Remarks on moduli spaces
of complete intersections

A. S. Libgober and J. W. Wood *

Introduction.

In this paper we give an algorithm to compute the dimensions of the space of moduli of a complete intersection. This extends the treatment of hypersurfaces given by Kodaira and Spencer [14]. One consequence, explained in §4, is that we can find diffeomorphic complete intersections of dimension three (and homeomorphic ones of dimension two) which lie in different dimensional components of the moduli space.

Our starting point is the results of E. Sernesi who showed that the family \( V \) of complete intersections of fixed multidegree \( d \) whose defining polynomials have coefficients close to those of a given variety \( V_n \), \( n \geq 2 \), is a complete complex analytic family of deformations of \( V \) except in the case of K-3 surfaces; \( n = 2 \) and \( d = (4), (3,2), \) or \( (2,2,2) \). It follows that sufficiently small deformation of \( V \) is again a global complete intersection of the same multidegree. We let

* Mathematics Subject Classification. Primary 14J15, 14M10, 20F, 32G13.

The authors were partially supported by the National Science Foundation.

© 1986 American Mathematical Society

0271-4132/86 $1.00 + $.25 per page
$\mathcal{V}_n(d_1, \ldots, d_r)$, or simply $\mathcal{V}_n$, denote a complete intersection of codimension $r$ in $\mathbb{P}^{n+r}$ with multidegree $d = (d_1, \ldots, d_r)$ and $d_i \geq 2$.

Proposition 1 (Sernesi). $H^1(\mathcal{V}_n, T_{\mathbb{P}^{n+r}}|\mathcal{V}_n) = 0$ for $n \geq 2$ except for the $K$-3 surfaces.

Sernesi used this in proving that the space of infinitesimal deformations of $\mathcal{V}$ is all of $H^1(\mathcal{V}, T_{\mathcal{V}})$. It follows that the dimension, denoted $m(\mathcal{V})$, of any effective, complete family of deformations is $\dim H^1(\mathcal{V}, T_{\mathcal{V}})$, [8, §§6 and 11].

To compute $H^1(\mathcal{V}, T_{\mathcal{V}})$ we use the exact cohomology sequence corresponding to the exact sequence of sheaves [3, p.182]

$$0 \rightarrow T_{\mathcal{V}_n} \rightarrow T_{\mathbb{P}^{n+r}}|\mathcal{V} \rightarrow N \rightarrow 0$$

where $N$ is the normal sheaf of $\mathcal{V}_n$ in $\mathbb{P}^{n+r}$. In §3 we prove

Proposition 2. For any complete intersection except for the quadratic hypersurface

$$H^0(\mathcal{V}, T_{\mathcal{V}}) = 0 \quad d \neq 2.$$  

In general if $\mathcal{V}$ is a compact complex analytic manifold and $H^0(\mathcal{V}, T_{\mathcal{V}}) = 0$, the Kuranishi space is a local space of moduli for $\mathcal{V}$. In our case, by Sernesi's result, its dimension is $m(\mathcal{V}) = \dim H^1(\mathcal{V}, T_{\mathcal{V}})$, cf. [16, Chapter 2], [17, Theorem 1.1]. It also follows from $H^0(\mathcal{V}, T_{\mathcal{V}}) = 0$ that $\mathcal{V}$ admits no continuous group of analytic automorphisms. Under the stronger assumption that $\mathcal{V}$ has ample canonical bundle, Kobayashi showed [9] that $\text{Aut } \mathcal{V}$ is finite and Narasimhan and Simha [12] showed the existence of a global moduli space. They show the set of isomorphism classes of complex structures on $\mathcal{V}$ with ample canonical bundle has a natural structure.
intersection of structure as a Hausdorff complex space given in a neighborhood of

\[ \cdots, d_r \] and particular structure \( V_t \) by the Kuranishi space of \( V_t \) modulo

\[ V_t. \]

or \( n \geq 2 \) except that \( \text{Aut } V \) is finite.

As a consequence of Proposition 2 we can extend Kobayashi's result in the case of complete intersections.

\textbf{Corollary.} If \( V \) is a complete intersection of dimension \( \geq 2 \) and

\( 1 \) for a K-3 surface or a quadratic hypersurface, then \( \text{Aut } V \) is

\( \text{finite.} \)

\textbf{Proof.} Let \( G = \text{Aut } P_{n+r} = \text{PGL}(n+r+1) \). The stabilizer \( N_G(V) = \text{G} : G \mid gV \subseteq V \) is an algebraic group, cf. [1, p.97]. The

\text{Proof of Theorem 8.2 in [10, I p.479] shows that except for K-3}

\( \rightarrow 0 \)

\( \text{surfaces, } \text{Aut } V = N_G(V). \) But Proposition 2 implies that \( \text{Aut } V \) is

\( \text{finite dimensional for degree } > 2. \) The Corollary follows.

\( \text{For a K-3 surface } \text{Aut } V \text{ may be infinite; in [15, p.288]}

\( \text{Ser} \) \text{gave an example of a surface of degree } 4 \text{ with infinite}

\( \text{Aut } V. \)

\( \text{In [11] Matsumura and Monsky give a more}

\( \text{algebraic proof, which holds also in nonzero characteristic, that}

\( \text{a K-3 manifold and } \text{Aut } V \text{ is finite for hypersurfaces (with the same exceptions.)}

\text{We also prove that for a generic hypersurface } V, \text{Aut } V = 0.

\text{In §2 we present a formula for } m(V). \) The proof of

\( \text{Proposition 2 is given in §3. Finally §4 contains examples in}

\text{which \( \text{Aut } V \) is finite and different classes of }

\text{components of different dimensions and a further survey of results on components.} \)
2. Computing \( m(V) \).

We will need the following facts about the cohomology of line bundles on \( V \). For \( d = (d_1, \ldots, d_r) \) and for any \( s \leq r \) define
\[
q(t, n, s, d) = \dim H^0(V_n(d_1, \ldots, d_s), O_V(t)).
\]

Lemma 1. Let \( V_n \) be a complete intersection of multidegree \( d = (d_1, \ldots, d_r) \) with \( d_1 \geq 2 \).

(a) \( H^i(V, O(t)) = 0 \) for \( i \neq 0, n \).

(b) The function \( q \) is determined by the recurrence relation
\[
q(t, n, r, d) = \begin{cases} 
0 & \text{if } t < 0 \\
\left( \begin{array}{c} n+1 \\ t \end{array} \right) & \text{if } r = 0 \\
q(t, n+1, r-1, d) - q(t-d_r, n+1, r-1, d) & \text{if } r > 0.
\end{cases}
\]

(c) \( \sum_{t=0}^\infty q(t, n, r, d) t^t = (1-t)^{-n-r-1} \prod_{i=1}^r (1-t^{d_i}) \)

Proof. (a) is found for example as an exercise in [3, p.231].
(c) is proved in [13, p.131]. (b) follows from (c) and conversely. Alternatively (a) and (b) follow by induction on \( r \) from the case of projective space [3, p.225] using the exact cohomology sequence coming from the exact sequence [4, 16.2.1]
\[
0 \longrightarrow O_X(t-d_r) \longrightarrow O_X(t) \longrightarrow O_XW(t) \longrightarrow 0
\]
where \( X = V_{n+1}(d_1, \ldots, d_{r-1}) \) and \( W = V_{n+r-1}(d_r) \).

We have emphasized the recurrence relation (b) because it is useful for computation. There is another interpretation of the function \( q \) which may help to clarify its behavior. Writing the power series (c) as
\[
(1+t+t^2+ \ldots)^{n+1} \prod_{i=1}^r (1+t+\ldots+t^{d_i-1}),
\]
we see that the coefficient of \( t^t \) is equal to the number of
negative integer solutions to the equation
\[ x_1 + \ldots + x_{n+r+1} = \ell \text{ with } 0 \leq x_i < d_i \text{ for } i = 1, \ldots, r. \]

The cohomology of \( \mathcal{O}_Y(\ell) \) of course equals the number of integer points inside an obviously defined polyhedron.

If multidegree \( d = (d_1, \ldots, d_r) \) with any \( s \leq r \) defining a \( \mathbb{Q} \)-equivariant symmetric polynomial. On the other hand if

\[ \ell > \Sigma(d_i + 1) - n - 1, \]

the recurrence relation given by the Kodaira vanishing theorem [4, 18.2.2]

\[ q(\ell, n, r, d) = x(V, \mathcal{O}(\ell)) \]

which by the Riemann-Roch formula is a polynomial in \( \ell + \frac{1}{2} c_1(V) \)

\[ n+1, r-1, d \] \( r > 0 \)

and the Pontryagin classes of \( V \) [4, p.150] and hence is polynomial in \( \ell \), the total degree, \( \Sigma d_i \), and the first \( n \) elementary symmetric functions of \( (d_1, \ldots, d_r) \).

Corollary (c) and Theorem. If \( V_n \) is a complete intersection of multidegree

\[ (d_1, \ldots, d_r) \] \( d_i \geq 2 \) and with \( n \) \( \geq 2 \) and not a K3 surface or a quadric hypersurface, then the number of moduli

\[ m(V) = 1 - (n + r + 1)^2 + \sum_i \epsilon_i q(d_i, n, r, d). \]

\( \ell \rightarrow 0 \)

Proof. We first find \( \dim H^0(V, T_{P_{n+r}}|V) \). Tensoring the exact sequence [3, p.182]

\[ 0 \rightarrow \mathcal{O}_{P_{n+r}} \rightarrow \mathcal{O}_{P_{n+r}}(1)^{n+r+1} \rightarrow T_{P_{n+r}} \rightarrow 0 \]

\( \mathbb{Q} \)-interpretation of the sheaf \( \mathcal{O}_V \), since \( T_P \) is flat we obtain

\[ 0 \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_V(1)^{n+r+1} \rightarrow T_{P_{n+r}}|V \rightarrow 0 \]

Writing the corresponding cohomology sequence gives

\[ 0 \rightarrow H^0(V, \mathcal{O}_V) \rightarrow H^0(V, \mathcal{O}_V(1))^{n+r+1} \rightarrow \]

\[ \ldots \]

the number of

\[ H^0(V, T_{P_{n+r}}|V) \rightarrow 0 \]
Because $V$ is not contained in any proper linear subspace of $\mathbb{P}^n$, we have $\dim H^0(V, O_V(1)) = n + r + 1$, in agreement with Lemma 1. Hence $\dim H^0(V, T_{\mathbb{P}^n+r}|_V) = (n+r+1)^2 - 1$.

In view of Propositions 1 and 2 the cohomology sequence corresponding to the sequence (1) becomes

(2) \[ 0 \rightarrow H^0(V, T_{\mathbb{P}^n+r}|_V) \rightarrow H^0(V, N) \rightarrow H^1(V, T_V) \rightarrow 0 \]

Since the normal sheaf $N$ is isomorphic to $O_V(d_1) + \ldots + O_V(d_r)$, the theorem follows from Lemma 1.

A table of values of $m(V)$ in cases of low degree is given at the end of §4.

For a hypersurface of degree $d$ we have

\[ q(d, n, 1, d) = \frac{(n+1)d}{d} - 1 \]

by Lemma 1 and so we obtain the formula

\[ m(V) = \left| \frac{n^2d+1}{d} \right| - (n+2)^2 \quad \text{for } d > 2 \]

of Kodaira and Spencer [8, (14.10)].

A quadratic hypersurface $V$ is rigid, $H^1(V, T_V) = 0$ [8, p.406], and our formula gives instead

\[ \dim H^0(V, T_V) = \frac{1}{4}(n+1)(n+2). \]

For the $K3$ surfaces the formula gives 19, the dimension of the image of $\delta^*$ in (2).

As the codimension $r$ increases a closed form expression for $m(V)$ becomes more complicated. For example if $r = 2$

\[ m(V) = -1 - (n+3)^2 + \left| \frac{n^2+2d_1}{d_1} \right| + \left| \frac{n^2+2d_2}{d_2} \right| - \left| \frac{n^2+2d_2-d_1}{d_2-d_1} \right| - \delta_{d_2} \]

assuming $d_2 \geq d_1$. Here $\delta$ is the Kronecker function.
ar subspace of $P_n$.

In this section we compute $H^0(V, T_V)$. By Serre duality this

is isomorphic to $H^n(V, \Omega_V^1(\Sigma(d_i - 1) - n - 1))$. If $V$ has

the canonical bundle, $\Sigma(d_i - 1) - n - 1 > 0$ and this group

vanishes by a criterion of Akizuki and Nakano [8, (11.12)]. We

use all but one of the remaining cases to their result using a

toem of Bott and an idea of Kodaira and Spencer

[Lemma 14.2].

**Theorem 2.** Let $V_n$ be a complete intersection. Then

$$H^q(V, \Omega_{V}^p(k)) = 0 \text{ for } p + q \geq n + 1 \text{ and } k > p - q.$$ 

**Proof.** For $k > 0$ this is the result of Akizuki and Nakano. The

proof is by induction on the codimension $r$ of $V$. For $r = 0$, $V =

2$ and our lemma follows from Bott's result [8, p. 405]. Note

that we may assume $q \geq 1$ since $\Omega^{n+1} = 0$.

$$V_n(T_V) = 0$$

For the inductive step assume $V_n$ is a hypersurface of degree

in the complete intersection $W_{n+1}$ of codimension $r-1$. The pair

of exact sequences

$$0 \rightarrow \Omega_{W}^{p+1}(j) \rightarrow \Omega_{V}^{p+1}(j) \rightarrow 0$$

Rodaira and Spencer (7, (3) and (4)) tensored with $\Omega_{W}(k+j)$ and

the dimension of

form expression for

$$H^q(W, \Omega_{W}^p(k+j)) \rightarrow H^q(W, \Omega_{W}^p(k+j)) \rightarrow H^q(W, \Omega_{W}^p(k+j))$$

for $r = 2$

$$H^q(W, \Omega_{W}^p(k+j)) \rightarrow H^q(V, \Omega_{V}^p(k)) \rightarrow H^{q+1}(W, \Omega_{W}^p(k)).$$

Since $\dim W = n+1$ and $j \geq 2$, the hypotheses are satisfied by the

first and last groups in the first sequence and the last group in

action.
the second sequence. Since \( W \) has lower codimension, the last groups are zero by induction. For large \( k \) (so \( k + j > 0 \)) the first group in the first sequence is zero by the criterion of Akizuki and Nakano. The lemma follows by induction on \(-k\).

Our lemma implies Proposition 2 provided

\[
\Sigma(d_i - 1) - n - 1 > 1 - n
\]

hence for all cases except \( d = (2), (3) \) and, \( (2,2) \). The case \( (3) \) uses a more complete form of Bott's theorem with \( W = P_1 \).

It is covered by [8, Lemma 14.2].

In the case of a complete intersection of two quadrics the vanishing results above are not sufficient. One can use the following alternative argument. We have a commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
H^0(V, O_V) \\
0 \longrightarrow H^0(V, O_V) \longrightarrow H^0(V, O_V(1))^{n+r+1} \longrightarrow H^0(V, T_{P^{n+r+1}} V) \longrightarrow 0 \\
\downarrow \\
H^0(V, O_V(d_1) + \ldots + O_V(d_r)) = H^0(V, N)
\end{array}
\]

The rightmost 0 comes from H&SL. \( (V, O_V) = 0 \) for \( n \geq 2 \) and the map \( \mu \) is given by

\[
(R_0, \ldots, R_{n+r}) \longrightarrow (\Sigma_{i=0}^{n+r} \Sigma_{i=0}^{n+r} \frac{3F_1}{3x_1}, \ldots, \Sigma_{i=0}^{n+r} \Sigma_{i=0}^{n+r} \frac{3F_r}{3x_1})
\]

where

\[
R_i = \Sigma_{i=0}^{n+r} r_ix_j \text{ are linear forms and } F_1 = 0, \ldots, F_r = 0
\]

are the defining equations of \( V_n(d_1, \ldots, d_r) \). The kernel of \( \mu \) is determined by the conditions

\[
\Sigma_{i=0}^{n+r} R_i \frac{3F_k}{3x_1} = \Sigma_{i=1}^{r} k/F_i \text{ for } k = 1, \ldots, r.
\]

For \( d = (2,2) \) we take
nction, the last

\[ \text{ker}(\mu) = 0 \] for \( k+j > 0 \) the \( \text{ker}(\mu) \) is trivial because \( c_i \neq c_j \) for \( i \neq j \). The equations imply \( r_{ij} = 0 \) for \( i \neq j \). Hence \( \text{dim} \ker(\mu) = 1 \). This implies \( H^0(V,T_Y) = 0 \). This completes the proof of Proposition 2.

In the case of a quadric hypersurface the computation of \( H^0(V,T_Y) \) yields

\[ \text{dim} H^0(V,T_Y) = \frac{1}{2}(n+1)(n+2) \]

agreement with the remark in §2.

two quadrics then we can use the nutative diagram

\[ \begin{array}{ccc}
1 & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow \\
\text{R} \rightarrow & \text{T}_{P_{n+1}} & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
\text{RP} & \rightarrow & \text{T}_{P_{n+1}} \\
\downarrow & \downarrow & \downarrow \\
0 & \rightarrow & \text{R} \\
\end{array} \]

Examples and remarks.

[5, 6] Horikawa gave examples which show that the moduli space of algebraic structures on a given smooth 4-manifold can have different components of different dimensions. In [10] the authors showed that in any complex dimension \( > 2 \) and for any positive integer \( k \), there are \( k \) distinct complete intersections, which are all diffeomorphic. In fact they have equivalent underlying almost complex structures. However they lie in different irreducible components of the moduli space of complex structures on the underlying smooth manifold. We obtained the same result in dimension two for structures on a homeomorphism type. Recent work of Catanese [2] provides for any \( k \) a homeomorphism type of real dimension \( 6 \) supporting complex structures lying in components of the moduli space of \( k \) different dimensions. It is natural to expect similar behavior among complete intersections.
For surfaces, the homotopy type is determined by the middle Betti number and the signature and type of the intersection pairing which can be computed from the total degree and the respective two symmetric functions of \( d \). By the work of Mike Freedman the homeomorphism type of these manifolds is determined by the homotopy type. We find the varieties \( V_2(6,6,6,2,2,2,2) \) and \( V_2(8,4,4,3,3,3) \) are homeomorphic but have \( m(V) = 7509 \) and 9546 respectively.

There is a way to generate larger sets of homeomorphic surfaces from pairs. Suppose \( d = (d_1, \ldots, d_r) \) and \( e = (e_1, \ldots, e_s) \) are two multidegrees with the same total degree and symmetric functions \( \sigma_1 \) and \( \sigma_2 \). Let \( de = (d_1, \ldots, d_r, e_1, \ldots, e_s) \). Then the three multidegrees \( dd, de, \) and \( ee \) also have the same invariants \( \sigma_1, \sigma_2, \) and total degree. Similarly for \( ddd, dde, dee, \) and \( eee, \) etc. It is reasonable to expect that the corresponding \( m(V) \)'s will all be different. Unfortunately we have no general way to prove this.

In case \( d = (10,10,4,3,3) \) and \( e = (12,6,5,5,2) \) which give homeomorphic surfaces we find

\[
\begin{array}{|c|c|}
\hline
\text{multidegree} & m(V_2) \\
\hline
\text{d} & 23356 \\
\text{e} & 27005 \\
\text{dd} & 1695364 \\
\text{de} & 2234720 \\
\text{ee} & 2758226 \\
\hline
\end{array}
\]

Computation of the four cases which come from juxtaposing three
REMARKS ON MODULI SPACES

...ined by the middle cases and of the five cases which come from juxtaposing four intersection forms gives four and five distinct moduli dimensions respectively.

Mike Freedman then mined by the \(2,2,2,2\) and \(2,2,2,2\) and \(7509\) and \(9546\) homeomorphic to the \(14,14,5,4,4,4\) and \(V_3(16,10,7,2,2,2)\) are diffeomorphic with moduli dimensions \(1028748\) and \(1191130\) respectively. As above homeomorphic sets of diffeomorphic varieties can be generated. We have checked that the moduli dimensions are all different through the use of five distinct but diffeomorphic complete intersections.

... and symmetric \(e = (e_1, \ldots, e_5)\). Then these same invariants \(de, dee,\) and \(eee\), corresponding \(m(V)\)'s do not have a general way to

<table>
<thead>
<tr>
<th>Multidegree</th>
<th>(m(V)) for some complete intersections of low degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>19 (K-3) 45</td>
</tr>
<tr>
<td>2,2</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>40</td>
</tr>
<tr>
<td>6</td>
<td>68</td>
</tr>
<tr>
<td>3,2</td>
<td>19 (K-3) 34</td>
</tr>
<tr>
<td>4,2</td>
<td>44</td>
</tr>
<tr>
<td>2,2,2</td>
<td>19 (K-3) 27</td>
</tr>
<tr>
<td>3,2,2</td>
<td>46</td>
</tr>
</tbody>
</table>

... juxtaposing three...
References.


Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, IL 60680