Vol. 49, No. 4 DUKE MATHEMATICAL JOURNAL © December 1982

ALEXANDER POLYNOMIAL OF PLANE ALGEBRAIC CURVES AND CYCLIC MULTIPLE PLANES

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0. Introduction. As part of the study of fundamental groups of the complement to plane algebraic curves, Zariski had undertaken an investigation of branched coverings of P^2. Zariski had shown that simple homological invariants of those coverings provide nontrivial invariants of the complement to the branching set. For example, if the fundamental group of the complement of a curve is cyclic, then all coverings branched over it have vanishing first Betti number. The branched coverings of the plane were called multiple planes by Italian algebraic geometers and rather detailed information was obtained by them for coverings of small degree ([6], [2]). Zariski devoted to the study of cyclic multiple planes works [20] and [21]. The main result of the latter can be formulated as follows:

ZARISKI'S THEOREM. *Let C be an irreducible algebraic curve of degree n in P^2 given in affine part by equation $f(x, y) = 0$. Let*

$$z^k = f(x, y) \tag{0.1}$$

be the system of cyclic coverings F_k, branched over C (and possibly over the line L in infinity). Assume that singularities of C are only nodes and cusps (i.e., locally given by the equation $u^2 + v^2 = 0$ or $u^2 + v^3 = 0$) and C is transversal to the line L. Then the first Betti number of the desingularisation $\tilde{F}_k$ of F_k vanishes unless both the degree n of the curve C and the degree k of the covering are divisible by 6.

In [21] Zariski also gives a condition for nonvanishing of the first Betti number in the case when k and n are divisible by 6 in terms of position of cusps, which enable him to give a condition under which the fundamental group of the complement will be nonabelian.

Since then cyclic branched coverings showed their usefulness in the study of the complements to knots and links (see [8] for a survey of this extensive subject). In particular, there was found a formula connecting the Alexander polynomial of knots and links with Betti numbers of cyclic coverings.

The purpose of this paper is to apply those methods from knot theory to the study of branched coverings of P^2. We extend Zariski's theorem in two directions. Firstly we consider coverings branching over an irreducible curve C which might possess any sort of singularities. Secondly the position of the line in

Received December 1, 1980. Revision received June 14, 1982. Research supported by NSF Grant.

infinity which is (for generic k) a part of the branching set can be arbitrary with respect to the curve C.

We define the global Alexander polynomial Δ_C of the complement to an irreducible curve C (relative to a line) which has a connection with homologies of cyclic branched coverings similar to one in classical knot theory. Our main result is the relationship between global Alexander polynomial and local Alexander polynomials of singularities of the curve.

THEOREM 1. *The global Alexander polynomial* Δ_C *divides the product* $\Delta_1\Delta_2 \ldots \Delta_N$ *of the local Alexander polynomials for all branches of singularities of the curve* C.

Note that we consider as singular also the points of C which are on L. The definition of the local Alexander polynomial in this case is given in Section 4. Furthermore we define the Alexander polynomial Δ_L of a line L relative to curve C, as the Alexander polynomial of the link obtained by intersection of C with the sphere S^3 which is the boundary of a small tubular neighborhood of the line L.

THEOREM 2. *The global Alexander polynomial* Δ_C *of curve* C *relative to the line* L *divides the Alexander polynomial* Δ_L *of the line* L *relative to the curve* C.

From Theorems 1 and 2, one can readily deduce a generalization of Zariski's theorem. We define an exponent of a singularity as the order of the primitive root of unity which is a root of its Alexander polynomial. The exponent of a line is the order of primitive root of unity which is a root of Δ_L.

THEOREM 3. *Let* $\epsilon_1^1, \ldots, \epsilon_1^{\alpha_1}, \ldots, \epsilon_N^1 \ldots \epsilon_N^{\alpha_N}$ *be exponents of singularities of the curve* C. *Let* $\epsilon_\infty^1, \ldots, \epsilon_\infty^{\alpha\infty}$ *be exponents of the line in infinity relative to the curve* C. *If none of the numbers* $g.c.d.(k, \epsilon_\infty^\alpha)$ *is divisible by one of* $\epsilon_1^1, \ldots, \epsilon_1^{\alpha_1}, \ldots, \epsilon_N^1 \ldots \epsilon_N^{\alpha_N}$, *then the first Betti number of cyclic k-fold branched covering* F_k *defined by* (0.1) *vanishes.*

For a cuspidal curve of degree n which is in general position with respect to the line in infinity one has $\epsilon_\infty = n$ and all local exponents are either 1 (for nodes and points in infinity) or 6 (for cusps). Therefore Theorem 3 implies Zariski's theorem mentioned above.

The content of this paper is the following. In Section 1 we describe biregular projective models with isolated singularities of cyclic multiple planes. In classical works (including Zariski's) the properties of those models contain only implicitly. (Note however [14] where a detailed desingularisation of double plane branched over curve of degree 8 is given). In Section 2 we discuss various notions connected with the Alexander polynomial, i.e., infinite cyclic coverings, homology of finite cyclic coverings, additivity, etc. In Section 3 a lemma on the relationship between homology of branched and unbranched coverings is proven which is analogous to the corresponding result in knot theory. Sections 4 and 5 contain the proofs of Theorems 1 and 2 respectively. The main tool in Section 4 is the Fulton–Deligne solution of Zariski's problem ([7], [3]) which is somewhat a reduction of global objects to local ones. In Section 6 we derive Theorem 3 and in Section 7 we compute the global Alexander polynomial in the case when the fundamental group of the complement is known.

Note that the topological approach to Zariski's theorem was suggested earlier by Artin and Masur [1]. (cf. also Mumford [23], p.231).

Finally, I would like to thank Louis Kauffman, Philip Wagreich and John Wood for helpful discussion during my work and reading of the manuscript.

I am mostly indebted to W. Fulton for his very important and stimulating comments on the earlier version of this work. In particular he pointed out the gap in the proof of Lemma 3.1.

1. Cyclic multiple planes. We are going to give here a construction of a projective surface with isolated singularities which contains as an open set the affine surface (0.1) and a mapping of it onto $\mathbf{P}^2$ branched over the curve $f(x, y) = 0$ and possibly over the line in infinity.

As in the introduction, let C be an irreducible curve of degree n given by equation

$$f(u,x,y) = 0 \tag{1.1}$$

and L be the line given by $u = 0$.

Note that $H_1(\mathbf{P}^2 - C, \mathbf{Z}) = \mathbf{Z}/n$ and $H_1(\mathbf{P}^2(C \cup L, \mathbf{Z}) = \mathbf{Z}$. In fact, for any curve with components $C_1, \ldots, C_l$ of degrees $d_1, \ldots, d_l$ respectively

$$H_1\left(\mathbf{P}^2 - \left(\bigcup_{i=1}^{l} C_i\right), \mathbf{Z}\right) = \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/(d_1, \ldots, d_l) \tag{1.2}$$

(Indeed one has the following sequence of the pair

$$\mathbf{Z} = H_2(\mathbf{P}^2, \mathbf{Z}) \xrightarrow{j} H_2\left(\mathbf{P}^2, \mathbf{P}^2 - \left(\bigcup C_i\right), \mathbf{Z}\right) \to H_1\left(\mathbf{P}^2 - \left(\bigcup C_i\right), \mathbf{Z}\right) \to H_1(\mathbf{P}^2, \mathbf{Z}) = 0$$

and if $T(\bigcup_{i=1}^{l} C_i)$ is a regular neighborhood of C_i which has retraction on $\bigcup_{i=1}^{l} C_i$, then by excision and Lefshetz duality

$$\begin{aligned} H_2\left(\mathbf{P}^2, \mathbf{P}^2 - \bigcup_{i=1}^{l} C_i, \mathbf{Z}\right) &= H_2\left(\mathbf{P}^2, \mathbf{P}^2 - T\left(\bigcup_{i=1}^{l} C_i\right), \mathbf{Z}\right) \\ &\to H_2\left(T\left(\bigcup_{i=1}^{l} C_i\right), \partial T\left(\bigcup_{i=1}^{l} C_i\right), \mathbf{Z}\right) \to H_2\left(T\left(\bigcup_{i=1}^{l} C_i\right), \mathbf{Z}\right) \\ &= \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{l \text{ times}} \end{aligned}$$

Image of j has the components $(d_1, \ldots, d_l)$.

Let F_k^u denote a k-fold unbranched covering of $\mathsf{P}^2 - (C \cup L)$ defined by the kernel of homomorphism $\pi_1(\mathsf{P}^2 - (C \cup L)) \to H_1(\mathsf{P}^2 - (C \cup L)\mathsf{Z}) = \mathsf{Z} \to \mathsf{Z}/k$, P^2 is a spread of $\mathsf{P}^2 - (C \cup L)$ in the terminology of [5], and by [5], F_k^u can be extended to the branched covering F_k of P^2, branched over $C \cup L$.

Riemann–Enriques–Grauert–Remmert theorem ([23] app. to Ch VIII) guarantees the unique algebraic structure on F_k in which F_k is normal variety. In more explicit terms F_k can be described as follows.

F_k^u is just affine hypersurface defined by (0.1). Now let us consider the case $k > n$. Let F_k' denote projective completion in P^3 of surface F_k^u (i.e., defined by (0.1)). F_k' given by

$$z^k = f(u, x, y)u^{k-n} \tag{1.4}$$

Projection from the point $(u, x, y, z) = (0, 0, 0, 1)$ (which does not belong to F_k') defines the regular map $F_k' \to \mathsf{P}^2$ with branching set $C \cup L$. This map is k to 1 away from the branching set and is 1 to 1 over it. Singularities of F_k' are at the points of the line $u = z = 0$ and at the points $(u, x, y, 0)$ such that (u, x, y) is a singular point of C. Let F_k denote the normalization of F_k' (see e.g. [9] or [11] for definitions).

LEMMA 1.1. *The normalization map $F_k \to F_k'$ is $1 \to 1$ away from the line $u = z = 0$ and has degree $g.c.d.(n, k)$ over it.*

Proof. It is sufficient to check that a general plane section transversal to the singular line $u = z = 0$ has $g.c.d.(n, k)$ branches at the intersection point with this line. Let us consider, for instance, the family of planes $x = ty$. For t general, such plane intersects F_k' at a curve which has at $u = z = 0$ local equation equivalent to $z^k = u^{k-n}$. This singularity has $g.c.d.(n, k)$ branches at $u = z = 0$. Therefore the degree of the normalization F_k over $u = z = 0$ is equal to $g.c.d.(n, k)$. All other singularities of F_k' are normal because they are isolated singularities of a hypersurface (cf. [11], pg. 35). Hence the normalization map is 1–1 away from $u = z = 0$. We obtain therefore, that $F_k \to \mathsf{P}^2$ which is normalization of F_k' followed by projection provides the model with isolated singularities of a cyclic multiple plane when $k > n$. It is $k \to 1$ over $\mathsf{P}^2 - C \cup L$ $1 \to 1$ over C and $g.c.d.(k, n) \to 1$ over $L - (C \cup L)$.

Now let us consider the case $k < n$. Let F_k'' denote the projective completion of (0.1) in $\mathsf{P}^2 \times \mathsf{P}^1$. (It is technically slightly more convenient then consider the blowing-up of (1.4) at the center of the projection.) In bihomogeneous co-ordinates $(z_0 : z_1, u : x : y)$ F_k'' given by equation

$$z_1^k u^n = z_0^k f(u, x, y) \tag{1.5}$$

The natural projection $\mathsf{P}^2 \times \mathsf{P}^2 \to \mathsf{P}^2$ defines the map $F_k'' \to \mathsf{P}^2$. This map is k-fold and unbranched over $\mathsf{P}^2 - (C \cup L)$. It is $1 \to 1$ over $C \cup L - (C \cap L)$ and collapses lines $(z_0 : z_1, 0, \alpha, \beta)$, where $(0, \alpha, \beta) \in C \cap L$, to the points. Singularities of F_k'' are preimages of singular points of C and the line $u = z_0 = 0$. Let $\bar{F}_k''$ denote the normalization of F_k''.

LEMMA 1.2. *The normalization map* $\bar{F}_k'' \to F_k''$ *is* 1–1 *away from the line* $u = 0$ *and has degree* $g.c.d.(n,k)$ *over it.*

Proof is similar to the proof of Lemma 1.1. We consider the family of $\mathbf{P}^1 \times \mathbf{P}^1$'s defined by $x = ty$. The local equation of the intersection has near $u = z_0 = 0$ an equation equivalent to $u^n = z_0^k$, which has $g.c.d.(n,k)$ branches.

In order to conclude the construction we consider the Stein factorization ([9]) of the composition map $\bar{F}_k'' \to \mathbf{P}^2$.

$$\begin{tikzcd} F_k'' \arrow[rr, "\varphi_1"] \arrow[dr, "\varphi''"'] & & F_k \arrow[dl, "\varphi"] \\ & \mathbf{P}^2 & \end{tikzcd} \tag{1.6}$$

Here φ_1 has connected fibres, and φ_2 is finite (i.e., all fibres are finite). Therefore φ_1 collapses to the points the lines which are preimages under φ'' of the points of $C \cap L$. Hence F_k has isolated singularities and provides the required model for cyclic multiple planes.

Note that Lemma 1.2 implies that if k divides n then F_k is unbranched over line in infinity.

In the remaining case, $k = n$, F_k is just projective completion in $\mathbf{P}^3$ of the affine surface (0.1). F_k has isolated singularities corresponding to the singularities of C and projection $F_k \to \mathbf{P}^2$ has C as branching set.

2. The Alexander polynomials. In this section we recollect necessary facts on Alexander polynomials of simplicial complexes. For a detailed survey see [4] or [8].

Let X be a finite simplicial complex. Let $\varphi : \pi_1(X) \to \mathbf{Z}$ be a homomorphism of fundamental group of X onto the group of integers. (We suppress base point because it does not play any role in the following.) Then $\ker \varphi$ defines the infinite cyclic covering $\tilde{X}_\varphi$. The group $\mathbf{Z}$ acts on the groups $H_i(\tilde{X}_\varphi, \mathbf{C})$ by deck transformations. Therefore $H_i(\tilde{X}_\varphi, \mathbf{C})$ are in a natural way modules over the group ring of $\mathbf{Z}$, i.e., over the ring of Laurent polynomials $\Lambda = \mathbf{C}[t, t^{-1}]$. $H_i(\tilde{X}_\varphi, \mathbf{C})$ are finitely generated over Λ because they are generated over Λ by preimages of cells of dimension i. The ring $\mathbf{C}[t, t^{-1}]$ is a principal ideals domain. In particularly any torsion module M can be represented as

$$M = \Lambda/\lambda_1 \oplus \Lambda/\lambda_2 \oplus \cdots \oplus \Lambda/\lambda_n \tag{2.1}$$

The λ's will be called Alexander factors and the polynomial $\Delta = \lambda_1 \ldots \lambda_n$ which is the "order" of Λ-module M is the Alexander polynomial of M. Δ is defined up to unite of the ring Λ i.e., up to αt^i. If $\Delta(1) \neq 0$ then we shall normalize it in such a way that

$$\Delta(1) = 1 \tag{2.2}$$

If M has free summand then it is convenient to put $\Delta = 0$. In this paper we are concerned only with $H_1(\tilde{X}_\varphi, \mathbf{C})$ and we shall refer to the order of $H_1(\tilde{X}_\varphi, \mathbf{C})$ as to the Alexander polynomial of X relative to φ.

Note that if $H_1(X, \mathbf{Z}) = \mathbf{Z}$ then $H_1(\tilde{X}_\varphi, \mathbf{C})$ is a torsion module. Recall briefly the argument ([13]). The exact sequence of chain complexes

$$0 \to C_*(\tilde{X}_\varphi) \xrightarrow{t-1} C_*(\tilde{X}_\varphi) \longrightarrow C_*(X) \to 0 \tag{2.3}$$

(t is the deck transformation) implies

$$\begin{array}{ccccccc} H_1(\tilde{X}_\varphi, \mathbf{C}) & \xrightarrow{t-1} & H_1(\tilde{X}_\varphi, \mathbf{C}) & \longrightarrow & H_1(X, \mathbf{C}) & \longrightarrow & H_0(\tilde{X}_\varphi, \mathbf{C}) \\ & & & & \| & & \| \\ & & & & \mathbf{C} & & \mathbf{C} \end{array}$$

$$\xrightarrow{t-1} H_0(\tilde{X}_\varphi, \mathbf{C}) \longrightarrow H_0(X, \mathbf{C}) \tag{2.4}$$

$(t-1)$ is the zero map on $H_0(\tilde{X}_\varphi, \mathbf{C})$. Therefore this exact sequence implies that $(t-1)$ on $H_1(\tilde{X}_\varphi, \mathbf{C})$ is a map onto $H_1(\tilde{X}_\varphi, \mathbf{C})$. Hence $H_1(\tilde{X}_\varphi, \mathbf{C})$ does not have free component and the claim follows. In particular, in this case Δ is well defined by (2.1) and can be normalized as in (2.2).

The Alexander polynomial of X can be derived directly from the fundamental group given by generators and relators using the Fox calculus ([4], [8]).

Let $\pi_1(X) = \{x_1, \ldots, x_n \mid r_1, \ldots, r_m\}$ be a presentation of the funadmental group with n generators and m-relators. Let F_n be free group on n generators. Let $\partial/\partial X_i$ denote the Fox derivative, i.e., the endomorphism of the group ring over $\mathbf{C}$ of free group F_n into itself, such that this is the linear map $\mathbf{C}[F_n] \to \mathbf{C}[F_n]$ defined by the following properties

$$\begin{gathered} \frac{\partial X_j}{\partial X_i} = \delta_{ij} \\ \frac{\partial(uv)}{\partial X_i} = \frac{\partial u}{\partial X_i} + u\frac{\partial v}{\partial X_i} \end{gathered} \tag{2.5}$$

Then the Alexander polynomial of $\pi_1(X)$ relative to $\varphi : \pi_1(X) \to \mathbf{Z}$ is the greatest common divisor of the minors of the order $(n-1)$ in the matrix (Jacobian matrix)

$$\varphi\left(\frac{\partial r_i}{\partial X_j}\right) \in \mathrm{Mat}_{n \times m}(\mathbf{C}[t, t^{-1}])$$

where by abuse of notations we denote by φ also the homomorphism

$\mathbf{C}[F_n] \to \mathbf{C}[t, t^{-1}]$ of group ring of F_n onto group ring of $\mathbf{Z}$ induced by the map $F_n \to \pi_1 \to \mathbf{Z}$.

A corollary of this procedure is that the Alexander polynomial of X depends only on $\pi_1(X)$ and on φ. Therefore the notation $\Delta_\varphi(G)$ for Alexander polynomial of a complex with fundamental group G relative to φ is well defined.

PROPOSITION 2.1. *Suppose we are given the following diagram*

$$\begin{tikzcd} G_1 \arrow[rr, "\varphi"] \arrow[dr, "\varphi_1"'] & & G_2 \arrow[dl, "\varphi_2"] \\ & \mathbf{Z} & \end{tikzcd} \tag{2.6}$$

Suppose that φ is onto. Then $\Delta_{\varphi_2}(G_2)$ divides $\Delta_{\varphi_1}(G_1)$.

Proof. The groups G_1 and G_2 can be represented by the same set of generators and G_2 can be obtained from G_1 by addition of new relators. Therefore the Jacobian matrix of G_2 is just an extension of Jacobian matrix of G_1. The description of Alexander polynomial via Jacobian matrices implies the proposition.

PROPOSITION 2.2. *Let X, X_1, X_2 be simplicial complexes such that $X = X_1 \cup X_2$. Let $X_0 = X_1 \cap X_2$ and φ is a homomorphism $\varphi : \pi_1(X) \to \mathbf{Z}$. Assume that the compositions of inclusions with φ*

$$\begin{aligned} \varphi_0 &: \pi_1(X_0) \to \pi_1(X) \to \mathbf{Z} \\ \varphi_1 &: \pi_1(X_1) \to \pi_1(X) \to \mathbf{Z} \\ \varphi_2 &: \pi_1(X_2) \to \pi_1(X) \to \mathbf{Z} \end{aligned} \tag{2.7}$$

are onto. Suppose that Alexander polynomials of X_0, X_1, X_2, X are well defined. Then $\Delta_\varphi(\pi_1(X))$ divides $\Delta_{\varphi_1}(\pi_1(X_1))\Delta_{\varphi_2}(\pi_1(X_2))$.

Proof. Let us consider the following Mayer–Vietoris sequence for infinite abelian coverings defined by $\varphi_0, \varphi_1, \varphi_2, \varphi$.

$$H_1(\tilde{X}_{0,\varphi_0}) \to H_1(\tilde{X}_{1,\varphi_1}) \oplus H_1(\tilde{X}_{2,\varphi_2}) \to H_1(\tilde{X}_\varphi) \to 0 \tag{2.8}$$

This sequence ends in 0 because for the covering $p : X_\varphi \to X$ defined by $\ker\varphi$ we have (by surjectivity of φ_0, φ_1 and φ_2) that $p^{-1}(X_0), p^{-1}(X_1), p^{-1}(X_2)$ are connected. Therefore in the beginning of the Mayer–Vietoris sequence we have the monomorphism $H_0(\tilde{X}_{0,\varphi_0}) \to H_0(\tilde{X}_{1,\varphi_1}) \oplus H_0(\tilde{X}_{2,\varphi_2})$. The proposition now follows from the standard property of orders of modules in exact sequences.

We conclude this section with the description of the relationship between the Alexander polynomial of simplicial complexes and finite cyclic coverings. Let φ_k denote the composition

$$\varphi_k : \pi_1(X) \to \mathbf{Z} \to \mathbf{Z}/k$$

We denote by $X^u_{k,\varphi}$ the (unbranched) covering defined by $\ker \varphi_k$, and suppress φ when it is clear which φ is used. Let

$$H_1(\tilde{X}_\varphi, \mathrm{C}) = \bigoplus_{i=1}^{l} \Lambda/\lambda_i$$

by representation of homology group of infinite cyclic covering as a sum of cyclic modules. Let α_i^k be the number of common roots of λ_i and $(t^k - 1)$. Then

$$rkH_1(X^u_k, \mathrm{C}) = \sum_{i=1}^{l} \alpha_i^k + 1 \tag{2.9}$$

The arguments in the proof of (2.9) use the sequences similar to (2.3) and (2.4) in which instead $(t-1)$ one takes $t^k - 1$. For details we refer to Sumner's paper [17].

3. The first Betti number of the cyclic multiple planes. In this section we derive the relationship between the first Betti number of a desingularization of the branched covering F_k constructed in Section 1 and the complement in F_k to the branching locus.

Let, as in Section 1, F^u_k denote the k-fold cyclic covering of $\mathrm{P}^2 - (C \cup L)$, $\tilde{F}_k$ be a desingularization of F_k. Let E denote the exceptional set, i.e., the subvariety of $\tilde{F}_k$ which maps to the singular points of F_k under the desingularization map. Let N denote a regular neighborhood of E and $\bar{F}_k$ be $\tilde{F}_k - E$.

LEMMA 3.1. *Let $t_1, \ldots, t_c$ be the intersection indices of C and L. Assume for simplicity that $g.c.d.(t_1, \ldots, t_c, k) = 1$.*
Then the following relation holds

$$rkH_1(F^u_k, \mathrm{C}) = rkH_1(\tilde{F}_k, \mathrm{C}) + 1.$$

We shall prove this lemma in several steps.

Step 1. The map

$$i : H_1(\partial N, \mathrm{C}) \to H_1(N, \mathrm{C}) \tag{$*$}$$

induced by inclusion is an isomorphism.

This is quite standard fact (cf. e.g., W. Hammond. The Hilbert modular surface of a real quadratic field. Math. Ann. 200 (25–45) 1971). Firstly we have $H_1(N, \partial N) = H_3(N) = 0$. Secondly, from exact sequence of the pair $(N, \partial N)$

$$\mathrm{Ker}(H_1(\partial N, \mathrm{C}) \to H_1(N, \mathrm{C})) = \mathrm{Coker}(H_2(N, \mathrm{C}) \to H_2(N, \partial N, \mathrm{C})) \tag{$**$}$$

The group $H_2(N, \partial N, \mathbf{C})$ can be identified with $\mathrm{Hom}(H_2(N, \mathbf{C}), \mathbf{C})$ and the map $H_2(N, \mathbf{C}) \to H_2(N, \partial N, \mathbf{C})$ can be identified with the map $g: H_2(N, \mathbf{C}) \to \mathrm{Hom}(H_2(N, \mathbf{C}), \mathbf{C})$ associated with the intersection form on $H_2(N, \mathbf{C})$. By Mumford's theorem (Publication Mathematiques, 1961, N. 9) the map g is an isomorphism and therefore both terms in (**) are zeroes and (*) follows from exact sequence of the pair $(N, \partial N)$.

Step 2. We have an isomorphism

$$H_1(\bar{F}_k, \mathbf{C}) = H_1(\tilde{F}_k, \mathbf{C}) \tag{3.1}$$

This follows from the Mayer–Vietories sequence

$$H_2(\tilde{F}_k) \to H_1(\partial N) \to H_1(N) \oplus H_1(\bar{F}_k) \to H_1(\tilde{F}_k) \to 0.$$

Step 3. Let $B = \varphi^{-1}(C \cup L - \mathrm{Sing}(C \cup L))$ where $\varphi: \bar{F}_k \to \mathbf{P}^2$ is the standard projection. Let $T(B)$ denote the tubular neighborhood of B in $\bar{F}_k$ and $\partial T(B)$ be the part of the boundary of $T(B)$ which is outside of ∂F_k. Let α be the number of connected components of B. If $t_1 \ldots t_c$ are the intersection indices of L and C then $\alpha = 1 + g.c.d.(t_1 \ldots t_c, k)$ and

$$H_i(\bar{F}_k, \bar{F}_k - T(B), \mathbf{C}) = \begin{cases} 0 & i = 1 \\ \mathbf{C}^\alpha & i = 2 \end{cases} \tag{3.2}$$

Indeed by excision theorem and Thom isomorphism, we have

$$H_i(\bar{F}_k, \bar{F}_k - T(B), \mathbf{C}) = H_i(T(B), \partial T(B), \mathbf{C}) = H_{i-2}(B, \mathbf{C}) \tag{3.3}$$

and (3.2) follows. To establish the formula for α we shall note that $L - (C \cap L)$ represents a sphere punctured in c points. If $\tau_1 \ldots \tau_c$ is a system of generators of $\pi_1(L - C \cap L, p_0)$ consisting of loops which circle exactly one point from $C \cap L$ then π_i acts on the set $\varphi^{-1}(p_0) = \{P_1 \ldots P_{g.c.d.(n,k)}\}$ by the formula $P_s \to P_{s+t_i}$ where we identify P_s and $P_{s+g.c.d.(n,k)}$ for any s. Because the connected components of $\varphi^{-1}(L - C \cap L)$ correspond to the orbits of this action of $\pi_1(L - C \cap L_1 p_0)$ on $\varphi^{-1}(p_0)$ the result follows.

Step 4. By our assumption $\alpha = 2$, i.e., $\varphi^{-1}(B)$ is irreducible. The cyclic group $\mathbf{Z}/k\mathbf{Z}$ acts trivially on $H_2(\bar{F}_k, \bar{F}_k - T(B), \mathbf{C})$. This implies that if $H_2(\bar{F}_k, \mathbf{C}) = \oplus T_{\omega^j}$ is the decomposition of $H_2(\bar{F}_k, \mathbf{C})$ into eigenspaces of the generator of $\mathbf{Z}/k\mathbf{Z}$ then the image of T_{ω^j} in $H_2(\bar{F}_k, \mathbf{C})$ is zero if $j \neq 0$. The eigenspace corresponding to $j = 0$ is the invariant subgroup of $H_2(\bar{F}_k, \mathbf{C})$ and can be identified with $H_2(\bar{F}_k/\mathbf{Z}/k, \mathbf{C}) = H_2(\mathbf{P}^2, \mathbf{C}) = \mathbf{C}$. Clearly this invariant part survives in

$H_2(\bar{F}_k, \bar{F}_k - T(B), \mathbf{C})$. Therefore from exact sequence

$$\begin{array}{c} H_2(\bar{F}_k, \mathbf{C}) \to H_2(\bar{F}_k, \bar{F}_k - T(B), \mathbf{C}) \to H_1(\bar{F}_k - T(B), \mathbf{C}) \\ \wr\wr \\ H_2(B, \mathbf{C}) \\ \to H_1(\bar{F}_k, \mathbf{C}) \to H_1(\bar{F}_k, \bar{F}_k - T(B)) \end{array}$$

and from formula (3.2) and the Step 2 the lemma follows in this case.

For the complement $\mathbf{P}^2 - (C \cup L)$ we can define the Alexander polynomial relative to the natural homomorphism $\pi_1(\mathbf{P}^2 - C \cup L) \to H_1(\mathbf{P}^2 - (C \cup L), \mathbf{Z}) = \mathbf{Z}$ (cf. (1.3)). Combining this lemma with the results mentioned in the end of the previous section we obtain the following.

COROLLARY 3.2. *Under the assumptions of the Lemma* 3.1, *the first Betti number of a cyclic multiple plane is equal to* $\sum_{i=1}^{\alpha} \alpha_i^k$ *where* α_i^k *is the number of common roots of* $t^k - 1$ *and ith Alexander factor of* $\mathbf{P}^2 - (C \cup L)$ *defined relative to the homomorphism* $\pi_1(\mathbf{P}^2 - (C \cup L)) \to H_1(\mathbf{P}^2 - (C \cup L), \mathbf{Z}) = \mathbf{Z}$. *In particular, lar, if* $(t^k - 1)$ *is relatively prime to the Alexander polynomial of* $\mathbf{P}^2 - (C \cup L)$ *then the first Betti number of the kth cyclic multiple plane is vanishing.*

4. Proof of Theorem 1. We start by introducing some notation. Let $n : \tilde{C} \to C$ denote the normalization of the curve C. By slight abuse of language we also denote by n, the composition of n with the natural inclusion $C \hookrightarrow \mathbf{P}^2$. Let $\operatorname{Sing} C$ be the set of singular points of C in which we also include the intersection points of C with the line L in infinity. Let $\widetilde{\operatorname{Sing}} C \subset \tilde{C}$ be $n^{-1}(\operatorname{Sing} C)$ and let C^* denote C with small open disks about points from $\widetilde{\operatorname{Sing}} C$ removed.

We apply now Fulton's connectedness theorem [7] to the map

$$(\mathbf{P}^2 - (C \cup L)) \times \tilde{C} \to \mathbf{P}^2 \times \mathbf{P}^2 \tag{4.1}$$

defined as restriction of the map $\delta : \mathbf{P}^2 \times \tilde{C} \to \mathbf{P}^2 \times \mathbf{P}^2$, (cf. [3]) which is the identity on the first component and which is the normalization map defined above on the second component. Let V_ϵ denote the ϵ-neighborhood of the diagonal Δ in $\mathbf{P}^2 \times \mathbf{P}^2$ then by Theorem 1.6 from [3] (a strengthened version of it appearing as Conjecture 1.3 in [3] is proven now by Goresky and McPherson) we have

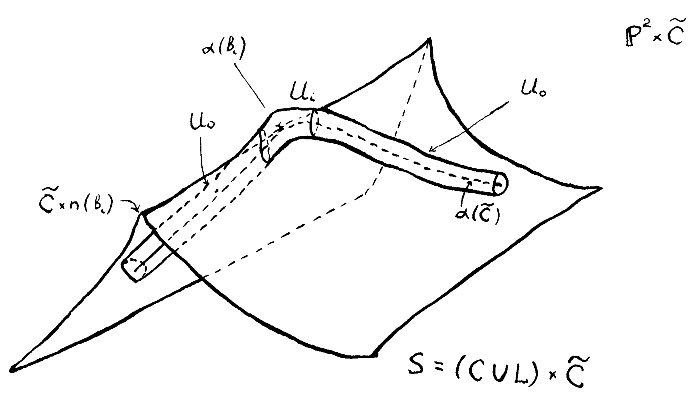
$$\pi_1(\delta^{-1}(V_\epsilon)) \twoheadrightarrow \pi_1((\mathbf{P}^2 - (C \cup L)) \times \tilde{C}) \twoheadrightarrow \pi_1(\mathbf{P}^2 - (C \cup L)) \tag{4.2}$$

We denote by S the image of $(C \cup L) \times \tilde{C}$ in $\mathbf{P}^2 \times \tilde{C}$ and by α the mapping $\tilde{C} \to \mathbf{P}^2 \times \tilde{C}$ defined by

$$\alpha(x) = (n(x), x)$$

Clearly $\alpha(C)=\delta^{-1}(\Delta)$. Therefore $\delta^{-1}(V_\epsilon)$ can be identified with the complement to S in tubular neighborhood $T(\alpha(\tilde{C}))$ of the curve $\alpha(\tilde{C})$ in $P^2\times\tilde{C}$:

$$\delta^{-1}(V_\epsilon)=T(\alpha(\tilde{C}))-S \tag{4.3}$$



The tubular neighborhood $T(\alpha(C))$ has a natural decomposition

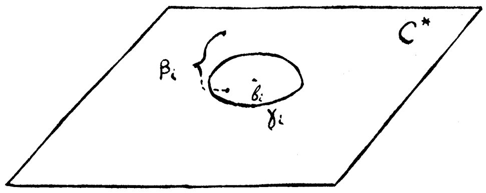
$$T(\alpha(\tilde{C}))=U_0\cup U_1\cup\cdots\cup U_N \tag{4.4}$$

where U_0 is D^4-disk bundle over C^* and U_i's are the 6-balls in $\mathsf{P}^2\times\tilde{C}$ about intersection points of $\alpha(C)$ and the singular set of $S=(\operatorname{Sing}C)\times\tilde{C}$. U_i's correspond to the branches B_i of singularities of the curve C.

Firstly we claim that U_0-S is equivalent to S^1-fibration over C^*. Indeed one can choose the local coordinates $(Z_1Z_2Z_3)$ near points $\alpha(x)$ for $x\in C^*$ in such a way that S (which is non-singular at $\alpha(X)$) is defined by the equation $Z_1=0$, and $\alpha(C)$ by the equations $Z_1=Z_2=0$. Hence the fibre of $T(\alpha(C))-S$ over x is equivalent to disk in C^2 with a deleted coordinate line which is homotopy equivalent to S^1.

Secondly U_i-S has the homotopy type of the complement to the link of singularity of the branch B_i of C to which corresponds the ball U_i. Indeed we can choose the local coordinates near $\alpha(b_i)$ (where b_i denotes the singular point of branch B_i) in such a way that U_i is given by $|Z_1|^2+|Z_2|^2+|Z_3|^2\leqslant\epsilon$ and $\tilde{C}\times n(b_i)$ given by $Z_2=Z_3=0$. Then the projection on (Z_2,Z_3)-plane restricted on U_i-S is D^2-fibration over the complement D^4-C where D^4 is a small ball on P^2 about the point $n(b)$. D^4-C has the homotopy type of the complement of the link of the singularity of B_i. At the singular points which are in the intersection will the line in infinity one obtains that U_i-S has the homotopy type of the complement of the link of the singularity of $C\cup L$.

Thirdly the intersection of U_i-S and U_0-S has the homotopy type of $S^1\times S^1$ where one generator of this torus corresponds to the loop γ_i in C^* which is the boundary of the disk in C about the point b_i and the second generator β_i corresponds to the fibre of fibration $T(\alpha(\tilde{C})) - S \to C^*$. The image of γ in $\pi_1(U_i - S)$ is zero and β_i linked with S with linking coefficient 1.



Let φ denote the homomorphism $\varphi : \pi_1(T(\alpha(\tilde{C})) - S) \to \mathbf{Z}$ which relates to any loop in $\pi_1(T(\alpha(\tilde{C})) - S)$ the linking number of it with S. Then the composition maps

$$\begin{aligned} \varphi_0 &: \pi_1(U_0 - S) \to \pi_1\big(T\big(\alpha(\bar{C})\big) - S\big) \to \mathbf{Z} \\ \varphi_i &: \pi_1(U_i - S) \to \pi_1\big(T\big(\alpha(\tilde{C})\big) - S\big) \to \mathbf{Z} \\ \varphi_{0\,i} &: \pi_1(U_0 \cap U_i - S) \to \pi_1\big(T\big(\alpha(\tilde{C})\big) - S\big) \to \mathbf{Z} \end{aligned} \tag{4.5}$$

are onto. Therefore we can apply N times the Proposition 2.2 and obtain that $\Delta_\varphi(\pi_1(T(\alpha(\tilde{C})) - S)$ divides

$$\Delta_{\varphi_0}(\pi_1(U_0 - S)) \prod_{i=1}^{N} \Delta_{\varphi_i}(\pi_1(U_i - S))$$

Note that $\Delta_{l_{0,i}}(\pi_1(U_0 \cap U_i - S))$ is well defined. In fact, $\widetilde{U_0 \cap U_i - S}$ is homotopy equivalent to S^1 with trivial action of t on $H_1(\widetilde{U_0 \cap U_i - S}, \mathbf{C})$. Hence Alexander polynomial of $\pi_1(U_0 \cap U_i - S)$ relative to $\varphi_{0,i}$ is $t - 1$.

Now $\Delta_{\varphi_i}(\pi_1(U_i - S))$ for $i = 1 \ldots N$, can be identified with the Alexander polynomial of the singularity of the branch B_i. For singular points which are intersection points of C and L, the local Alexander polynomial should be understood in the following sense.

Definition. The local Alexander polynomial of branch B of a singular point on the line L is the characteristic polynomial of monodromy operator of singularity $C \cup L$ to which B belongs.

The Λ-module $H_1(\widetilde{U_0 - S}, \mathbf{C})$ is isomorphic to

$$\underbrace{\Lambda/(t-1) \oplus \cdots \oplus \Lambda/(t-1)}_{2g + N - 1 \text{ times}} \tag{4.6}$$

where g is genus of C because $\widetilde{U_0 - S}$ has homology of surface of genus g with

N-holes and action of t on homologies is trivial. Therefore $\Delta_\varphi(\pi_1(T(\alpha(\tilde{C})) - S))$ divides

$$\Pi\Delta_i(t-1)^{2g+N-1} \tag{4.7}$$

where Δ_i runs through all local Alexander polynomials.

Now because of (4.2) the Proposition 2.1 implies that the Alexander polynomial of $\pi_1(\mathsf{P}^2 - (C \cup L))$ divides the polynomial (4.7). By (1.2), $H_1(\mathsf{P}^2 - (C \cup L)\mathsf{Z}) = \mathsf{Z}$ hence as noted in the Section 2, $\Delta(\pi_1(\mathsf{P}^2 - (C \cup L))$ can be normalized so that it will satisfy (2.2). Therefore $\Delta(\pi_1(\mathsf{P}^2 - (C \cup L)))$ in fact divides $\Pi_{i=1}^N \Delta_i$ Theorem 1 is proven.

Remark. The Alexander polynomial can be defined in a similar way for immersions of surfaces in CP^2. However, Theorem 1 is not valid any more for non-algebraic immersions. In fact, the Alexander polynomial fails to be even cyclotomic. For given any immersion, say an algebraic one, by taking a knotted $S^2 \hookrightarrow S^4$ with non-cyclotomic Alexander polynomial (by [10] any polynomial f with $f(1) = 1$ can be used) and forming the connected sum (P^2, C) (S^4, S^2) one obtains a new immersion of C with Alexander polynomial $\Delta_C \Delta_{S^2}$. In fact any polynomial f with $f(1) = 1$ can be the Alexander polynomial of an embedded surface. It is easy to see that the Alexander polynomial of $C \hookrightarrow \mathsf{P}^2$ coincides with the Alexander polynomial of a certain immersion $C \hookrightarrow S^4$.

5. Alexander polynomial relative to the line at infinity. Now we turn to the proof of Theorem 2. Let A^2 denote the complement of a small tubular neighborhood of the line L at infinity. Note that $\pi_1(\mathsf{P}^2 - (C \cup L)) = \pi_1(\mathsf{A}^2 - C)$. We are going to prove that the Alexander polynomial of $\pi_1(\mathsf{A}^2 - C)$ divides the Alexander polynomial of $\pi_1(\partial \mathsf{A}^2 - C)$.

LEMMA 5.1. *There exists a differentiable function f on A^2 which has only one minimum, such that the restriction f to C does not have maxima inside A^2 and all critical values of it are distinct.*

Note that by critical values we also mean the values at singular points of C. Without loss of generality we may assume that $(0,0) \notin C$.

Proof. Let $F(z_1, z_2) = \max(|z_1|, |z_2|)$. Clearly F has a unique minimum on A^2. F does not have maxima inside A^2. Indeed neither of the functions $|z_1|$ or $|z_2|$ has maxima on C by the maximum modulus theorem. F coincides with $|z_1|$ or $|z_2|$ at the points where $|z_1| \neq |z_2|$. But the maximum of two functions, neither of which has a maximum, can not have a maximum.

Now a small perturbation of F makes it differentiable and satisfies all conditions of the lemma.

It follows from the lemma that all level varieties of f, i.e., $\{(z_1 z_2) \in \mathsf{A}^2 \mid f(z_1, z_2) = a\}$ are spheres which we denote by S_a^3. Let G_A denote the fundamental group of the link $S_a^3 \cap C$ in S_a^3. We may assume that $\partial \mathsf{A}^2 = S_A^3$ for large A.

LEMMA 5.2. *If $f|_C$ does not have critical value bigger than A, then*

$$G_A \twoheadrightarrow \pi_1(\mathbf{P}^2 - (C \cup L))$$

Proof. The idea is similar to one usually used in the theory of knotted $S^2 \hookrightarrow S^4$ (cf [10]). Let $0 < a_1 < \cdots < a_i < A$ be real numbers such that $f|_C$ has only one critical value on each interval $[a_i, a_{i+1}]$. First observe that

$$\begin{tikzcd}[row sep=small] G_{a_i} \arrow[dr, two heads] & \\ & \pi_1(f^{-1}[a_i, a_{i-1}] - C) \\ G_{a_{i-1}} \arrow[ur, two heads] & \end{tikzcd} \tag{$*$}$$

Indeed if c is critical value, then in Wirtinger presentation [4], one has

$$G_{a_i} = \pi_1(f^{-1}(c) - C)/r_1, \ldots, r_{k_r}$$

and

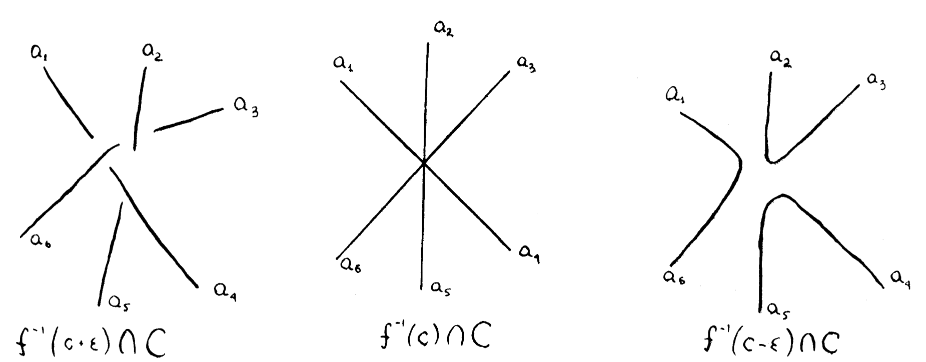
$$G_{a_{i-1}} = \pi_1(f^{-1}(c) - C)/s_1, \ldots, s_{k_s}$$

Then standard argument using the van Kampen theorem shows that

$$\pi_1(f^{-1}[a_i, a_{i-1}] - C) = \pi_1(f^{-1}(c) - C)/r_1, \ldots, r_{k_r}, s_1, \ldots, s_{k_s}$$

This proves ($*$).

As an example of this step consider the case when $f^{-1}(c + \epsilon, c - \epsilon)$ contains one cuspidal point (i.e., locally defined by $x^2 = y^3$) Here the Wirtinger presentation for $f^{-1}(c + \epsilon) \cap C$ has the form



$$\{\ldots a_1, a_2, a_3, a_4, a_5, a_6, \ldots\}/R(a_1, a_6, a_4) = 1,$$

$$R(a_2, a_6, a_3) = 1, \qquad R(a_4, a_2, a_5) = 1$$

where R is the standard relation corresponding to an overcrossing, for example, $a_6a_1a_6^{-1}a_4^{-1} = 1$.

For $S^3_{c_1} \cap C$ there are no relations among $a_1, a_2, a_3, a_4, a_5, a_6$.

For $S^3_{c-\epsilon} \cap C$ one obtains a presentation

$$\{\ldots a_1, a_2, a_3, a_4, a_5, a_6, \ldots\}/a_1 = a_6,\ a_2 = a_3,\ a_4 = a_5$$

Finally $\pi_1(f^{-1}(c-\epsilon, c+\epsilon) \cap C)$ has the presentation

$$\{\ldots a_1, a_2, a_3, a_4, a_5, a_6, \ldots\}/a_1 = a_6,\ a_2 = a_3,\ a_4 = a_5,$$
$$R(a_1, a_6, a_4) = R(a_2, a_6, a_3) = R(a_4, a_2, a_5) = 1.$$

Now the lemma can be deduced by repeated use of van Kampen's theorem. We have

$$\pi_1\big(f^{-1}(A, a_{n-1}) - C\big) = \pi_1\big(f^{-1}(A, a_n) - C\big) \underset{\pi_1(f^{-1}(a_n) - C)}{*} \pi_1\big(f^{-1}(a_n, a_{n-1}) - C\big)$$

Because $\pi_1(f^{-1}(a_n) - C) \twoheadrightarrow \pi_1(f^{-1}(a_n, a_{n-1}) - C)$ it follows that $\pi_1(f^{-1}(A, a_n) - C) \twoheadrightarrow \pi_1(f^{-1}(A, a_{n-1}) - C)$ and hence $\pi_1(f^{-1}(A) - C) \twoheadrightarrow \pi_1(f^{-1}(A, a_{n-1}) - C)$. Arguing in the same vein we obtain the conclusion of the lemma.

Theorem 2 follows from the Lemma 2 and Proposition 2.1.

6. Exponents and the proof of Theorem 3. In this section we compute the exponents of singularities and lines in important special cases and deduce Theorem 3.

Definition. The set of exponents of singularity is the set of highest orders of primitive roots of unity which are roots of the local Alexander polynomial of singularity. The set of exponents of the line relative to the curve C is the set of the orders of primitive roots of unity which are roots of Alexander polynomial of this line relative to the curve C.

Examples. (1) The exponent of a node is 1. Indeed the Alexander polynomial of $x^2 = y^2$ is $(t-1)$.

(2) The exponent of a cusp is 6. Indeed the Alexander polynomial of singularity $x^2 = y^3$ is $t^2 - t + 1$ ([12]).

(3) Exponent of the line L transversely intersecting curve C of degree n is equal to n.

Proof. It is easy to see that link $\partial T(c) \cap C$ is the link of singularity $z_1^n = z_2^n$. (Blow down line L using $\bar{\sigma}$-process.) Then the characteristic polynomial of monodromy operator is equal to (cf. [12]).

$$\prod_{\substack{\omega_i^r=1\\ \omega_j^r=1\\ \omega_i\neq 1\\ \omega_j\neq 1}} (t-\omega_i\omega_j) = (t^n-1)^{n-2}(t-1)$$

The highest primitive root of unity has order n.

(4) Let C given by $y^2u = x^3$. Then the Alexander polynomial of the line $u = 0$ relative to C is $t^2 - t + 1$. Hence the exponent of line in infinity is 6.

(5) The exponents of unibranched singularity can be effectively found from the Puiseux expansion.

Indeed there is the classical algorithm for finding the Alexander polynomial using Puiseux expansion (see [18]). Let

$$y = a_1 x^{m_1/n_1} + a_2 x^{m_2/n_1n_2} + \cdots + a_s x^{m_s/n_1\cdots n_s} \tag{6.1}$$

where $(m_i n_i = 1,\ 1 \leqslant i \leqslant s,\ m_1 > n_1,\ m_2 > n_2 m_1, \ldots,\ m_s > n_s m_{s-1}$ and let w_i, $1 \leqslant i \leqslant s$ defined by

$$\begin{aligned} w_1 &= m_1 \\ w_i &= m_i - m_{i-1}n_i + w_{i-1}n_{i-1}n_i \end{aligned} \tag{6.2}$$

Then the link L of singularity of this branch is the iterated torus knot of type $\{K_1, \ldots, K_s\}$, where K_i is torus knot of type $\{w_i, n_i\}$. This means that L constructed as follows. One starts from the knot K_1. Then the knot $\{K_1, K_2\}$ is the knot on the boundary of a small tubular neighborhood of K_1 and winding w_2-times over meridian of K_1 and n_2 times over longitude. Next iterations one performs similarly. The Alexander polynomial of iterated torus knots can be found using the following.

LEMMA 6.1. ([16], [18]). *Let L and K be knots. Let L', obtained by iteration about L via K, and let γ denote the longitude winding number of K. Then*

$$\Delta_{L'}(t) = \Delta_L(t^\gamma)\Delta_K(t)$$

Because the roots of the Alexander polynomial of torus knot of type (m, n) (which is the link of singularity $z_1^m = z_2^w$) are roots of degree mn, one obtains the following.

COROLLARY 6.2. *The exponents of the branch given by* (6.1) *are*

$$(w_1 n_1 \ldots n_s, w_2 n_2 \ldots n_s, \ldots, w_s n_s).$$

Now we proceed to the proof of the Theorem 3.

If neither of the numbers $g.c.d.(k, \delta_\infty^{d_i})$ is divisible by one of $\epsilon_1^1 \ldots \epsilon_1^{\alpha_1}, \ldots, \epsilon_N^1 \ldots \epsilon_N^{\alpha_N}$ then neither of the roots of $\Delta_1, \ldots, \Delta_N$ can be a root of Δ_∞

and $t^k - 1$. Hence $g.c.d.(\Delta_1 \ldots \Delta_N, \Delta_\infty)$ is relatively prime to $t^k - 1$. Therefore by the Theorems 1 and 2 the global Alexander polynomial of C is relatively prime to $(t^k - 1)$ and Theorem 3 follows from the Corollary 3.2.

COROLLARY 6.3. *If neither of $\epsilon_\infty^{\alpha_i}$ is divisible by one of the local exponents then Alexander polynomial of the curve is equal to* 1.

Proof. The condition of the corollary implies that Δ_∞ and $\Delta_1 \ldots \Delta_N$ are relatively prime.

Note finally that Zariski's theorem (from Introduction) follows from Theorem 3 because the exponents of cusps are equal 6 (see example 2) and exponent of transversal line is equal n (example 3).

7. Examples. We conclude with several examples of computation of the Alexander polynomial which are based on the known results of computations of the fundamental group.

Recall first the connection between $\pi_1(\mathrm{P}^2 - C)$ and $\pi_1(\mathrm{P}^2 - (C \cup L))$. According to the van Kampen theorem, $\pi_1(\mathrm{P}^2 - C)$ has the system of generators $g_1, \ldots, g_n$ which are the loops on straight line l in general position with respect to C surrounding points $C \cap l$. The base point of the fundamental group defines the pencil of lines passing through this point. A complete system of relators can be obtained by

(a) moving the line l in this pencil around the lines tanget to C and passing through the singular points of C.

(b) Adjoining the relator $g_1 \ldots g_n = 1$ ([19], [23]). If L and C are in general position then, the group $\pi_1(\mathrm{P}^2 - (C \cup L)$ is the central extension of $\pi_1(\mathrm{P}^2 - C)$ by Z and has presentation with generators $g_1, \ldots, g_n, \gamma$ and relators:

$$\begin{array}{ll} \text{(A)} & \text{Relators (a)} \\ \text{(B)} & g_i\gamma = \gamma g_i \\ \text{(C)} & g_1 g_2 \ldots g_n \gamma = 1 \end{array} \tag{7.1}$$

(see [21]).

Example 1. Let C be a curve of degree 6 with 6 cusps on conic. Such a curve can be obtained as either branching curve of generic projection of a non-singular cubic surface in P^3 [19], or as dual to rational nodal quartic* ([22].) Then $\pi_1(\mathrm{P}^2 - C)$ has the presentation by generators and $u = g_1 g_2 g$, $v = g_1 g_2$, where g_1, g_2 are van Kampen generators and relators $u^2 = 1$, $v^3 = 1$. Therefore the group $\pi_1(\mathrm{P}^2 - (C \cup L))$ can be defined by generators u, v, γ and relators

$$1)\ u^2 = v^3 \qquad 2)\ \gamma v^3 = 1 \qquad 3)\ u\gamma = \gamma u \qquad 4)\ \gamma v = v\gamma \tag{7.2}$$

Clearly this group is isomorpic to the fundamental group of the trefoil knot which has the Alexander polynomial

$$\Delta = t^2 - t + 1 \tag{7.3}$$

*In this case C also has as singularities 4 nodes.

This polynomial is hence the Alexander polynomial of 6-cuspidal sextic with cusps on quadric. The Corollary 3.2 allows in this case to find the values of the first Betti number of multiple planes. We obtain

$$rkH_1(\tilde{\mathsf{F}}_k,\mathsf{C}) = \begin{cases} 0 & k \not\equiv 0\ (6) \\ 2 & k \equiv 0\ (6) \end{cases} \tag{7.4}$$

This of course agrees with Zariski's result which gives the first Betti number in terms of superabundance of the linear system of quadrics passing through the cusps of the curve.

Example 2. The Alexander polynomial of $x^2u = y^3$ relative to the line in infinity $u = 0$ is equal to $1 - t + t^2$ because the complement to this curve is a retract of the complement of the trefoil knot in S^3. The first Betti number is given by the formula (7.5). Similarity for the curve $x^2u = y^5$ one obtains

$$\begin{gathered} \Delta_c = t^4 - t^3 + t^2 - t + 1 \\ rkH_1(F_k,\mathsf{C}) = \begin{cases} 0 & k \neq 0\ (10) \\ 4 & k \equiv 0\ (10) \end{cases} \end{gathered} \tag{7.5}$$

Example 3. M. Oka [15] constructed the curve $C_{p,q}$ (p,q-relatively prime) with pq singularities locally defined by

$$x^p + y^q = 0$$

such that $\pi_1(\mathsf{P}^2 - C_{p,q}) = \mathsf{Z}_p * \mathsf{Z}_q$. In fact this curve is given by equation

$$(X^p + Y^p)^q + (Y^q + U^q)^p = 0 \tag{7.6}$$

The presentation for $\pi_1(\mathsf{P}^2 - (C_{p,q} \cup L))$ can be given with generators u, v, y and relators

$$1)\ u^p = v^q \qquad 2)\ \gamma v^q = 1 \qquad 3)\ \gamma u = u\gamma \qquad 4)\ \gamma v = v\gamma \tag{7.7}$$

Clearly this group is isomorphic to the fundamental group of the torus knot of the type (p,q). Therefore the Alexander polynomial of $C_{p,q}$ relative to the line in infinity which is in general position with respect to $C_{p,q}$ given by (7.6) equals to

$$\Delta_{C_{p,q}} = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)} \tag{7.8}$$

Again the Corollary 3.2 yields

$$rkH_1(\tilde{F}_k,\mathsf{C}) = \begin{cases} 0 & \text{if } k \not\equiv 0(pq) \\ (p-1)(q-1) & \text{if } k \equiv 0(pq) \end{cases}$$

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