedding $s_{2,2}: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^8$. The Segre embeddings are natural generalizations of Zariski's examples; however, except some trivial cases, the above cases are the only ones where we were able to succeed in computations. We refer to the paper [L] of the second author in which the case $v_3: \mathbb{P}^3 \to \mathbb{P}^{19}$ is discussed. This case was the main stimulus of our work.

$\tilde{V} = V \times_{\mathbb{P}(E^*)} Z$, $p:\tilde{V} \to \mathbb{P}(E)$ the composition of the second projection $\tilde{V} \to Z$ and the projection p_2. The variety $\tilde{V}$ is nonsingular (it is isomorphic to the projective bundle associated with the tangent bundle $T_{\mathbb{P}(E^*)}$ restricted to $i(V)$). Under the first projection $\tilde{V} \to V$ the fibres of p are isomorphic to hyperplane sections of $i(V)$. The set S of all points where p is not smooth is a smooth closed subvariety of $\tilde{V}$ of dimension $\dim \mathbb{P}(E)-1$ (it is isomorphic to the projective bundle associated with the normal bundle to $i(V)$). Its projection into $\mathbb{P}(E)$ is the dual variety of $\check{i(V)}$. In the majority of cases the dual variety $\check{i(V)}$ is a hypersurface in the projective space $\mathbb{P}(E)$ (see some exceptional examples in [KL] p.360). In the sequel we will always assume that $i(V)$ is a hypersurface. Its degree d is computed by the formula (see [KL] p. 361 or [K] 5.5.1)

$$\check{d} = \sum_{i=0}^{r} (i+1)\deg(c_{r-i}(\Omega_V^1)c_1(\underline{L})^i) \tag{2.1}$$

where $r = \dim V$, other notation is standard.

The dual variety $\check{i(V)}$ is always irreducible. Its nonsingular points correspond to those hyperplanes $H_x (x \in \mathbb{P}(E))$ which are tangent to $i(V)$ at exactly one point and $H_x \cap i(V)$ has a non-degenerate quadratic singularity at this point. Equivalently, the set of nonsingular points of $i(V)$ is the largest open set over which the projection $p:S \to i(V)$ is an isomorphism (see [K], prop. 3.5). The set of all points $s \in S$ where p is not an isomorphism is the double locus of the induced map $p:S \to P(E)$ in sense of [K]. Its image D in $i(V)$ is the singular locus of $\check{i(V)}$. Let D_c (resp. D_n) be the set of points $x \in \check{i(V)}$ such that $H_x \cap i(V)$ has a unique singular point and the quadratic form of the local defining equation has rank $r-1$ (resp. $H_x \cap i(V)$ has two non-degenerate quadratic singularities). Then $D = \bar{D}_c \cup \bar{D}_n$ and $\operatorname{codim}(D-D_c-D_n, \check{i}(V)) \geq 2$. In the case where $\bar{D}_c$ and $\bar{D}_n$ have codimension 2 in $\mathbb{P}(E)$ (or 1 in $\check{i}(V)$) their degrees are given by the following formulas

$$k = \deg \bar{D}_c = r\binom{r+2}{2}\deg(i(V)) + \sum_{i=1}^{r}\binom{r-i+2}{2}\deg(rc_1(\Omega_X^1) + 2c_1(\Omega_X^1)c_{i-1}(\Omega_X^1))c_1(\underline{L})^{r-1} \tag{2.2}$$

$$\delta = \deg \bar{D}_n = \tfrac{1}{2}\Big(\check{d}(\check{d}-n-1) + (-1)^r c\sum_{i=c-2}^{n-2}\binom{n-2}{i}\deg(b_{i-c+2}c_1(L)^{n-2-i}) - {} + (-1)^r\sum_{i=c-1}^{n-2}\deg\Big(b_{i-c+1}c_1(\underline{L})^{n-2-i}(c_1(\Omega_X^1)+c_1(N))\Big)\binom{n-2}{i}\Big) - k \tag{2.3}$$

where N is the normal bundle to $i(V)$ in $P(E^*)$, $n = \dim P(E)$, $c = \operatorname{codim}(i(V), P(E)) = n - r$, $c(N)^{-1} = \left(\Sigma\, c_i(N)t^i\right)^{-1} = \sum_{j=0}^{r} b_j t^j$.

Formula (2.2) is given in [R_1], th.2. Formula (2.3) is based on formula (V.9) of [KL] (or cor. 4.2 of [R_2]) and simple computations similar to [K], §5.

Let P be a general 2-plane in $P(E)$, then it intersects D_c at k points and D_n at δ points. The curve P $P \cap \check{i}(V)$ is a plane irreducible curve of degree $\check{d}$ which has k cusps and δ nodes as its singularities.

Examples. 1. Veronese embeddings.

Let $V = P^r$ and $v_m : P^r \to P^n$ $\left(n = \binom{r+m}{m} - 1\right)$ be the Veronese map given by the complete linear system $P(H^0(P^r, \underline{O}_{P^r}(m))$. In this case $c(\Omega_V^1) = (1-ht)^{r+1}$, $h = c_1(\underline{O}_{P^r}(1))$. Also, $c(N) = (1+mht)^{n+1}/(1+ht)^{r+1}$. Plugging this into formulas (2.1)-(2.3) we easily obtain

$$\check{d} = \sum_{i=0}^{r} (-1)^{r-i}(i+1)\binom{r+1}{r-i} m^i ,$$

$$k = r\binom{r+2}{2}m^r + \sum_{i=1}^{r}\binom{r-i+2}{2}(-1)^i\left(r\binom{r+1}{i} + 2(r+1)\binom{r+1}{i-1}\right)m^{r-i} ,$$

$$\delta = \frac{1}{2}(\check{d}(\check{d}-n-1)+(-1)^r \sum_{i=n-r-1}^{n-2} a_{i-n+r+1} m^{n-2-i}(m(n+1)-2(r+1))\binom{n-2}{i}$$

$$+ (-1)^r(n-r) \sum_{i=n-r-2}^{n-2} \binom{n-2}{i} a_{i-n+r+2} m^{n-i-2}) - k \qquad (2.4$$

where $n = \binom{m+r}{m} - 1$, $(1+t)^{r+1}/(1+mt)^{n+1} = \sum_{j=0}^{\infty} a_j t^j$.

For example, if $r = 1$ we get

$\check{d} = 2m - 2$, $k = 3m - 6$, $\delta = 2(m - 2)(m - 3)$;

if $r = 2$, $m=2$

$\check{d} = 3$, $k = \delta = 0$;

if $r = 2$, $m = 3$

$\check{d} = 12$, $k = 24$, $\delta = 21$.

2. V is an algebraic curve of genus g, $d = \deg(\underline{L})$, $n = \dim H^0(V,\underline{L}) - 1$. In this case formulas (2.1) - (2.3) give

$\check{d} = 2d + 2g - 2$

$k = 3(d + 2g - 2)$

$\delta = \frac{1}{2}(\check{d}(\check{d}-n-1) + (n-3)(2g-2) + (2n-4)d) - 3(d+2g-2)$

If $d > 2g - 2$, then by Riemann-Roch we have $n = -g + d$ and we get

$\check{d} = 2d + 2g - 2,$

$k = 3(d + 2g - 2)$

$\delta = 2(d - 2)(d - 3) + 2g(2d + g - 7)$

compare $[Z_2]$ p. 335)

3. V is a surface, D a very ample divisor, $\underline{L} = \underline{O}_V(D)$, $n = \dim H^0(V, O_V(D))$. Let $p_a(V) = \chi(V, O_V(D)) - 1$, $K_V = c_1(\Omega_V^1)$, $\pi = \dim H^1(D, O_D)$, $c_2 = c_2(X) = \chi(V, C)$.

In this case, simple computations yield

$\check{d} = c_2 + 4\pi - 4 + D^2$

$k = 24(p_a + \pi)$ (compare $[Z_4]$ p.236)

$$\delta = \frac{1}{2}(\check{d}(\check{d}-n-1) + (3n-15)D^2 + (2n-13)(DK) - 2K^2 + (n-2)c_2) - k \tag{2.5}$$

Special cases:

a) $V = \mathbb{P}^2$, D a cubic curve

$\check{d} = 12$, $k = 24$, $\delta = 21$

b) $V = \mathbb{P}^1 \times \mathbb{P}^1$, D is a curve of degree (2,2)

$\check{d} = 12$, $k = 24$, $\delta = 22$

c) V is a Del Pezzo surface of degree d (a nonsingular surface of degree d in $\mathbb{P}^d$ with $\underline{O}_V(1) \sim \underline{O}(-K_V)$),

$D = -K_V$, $3 \le d \le 9$, $\check{d} = 12$, $k = 24$, $\delta = 30 - d$

3. Zariski's examples.

As was mentioned in the introduction they correspond to the case of a linear system on an algebraic curve.

Let $\underline{L}$ be an invertible sheaf or degree d on a nonsingular algebraic curve V of genus g, $E \subset H^0(V, \underline{L})$ a linear subspace. Assume that the linear system E determines a closed embedding $i:V \to P = \mathbb{P}(E^*)$. Let α be the map that sends a point $x \in \check{P} = P(E)$ to the corresponding divisor $D_x = H_x \cap i(V)$, where H_x is the hyperplane section corresponding to x. In the sequel we will identify the set of all positive divisors of degree d on V with the symmetric product $V^{(d)} = V^d/S_d$, where S_d is the symmetric group on d letters. Thus, we have a map $\alpha:\check{\mathbb{P}} \to V^{(d)}$. Let $\mu:V^{(d)} \to J_d(V)$ be the map which sends a divisor D to the isomorphism class of invertible sheaves $\underline{L} \simeq \underline{O}_V(D)$. The set $J_d(V)$ is a principal homogeneous space over the Jacobian variety $J(V) = J_0(V)$. Clearly the composition $\mu \circ \alpha$ is a constant map, thus $\alpha(\check{P})$ is a closed subvariety of the fiber $\mu^{-1}(cl(\underline{L}))$. Let

$$\Delta = \left\{(v_1,\ldots,v_d) \in V^d:\ v_i = v_j \text{ for some } i \neq j\right\} / S_d$$

Then D_x $(x \in P)$ is a singular divisor if and only if $\alpha(x) \in \Delta$. Thus

$$\mathrm{Disc}(E,\underline{L}) \simeq \Delta \cap \alpha(P) .$$

Let $\tilde{V}(d) = V^{(d)} - \Delta$, then we have a close embedding

$$\alpha:\check{P} - \mathrm{Disc}(E,L) \longrightarrow \tilde{V}^{(d)}$$

Choosing a point $x_o \in P - \mathrm{Disc}(E,\underline{L})$ the map α induces the homomorphism

$$\alpha_*:\pi_1(\check{P} - \mathrm{Disc};\ x_o) \longrightarrow \pi_1(\tilde{V}^{(d)};\ \alpha(x_o)) .$$

Since $\alpha(\check{P})$ lies in a fiber, the image of α_* is contained in the kernel of the map $\mu_*:\pi_1(V^{(d)};\alpha(x_o)) \longrightarrow \pi_1(J_d(V);\ \mu\circ\alpha(x_o)) \simeq Z^{2g}$.

In general, one hardly can say anything about the homomorphism

$$\alpha_*:\pi_1(\check{\mathbb{P}} - \mathrm{Disc}(E,\underline{L})) \longrightarrow \mathrm{Ker}\ \mu_*$$

The following cases are the only known cases to us where α_* is an isomorphism.

Case I. $V = \mathbb{P}^1$, $E = H^0(V,\underline{L})$, $\underline{L} = \underline{O}_V(d)$. In this case $J_d(V)$ is a single point, α defines an isomorphism $\mathbb{P}(E) \simeq V^{(d)}$ which induces an isomorphism

$$\pi_1(\check{\mathbb{P}} - \mathrm{Disc}(\underline{L})) \xrightarrow{\sim} \pi_1(\tilde{V}^{(d)}) = \mathrm{Ker}\ \mu_* .$$

Case II. $g \geq 1$, $\deg(\underline{L}) > 2g - 2$, $|E|$ a complete linear system.

Let $\mathbb{L}$ be a universal invertible sheaf of degree d on $V \times J_d$ (i.e. $\mathbb{L}\ |V \times \{x\} = L_x$ and $\mathrm{cl}(L_x) = x$), $\underline{E} = p_{2*}(\mathbb{L})$. Since $d > 2g$ $d > 2g - 2$ the sheaf $\underline{E}$ is locally free of rank $-g + d + 1$ and $V^{(d)} \simeq \mathbb{P}(\underline{E})$, the projective bundle over J_d (see details in). Also, we have a universal embedding of the J_d-schemes

$$V_{J_d} = V \times J_d \longrightarrow \mathbb{P}(E^*)$$

whose fiber over a point $x \in J_d$ is the embedding $V \to \mathbb{P}(H^0(V,L_x))$.

Repeating the definition of the dual variety in this relative situation we easily get that it coincides with Δ and it is locally trivial over J_d. This shows that $V^{(d)} - \Delta \to J_d$ is a Serre fibration, hence, the exact homotopy sequence gives the needed isomorphism.

To compute $\mathrm{Ker}\ \mu_*$ we first compute the group $B_d(V) = \pi_1(\tilde{V}^{(d)})$, called the d-th braid group of V. Then knowing its generators and defining relations we can determine those for $\mathrm{Ker}\ \mu_*$ using the standard process of Schreier ([MKS]).

To compute $B_d(V)$ we represent it as the extension of the symmetric group S_d, corresponding to the natural non-ramified covering $p: V^d \to V^{(d)} = V^d/S_d$ restricted over $\tilde{V}^{(d)}$. The kernel of the homomorphism $B_d(V) \to S_d$ $(= \pi_1(V^d - p^{-1}(\Delta))$ is called the <u>pure d-th braid group</u> and is denoted by $F_d(V)$. The structural analysis of this group is based on the homotopy exact sequence associated with the fibration

$$V^d - p^{-1}(\Delta) \to V^{d-1} - p^{-1}(\Delta),\ (v_1,\dots,v_d) \to (v_1,\dots,v_{d-1})$$

with the typical fiber isomorphic to $V - (d-1)$ points. The analysis of this sequence gives a set of generators and defining relations for $F_d(V)$ (see [B], [S]). In the case $g = 0$ we get for $B_d(V)$:

Generators: $g_1,\dots,g_{d-1}$

Relations:

$$g_i g_j = g_j g_i \quad \text{if} \quad |i-j| \geq 2,\ 1 \leq i,\ j \leq d-1$$

$$g_i g_{i-1} g_i = g_{i-1} g_i g_{i-1} \quad 2 \leq i \leq d-1$$

$$g_1 \cdots g_{d-2} g_{d-1}^2 g_{d-2} \cdots g_1 = 1 \tag{3.1}$$

Here the group $F_d(V)$ is the normal subgroup generated by the squares of the g_i's, the cosets of the g_i's correspond to the standard generators of S_d, considered as a Coxeter group.

Summing up we conclude that the group $\pi_1(\check{P} - \mathrm{Disc}(\underline{L}))$ is computable for any curve V of genus g and an invertible very ample sheaf of degree $d > 2g-2$. Taking a general plane section we get the fundamental group of the complement to a certain plane curve of degree $2(d-g-1)$ with $3(d-2g-2)$ cusps and $2(d-2)(d-3)-2g(2d+g-7)$ nodes (see example 2 in $n^o 2$). In the simplest nontrivial case where $g = 0$, $d = 3$ one can make all calculations as an exercise and get the fundamental group of the complement to a cuspidal quartic. It turns out to be a metacyclic group of order 12. Notice that the family of plane curves above can be also characterized as dual curves to singular plane curves of genus g with maximal number of nodes.

The braid groups of Riemann surfaces $B_d(V)$ have a close relation to the mapping class groups which play an important role in the uniformization theory (see [MA]).

Let $\mathrm{Diff}^+(M)$ be the group of orientation preserving diffeomorphisms of a smooth manifold M endowed with the Whitney C^∞-topology. Then the group $\mathrm{Diff}^+(V)$ acts transitively on the set of d distinguished points on a Riemann surface V. Thus, fixing a point $x_o \in \tilde{V}^{(d)}$ we can identify the space $\tilde{V}^{(d)}$ with the coset space $\mathrm{Diff}^+(V)/\mathrm{Diff}^+(V,x_o)$, where $\mathrm{Diff}^+(V,x_o)$ denotes the subgroup of diffeomorphisms which leave x_o invariant. Applying the exact homotopy sequence we get the exact sequence of groups

$$\pi_1(\mathrm{Diff}^+(V);\mathrm{id}) \xrightarrow{\beta} \pi_1(\tilde{V}^{(d)};x_o) \xrightarrow{\gamma} \pi_o(\mathrm{Diff}^+(V,x_o);\mathrm{id})$$

$$\longrightarrow \pi_o(\mathrm{Diff}^+(V);\mathrm{id}) \to \{1\} \tag{3.2}$$

The group $\pi_o(\mathrm{Diff}^+(M);\mathrm{id})$ is the group of isotopy classes of orientation-preserving diffeomorphisms. In the case of Riemann surfaces it is called the mapping class group and is denoted by $M(g)$. Its subgroup $\pi_o(\mathrm{Diff}(V,x_o);\mathrm{id})$ is denoted by $M(g,d)$. In this case one can change diffeomorphisms by homeomorphisms without changing the groups.

Using the sequence (3.2) one can prove that

$$\beta(\pi_1(\mathrm{Diff}(V);\mathrm{id}) = \text{Center } \pi_1(\tilde{V}^{(d)};x_o) \tag{3.3}$$

Except the trivial cases $g = 1,\ d = 1$ and $g = 0,\ d \leq 2$ we have (see [B], 4.1):

$$1 \longrightarrow \mathbb{Z}/3 \longrightarrow SL_3(\mathbb{C}) \longrightarrow PSL_3(\mathbb{C}) \longrightarrow 1$$

In order to describe those groups more explicitly we first consider the extension K

$$1 \longrightarrow \mathbb{Z}/3 \longrightarrow K \longrightarrow \mathbb{Z}/3 \oplus \mathbb{Z}/3 \longrightarrow 1 \qquad (4.3)$$

of the group $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ of translations by the points of order 3. For the curves given in the canonical Hesse form:

$$x^3 + y^3 + z^3 - 3\mu\, xyz = 0$$

the group of translations consists of the matrices ([M], [E])

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \qquad (\omega^3 = 1)$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{pmatrix} \qquad (4.4)$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 1 \\ 0 & \omega^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 1 \\ 0 & \omega & 0 \end{pmatrix}$$

All those matrices are of order 3 (as elements of $SL_3(\mathbb{C})$ and not just $PSL_3(\mathbb{C})$!). Therefore K is the group of exponent 3 and order 27 (obviously non-abelian). There is only one such group ([H]) and it is isomorphic to the Heisenberg group of unipotent 3×3 matrices over $\mathbb{Z}/3$:

$$\text{Center}(B_d(V)) = \begin{cases} \{1\} & \text{if } g \geq 2 \\ \mathbb{Z}^2 & \text{if } g = 1,\ d \geq 2 \\ \mathbb{Z}/2 & \text{if } g = 0,\ d \geq 3 \end{cases}$$

In the case $g = 0$, $\pi_1(\text{Diff}(V);\text{id}) = \pi_1(\text{Diff}(S^2);\text{id}) \simeq \mathbb{Z}/2$ according to Smale [S].

In the case $g \geq 1$, $\pi_1(\text{Diff } V, \text{ id}) = \pi_1(\text{Aut}_o(V),\text{id}) \simeq \mathbb{Z}^2$ $(g = 1)$ or $\{0\}$ $(g > 1)$, where $\text{Aut}_o(V)$ is the connected group of automorphisms of this complex manifold V ([EE]).
Thus we get

$$\text{Center } B_d(V) \simeq \pi_1(\text{Diff}(V);\text{id}) \qquad (3.4)$$

In the case $g = 0$ the group $M(0)$ is trivial and $M(0,d) \simeq B_d(P^1)/\text{Center}$ can be computed using presentation (3.1) of $B_d(\mathbb{P}^1)$. In this presentation the center is generated by the element $(g_1 \ldots g_{d-1})^d$. In the case $g > 2$ only generators of $M(g,d)$ are known (the so called Dehn twists).

4. Fundamental groups arising from some systems of elliptic curves

In this section we compute the fundamental group of the complement to the plane curves of examples 3a,b of §2. First we consider the case $V = \mathbb{P}^2$ and $\underline{L} = \underline{O}_{\mathbb{P}^2}(3)$. We start by defining some groups associated with elliptic curves in terms of which the answer will be given.

The group of biregular authomorphism of a cubic curve is the semi-direct product of the group of translations and the group of automorphism of the abelian variety associated to the cubic curve by fixing a point as its zero point. The latter group for different values of the j-invarinat is given in [C].

$$\operatorname{Aut} C_j = \begin{cases} \mathbb{Z}/4 & j = 0 \\ \mathbb{Z}/6 & j = 1 \\ \mathbb{Z}/2 & j \neq 0,1 \end{cases} \tag{4.1}$$

Any automorphism of C_j is determined by its action on the points of order 3. This identifies each $\operatorname{Aut} C_j$ with the subgroup of $SL_2(\mathbb{Z}/3)$. Any element of $\operatorname{Aut} C_j$ is represented by a projective transformation of $\mathbb{P}^2$. The translations which are projective transformations are translations by points of order 3. Indeed, a translation by point x does not change the class of linear equivalence of an effective divisor of degree 3 which provides the embedding if and only if x has order 3. We denote the group of projective automorphisms of a cubic curve by G_0, G_1, G_2 according to the values $j = 0$, 1, or $j \neq 0$, 1 of the j-invariant of the corresponding elliptic curve. Those groups are the extensions of subgroups of $SL_2(\mathbb{Z}/3)$ defined by the representation of the affine linear group $SA(\mathbb{Z}/3)$ as an extension

$$1 \longrightarrow \mathbb{Z}/3 \oplus \mathbb{Z}/3 \longrightarrow SA_2(\mathbb{Z}/3) \longrightarrow SL_2(\mathbb{Z}/3) \longrightarrow 1$$

In other words, G_0, G_1, G_2 are the subgroups of $SL_3(\mathbb{Z}/3)$ of matrices of the form

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \tag{4.2}$$

where $x \in \operatorname{Aut} C_j \subset SL_2(\mathbb{Z}/3)$ and $y \in \mathbb{Z}/3 \oplus \mathbb{Z}/3$.

Let $\tilde{G}_0$, $\tilde{G}_1$, $\tilde{G}_2$ be the central extensions of the groups G_0, G_1, G_2 respectively, induced by the extension

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad a,b,c \in \mathbb{Z}/3 \qquad (4.5)$$

The group $\tilde{G}_j$ now can be determined from the exact sequence

$$1 \longrightarrow K \longrightarrow \tilde{G}_j \longrightarrow \mathrm{Aut}\ C_j \longrightarrow 1$$

which is the semidirect product. The homomorphism $\mathrm{Aut}\ C_j \longrightarrow$ Outer Aut K/ Inner Aut K defining this extension is given by

$$\varphi \longrightarrow \left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \varphi_1(a,c) & b \\ 0 & 1 & \varphi_2(a,c) \\ 0 & 0 & 1 \end{pmatrix} \right) \qquad (4.6)$$

where $(a.c) \xrightarrow{\varphi} (\varphi_1,(a,c),\ \varphi_2(a,c))$ is an automorphism of $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ obtained from the identification Aut C_j with a subgroup of $(SL_2(\mathbb{Z}/3)$ (Indeed Aut C_j acts trivially on the center of K because it is also the center of $\tilde{G}_j$).

Now we are ready to describe the fundamental group $\pi_1(U_3)$ of the complement to the discriminant variety , where

$$U_3 = \mathbb{P}(H^0(\mathbb{P}^2,\ 0_{\mathbb{P}^2}(3)) - \mathrm{Disc}.$$

We can identify U_3 with the space of non-singular plane cubics. Let $\mathrm{Inf} \subset \mathbb{P}(H^0(\mathbb{P}^2, 0_{\mathbb{P}^2}(3)) \times \mathbb{P}^2$ be the graph of the incidence correspondence, consisting of the pairs (C,x) where C is a cubic curve and x is an inflection point of C. As usual p_1, p_2 be the projections of Inf to the factors. Let $\bar{U}_3 = p_1^{-1}(U_3)$. The group $PGL(3,\mathbb{C})$ acts on the both $\bar{U}_3$ and $\bar{U}_3$. We have

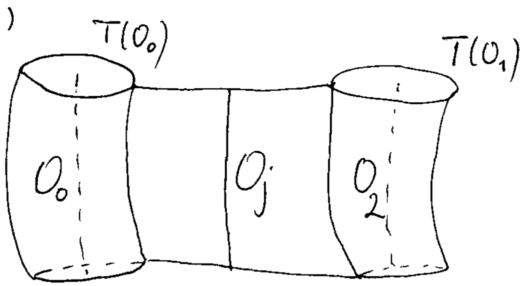
$$\bar{U}_3/PGL(3,\mathbb{C}) = U_3/PGL(3,\mathbb{C})$$

because $PGL(3,\mathbb{C})$ acts transitively on the set of inflection points of any cubic curve. Indeed inflection points correspond to the points of order 3 and $PGL(3,\mathbb{C})$ contains all translations by points of order 3.

On the other hand

$$\bar{U}_3 \ / \ PGL(3,\mathbb{C}) = \mathbb{C}$$

and isomorphism is given by the j-invariant (see [C]). The action of $PGL(3,\mathbb{C})$ on U_3 is proper ([M2]) with finite isotropy groups. Therefore ([H]) the orbits O_o and O_1, consisting of the curves with the j-invariant equal to 0 or 1 respectively, have slice neighborhoods $T(O_o)$ and $T(O_1)$ each isomorphic to $(\mathbb{C} \times PGL(3,\mathbb{C}))/G_j$. Clearly U_3 can be retracted on the union of $T(O_o)$ and $T(O_1)$ and the intersection is homotopy equivalent to $T(O_j)$ ($j \neq 0,1.$)



By the Van Kampen theorem we deduce

$$\pi_1(U_3) = \pi_1(T(O_o)) \underset{\pi_1(T(O_2))}{*} \pi_1(T(O_1))$$

The groups $\pi_1(T(O_o))$, $\pi_1(T(O_1))$, $\pi_1(T(O_2))$ are isomorphic to $\tilde{G}_o$, $\tilde{G}_1$, $\tilde{G}_2$ respectively. This gives

$$\pi_1(U_3) = \tilde{G}_o \underset{\tilde{G}_2}{*} \tilde{G}_1 . \qquad (4.7)$$

Now let us consider the monodromy map

$$m: \pi_1(U_3,p) \longrightarrow \mathrm{Aut}(H_1(C_p,\mathbb{Z})) = SL_2(\mathbb{Z})$$

(Here p is an arbitrary point in U_3 and C_p is the corresponding cubic curve)

In each group $\tilde{G}_j$ (which is the subgroup of $\pi_1(U_3,p)$, cf [Se]) this map takes subgroup K into the identity. Indeed the elements of K induce by monodromy the diffeomorphisms on C_p which corresponds to the translation. But clearly they are homotopy to the identity map and hence induce the identity on $H_1(C_p,\mathbb{Z})$. (The homotopy for the translation $x \to x + a$ is the family of the translations $x \to x + a_t$ where a_t is a path in C_p connecting a with the zero on C_p). All other elements of $\tilde{G}_j$ act nontrivally on $H_1(C_p,\mathbb{Z})$. Hence m takes $\tilde{G}_0 \underset{\tilde{G}_2}{*} \tilde{G}_1$ onto $Z/4 \underset{Z/2}{*} Z/6 = SL_2(Z)$ and we obtain the exact sequence

$$1 \longrightarrow K \longrightarrow \pi_1(U_3,p) \longrightarrow SL_2(\mathbb{Z}) \longrightarrow 1 \tag{4.8}$$

This sequence allows in particular to compute the center of $\pi_1(U_3,p)$. The only non-trivial central element of $SL_2(\mathbb{Z})$ is the image of the element in $\tilde{G}_2$ represented in form (4.2) by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is not in the center of $\tilde{G}_2$.

Thus we have

$$\mathrm{Center}\ (\pi_1(U_3,p)) = \mathrm{Center}\ (K) = \mathbb{Z}/3 \tag{4.9}$$

Now we turn to the computation of the fundamental group of the complement to the discriminant in the second case of example 3 i.e. $V = \mathbb{P}^1 \times \mathbb{P}^1$ and $\underline{L} = \underline{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2)$. In this case we can identify $U_{2,2} = P(H^0(P^1 \times P^1, O_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2))$ - Disc with the space of non-singular elliptic curve lying on a quadric $V = \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. Our computation in this case is rather similar to the described above. However, several words should be said about the choice of analogue of the group $PGL(3,\mathbb{C})$. The natural candidates $PGL(4,\mathbb{C})$ and $PSO(4,\mathbb{C}) = \text{Aut } V$ are not appropriate here. The former group is too big, because it does not leave stable the set of curves lying on a quadric. The latter group $PSO(4,\mathbb{C}) = \text{Aut } V = PGL(2,\mathbb{C}) \times PGL(2,\mathbb{C})$ is two small, because it does not act transitively on the set of biregular equivalent elliptic curves on V. We use the group P instead, which is the subgroup of $PGL(4,\mathbb{C})$ consisting of transformations which take a fixed elliptic curve on V into a curve lying on the same surface V. P can be described also as a subgroup of $PGL(4,\mathbb{C})$ which in the natural representation on the space $\mathbb{P}(H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(2)))$, takes a fixed line ℓ passing through a fixed point α into a line through the same point α. Those two definitions of P are equivalent, because quadrics in $\mathbb{P}^3$ are identified with the elements in $\mathbb{P}(H^0(\mathbb{P}^3, O(2))$ and elliptic curves in $\mathbb{P}^3$ are represented by lines in this space. The group P acts transitively on the points of ℓ by the formula $a \xrightarrow{\varphi} \varphi^{-1}(a)$. The isotropy group α under this action is $PSO(4,\mathbb{C})$ and the fibration $P \xrightarrow{PSO(4,\mathbb{C})} \mathbb{P}^1(\mathbb{C}) = S^2$ admits a section. Hence

$$\pi_1(P) = \pi_1(PSO(4,\mathbb{C}) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad ,$$

The group of translations of an elliptic curve on V which are in P is the group of translations by the points of order 4 (only those translations are induced by projective transformations). For the curves written in the canonical form:

$$X_1^2 + X_3^2 = 2\lambda\, X_o\, X_2$$

$$X_0^2 + X_2^2 = 2\lambda\, X_1\, X_3$$

this group generated by the matrices [M1]

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \quad (i^2=-1) \qquad (4.11)$$

The central extension of the translation group is the subgroup Q of the universal covering $\tilde{P}$ of P, generated by matrices (4.11) and

$$\epsilon_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \epsilon_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (4.12)$$

Therefore Q is a non-abelian group of order 64 and exponent 4, which can be represented as an extension of the form

$$1 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow Q \longrightarrow \mathbb{Z}/4 \oplus \mathbb{Z}/4 \longrightarrow 1$$

As it was kindly explained to us by R. Griess and N. Ito these properties define uniquely the group. We have

$$Q = \mathbb{Z}/2 \times H$$

where

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in GL(3,\mathbb{Z}/4),\ a,b,c \in \mathbb{Z}/4 \right\} \Big/ \left\{ \begin{pmatrix} 1 & 0 & 2c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Note that

$$\text{Center}(H) = (\mathbb{Z}/2)^3$$

and hence

$$\text{Center } Q = (\mathbb{Z}/2)^4 \qquad (4.13)$$

Let G_0', G_1', G_2' denote the groups of automorphisms of an elliptic curve in $\mathbb{P}^3$ induced by projective transformations. They are the semidirect products of the group of translations $\mathbb{Z}/4 \oplus \mathbb{Z}/4$ and groups Aut C_j defined by (4.1) with the matrix presentation (4.2) where $x \in SL_2(\mathbb{Z}/4)$ and $y \in \mathbb{Z}/4 \oplus \mathbb{Z}/4$.

Let $\tilde{G}_0'$, $\tilde{G}_1'$, $\tilde{G}_2'$ be the central extensions of the group G_0', G_1', G_2' induced by the universal extension of P. They admit a description which is similar to the given above for $\tilde{G}_j$. By considering the action of P on $U_{2,2}$ and applying again the slice theorem and Van Kampen theorem, we deduce

$$\pi_1(U_{2,2},p) = \tilde{G}_0' \underset{\tilde{G}_2'}{*} \tilde{G}_1'$$

Moreover the monodromy map yields the following exact sequence

$$1 \longrightarrow Q \longrightarrow \pi_1(U_{2,2},p) \longrightarrow SL_2(\mathbb{Z}) \longrightarrow 1 \qquad (4.14)$$

where Q is defined above.

Note that the similar arguments as in the case of plane cubics show that the center of $\pi_1(U_{2,2},p)$ is the same as the center of Q i.e. isomorphic to $(\mathbb{Z}/2)^4$.

Finally we remark that the groups $\pi_1(U_{2,2})$ and $\pi_1(U_3)$

both have the commutator subgroups with torsion elements. For example, for $\pi_1(U_3,p)$ we have

$$\pi_1(U_3,p)/\left[\pi_1(U_3,p),\ \pi_1(U_3,p)\right] = \mathbb{Z}/12 \qquad (4.15)$$

because the degree of the discriminant variety is 12. Therefore the homomorphism $K \longrightarrow \pi_1(U_3)/\left[\pi_1(U_3),\pi_1(U_1)\right]$ is not injective. This provides an answer to a question in [O].

5. Variations on the theme of the mapping class groups.

Here we speculate on possible relations between computations of the fundamental group of the complement to a discriminant variety and the homotopy groups of diffeomorphisms groups. As we saw in §3 there is such a relation in the cases of curves.

Sequence (3.2) can be easily generalized as follows. Instead of $\tilde{V}^{(d)}$ we may consider the space $Im(W,V)$ of all smooth orientation preserving embeddings of a smooth compact manifold W into a smooth compact manifold V endowed with the Whitney C^∞-topology. Fixing an immersion $i_o: W \to V$ we consider the connected component $Im(W,V)_O$ of $Im(W,V)$ containing i_o. Then the group $Diff^+(V) = Im(V,V)$ acts transitively by compositions on $Im(W,V)_O$ ([CE], p.116) identifying the latter with the coset space $Diff^+(V)/Diff^+(V,i_o(W))$, where the second group is the subgroup of diffeomorphisms which leave $i_o(W)$ invariant. Now, the exact homotopy sequence yields

$$\pi_1(Diff^+(V);id) \overset{\alpha}{\to} \pi_1(Im(W,V);i_o) \overset{\varphi}{\to} M(V,W) \to M(V) \to \{1\} \qquad (5.1)$$

Here $M(V,W) = Diff^+(V,i_o(W)/$ isotopy, $M(V) = M(V,\emptyset)$ are the generalized mapping class groups.

Let V be a nonsingular algebraic variety and $i: V \to \mathbb{P}^n$

its closed embedding. Fixing a point $x_o \in \check{\mathbb{P}}^n - \check{i}(V)$ we have an immersion $i_o: W \to V$, where $W = D_{x_o} = H_{x_o} \cap i(V)$. Varying x in $\check{\mathbb{P}}^n - \check{i}(V)$ we get an injective map

$$\check{\mathbb{P}}^n - \check{i}(V) \longrightarrow \mathrm{Im}(W,V)_0$$

It is not difficult to prove that this map is continuous with respect to the usual Hausdorff topology of $\mathbb{P}^n$ and the Whitney C^∞-topology of $\mathrm{Im}(W,V)$. Thus, we obtain a homomorphism of groups

$$\gamma: \pi_1(\check{\mathbb{P}}^n - \check{i}(V);\ x_o) \longrightarrow \pi_1(\mathrm{Im}(W,V)_{i_o}; i_o)$$

<u>Question 1.</u> What one can say about this homomorphism?

For example, suppose that we know that the map γ is surjective. Consider the universal monodromy map $\rho: \pi_1(\check{\mathbb{P}}^n - \check{i}(V);\ x_o) \to M(W)$. Let $r: M(V,W) \to M(W)$ be the restriction homomorphism. Since it is always surjective ([CE],p.114), exact sequence (5.1) shows that ρ will be surjective as soon as the group $M(V)$ is trivial.

<u>Question 2.</u> Is every orientation-preserving diffeomorphism of the complex projective space $\mathbb{C}P^n$ is isotopical to the identity map (that is, $\pi_0(\mathbb{C}P^n, \mathrm{id}) = \{1\}$)?

The positive answer to this question will certainly agree with computations of section 4. Returning to sequence (5.1) we may ask the following question (keeping in mind the analogy with the case of the braid group (3.4)).

<u>Question 3.</u> Is it true that

$$\alpha(\pi_1(\mathrm{Diff}(V), \mathrm{id}) = \text{Center } (\pi_1(\mathrm{Im}(W,V);\ i))?$$

Suppose that γ is injective. Then $\alpha(\pi_1(\mathrm{Diff}\ V); \mathrm{id}) \cap \pi_1(\check{\mathbb{P}}^n - i(\check{V});\ x_o)$ lies in the kernel of the universal monodromy map

$\rho : \pi_1(\check{\mathbb{P}}^n - i(\check{V}); id) \to M(W)$. The positive answer to the question 3 would imply that $\alpha(\pi_1(\mathrm{Diff}\ V); id) \cap \pi_1(\check{\mathbb{P}}^n - i(\check{V}); x_o)$ lies in the center of $\pi_1(\check{\mathbb{P}}^n - i(\check{V}), x_o)$.

Let $\mathrm{Aut}_o(V)$ be the subgroup of Diff^+V consisting of automorphisms of V as a complex manifold. In the case $\dim V = 1$, $\mathrm{Aut}_o(V)$ is a retract of $\mathrm{Diff}(V)$ ([EE]) and hence the natural map

$$\beta : \pi_1(\mathrm{Aut}_o(V); id) \longrightarrow \pi_1(\mathrm{Diff}^+(V); id)$$

is an isomorphism. In general the map i is not surjective anymore ([ABK]). Composing β with α we get a map (assuming question 3 is solved postively)

$$\pi_1(\mathrm{Aut}_o(V); id) \longrightarrow \mathrm{Center}\ \pi_1(\check{\mathbb{P}}^n - i(\check{V}), x_o)$$

In the examples considered in §4 we have

$$\pi_1(\mathrm{Aut}_o(V)) = \begin{cases} \mathbb{Z}/3 & \text{(the first example)} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{(the second example)} \end{cases}$$

and

$$\mathrm{Center}\ \pi_1(\check{\mathbb{P}}^n - i(\check{V}); x_o) = \begin{cases} \mathbb{Z}/3 & \text{(the first example)} \\ (\mathbb{Z}/2)^4 & \text{(the second example)} \end{cases}$$

Here there exists a non-trivial homomorphism from one group to another This gives a certain evidence to question 3. Also it poses another question

Question 4. Is it true that the map

$$\pi_1(\mathrm{Aut}_o(\mathbb{CP}^2);\mathrm{id}) = \mathbb{Z}/3 \longrightarrow \pi_1(\mathrm{Diff}\ \mathbb{CP}^2);\mathrm{id})$$

is non-trivial.

Notice that the answer is positive if we replace $\mathbb{CP}^2$ by $\mathbb{CP}^1 \times \mathbb{CP}^1$.

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