Abelian Branched Covers of Projective Plane

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to Professor C.T.C. Wall

Abstract: *This note outlines a relationship between the fundamental groups of the complements to reducible plane curves and certain geometric invariants (polytopes of quasi-adjuntion) depending on the local type and* configuration *in the plane of singularities of the curves. This generalizes the relationship between the Alexander polynomial of plane curves and the position of singularities (cf. [Z],[L1],[LV]). Complete details will appear elsewhere.*

A relationship between the homology of cyclic covers of $\mathbf{CP}^2$ branched over a singular plane curve and the position of singularities was discovered by Zariski. In [L1] it was described how a certain portion of the fundamental group of an *irreducible* curve depends on the degree, the local type and the position of singularities. One associates to a curve C its Alexander module $A(C)$ which can be described as

$$A(C) = H_1(\mathcal{M}_f, \mathbf{Q}) \tag{1}$$

where $\mathcal{M}_f$ is the Milnor fiber of the (non-isolated if C is singular) singularity $f(x, y, z) = 0$ at the origin where f is the defining equation of C. $A(C)$ has the structure of a $\mathbf{Q}[t, t^{-1}]$-module where t acts on $H_1(\mathcal{M}_f, \mathbf{Q})$ as the monodromy operator of the singularity. Alternatively

$$A(C) = G'/G'' \otimes \mathbf{Q} \tag{2}$$

where $G = \pi_1(\mathbf{C}P^2 - C), G' = [G, G]$ is the commutator of $G, G'' = [G', G']$ is the second commutator, and t acts as the generator of $\mathbf{Z}/(\deg C)\mathbf{Z} = G/G'$ with G/G' acting on G'/G'' in the standard way. It turns out that $A(C) \otimes \mathbf{C}$ also can be described as follows. There exist a collection of rational numbers (constants of quasiadjunction) $\kappa_1, \kappa_2, ..., \kappa_N$

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depending on the local type of the singularities of C such that inequality $A(C) \neq 0$ is possible only if $\deg C \cdot \kappa_i \in \mathbf{Z}$ for all i (these constants of quasiadjunction κ_i were related to the Arnold-Steenbrink spectrum [A],[S] of singularities in [LV]). Moreover with the array κ_i is associated a collection of sheafs of ideals $\mathcal{J}_{\kappa_1}, ..., \mathcal{J}_{\kappa_N}$ such that for each κ_i the quotient $\mathcal{O}_{\mathbf{C}P^2}/\mathcal{J}_{\kappa_i}$ is supported at the union of singular points of C such that

$$A(C) \otimes \mathbf{C} = \oplus_{\kappa_i} \left(\mathbf{C}[t,t^{-1}]/(t-e^{2\pi i \kappa_i})(t-e^{-2\pi i \kappa_i})\right)^{dim H^1(\mathbf{C}P^2, \mathcal{J}(d-3-d\cdot\kappa_i))} \tag{3}$$

For example if $d = \deg C$ and C has nodes and ordinary cusps as the only singularities then $A(C) = 0$ if 6 does not divide d and $A(C) = \oplus_s \mathbf{C}[t,t^{-1}]/(t^2-t+1)$ where s is the difference between the dimension of the space of curves of degree $d-3-\frac{d}{6}$ passing through the cusps of C and the expected dimension of this space (i.e. $\frac{1}{2}(d-2-\frac{d}{6})(d-1-\frac{d}{6})$). (In particular if $\deg C = 6$ and the number of cusps is 6, then $A(C)$ is either zero or $\mathbf{Q}[t,t^{-1}]/(t^2-t+1)$ depending on whether the cusps belong to a curve of degree $6-1-\frac{6}{1} = 2$ or not.

In the case when C is reducible, one still can define a $\mathbf{Q}[t,t^{-1}]$-module $A_\ell(C)$ similarly to (1) or to (2) (and $A_\ell(C)$ can be related to the position of singularities ([LV])), which depends on the complement and on the homomorphism $\ell : G = \pi_1(\mathbf{C}P^2 - C) \to \mathbf{Z}/(\deg)\mathbf{Z}) \cdot \mathbf{Z}$ given by the linking number with C. This linking number of a loop $\alpha \in \pi_1(\mathbf{C}P^2 - C)$ is equal to the intersection number of a 2-chain σ such that $\partial\sigma = \alpha$ with C which is well defined as an integer modulo d: if σ^1 is another cochain with $\partial\sigma = \alpha$ then $(\sigma - \sigma^1, C) \equiv 0($ $mod\ d)$. In this case $\mathbf{Z}/d\mathbf{Z}$ acts on K/K' where $K = \ker \ell : G \to \mathbf{Z}/d\mathbf{Z}$. In particular $A_\ell(C)$ is an invariant of the pair $(G, \ell : G \to \mathbf{Z}/(\deg d)\mathbf{Z})$ rather than an invariant of the fundamental group alone.

To obtain an invariant of the fundamental group it is convenient to look at the complement to C in an affine plane $\mathbf{C}^2$ where the line at infinity is transversal to C. Then $\pi_1(\mathbf{CP}^2 - C)$ and $\pi_1(\mathbf{C}^2 - C)$ are related via the central extension (cf. [L5]):

$$1 \to \mathbf{Z} \to \pi_1(\mathbf{C}^2 - C) \to \pi_1(\mathbf{C}P^2 - C) \to 1. \tag{4}$$

Moreover, we have the isomorphism

$$\pi_1(\mathbf{C}^2 - C)/\pi_1(\mathbf{C}^2 - C)' = \mathbf{Z}^n \tag{5}$$

where n is the number of irreducible components of C. Identification (5) with $\mathbf{Z}^n$ is obtained by assigning to an element in $\pi_1(\mathbf{C}^2 - C)$ the collection of its linking numbers with the components of C. If $G = \pi_1(\mathbf{C}^2 - C)$ then one defines $A(C)$ as a group by (2) but views $A(C)$ is a module over the ring $R = \mathbf{Q}[G/G'] = \mathbf{Q}[\mathbf{Z}^n]$.

Since the structure of such modules for $n \geq 1$ is rather complicated even after extending the field from $\mathbf{Q}$ to $\mathbf{C}$ we consider only certain invariants of $A(C)$. Let $E_i(A(C)) \subset \mathbf{Q}[\mathbf{Z}^n]$ be the determinantal ideal generated by $(N-i) \times (N-i)$ minors of the matrix of the map $\Phi : R^M \to R^N$ where $A(C) = \text{Coker }(\Phi)$. Then each $E_i(A(C))$ defines a subvariety $V(E_i(A(C))$ of Spec $(\mathbf{Q}[\mathbf{Z}^n])$ which is the torus $\mathbf{T}^n$. Let $Char_i(C)$ (cf. [L3]) be the support of $V(E_i(A(C)))$ which is the reduced subvariety of $\mathbf{T}^n$ with the same set of zeros as $V(E_i(A(C)))$.

According to the Sarnak-Laurent theorem ([AS],[Lau]) points of finite order on any algebraic subvariety X of a torus belong to a finite union of translated subgroups embedded in this subvariety X (a translated subgroup in $\mathbf{T}^n$ is the set of solutions of a system $t_1^{\alpha_1} \cdots t_1^{\alpha_n} = \omega_\beta$ where $\alpha \in \mathbf{Z}$ and ω_β is a root of degree β). These translated subgroups are uniquely determined by X. Hence the collection of translated subgroups of $Char_i(C)$ affording the points of finite order of the latter are the invariants of the fundamental group of the complement and below is an explicite description of these subgroups.

The translated subgroups of charactersitic varieties can be calculated from a presentation using Fox Calculus (cf. [H]). For the Alexander module of links of algebraic singularities a calculation of these subgroups is contained implicitly in [Y]. Finally if $N = 1$ the union of these translated subgroups coincides with $Char_i(C)$ as a consequence of the fact that the Alexander polynomial is cyclotomic ([L2]).

In order to describe the translated subgroups of $Char_i(C)$ we shall calculate for any homomorphism $H_1(\mathbf{C}^2 - C) \to \mathbf{Z}/m_1 \oplus \cdots \oplus \mathbf{Z}/m_n$ (which sends a loop into the collection of residues of its linking numbers with the components of C) the first Betti number of a resolution of singularities $\widetilde{V_{m_1 \cdots m_n}}(C)$ of the corresponding branched abelian cover $V_{m_1,\ldots,m_n}(C)$ of $\mathbf{C}P^2$. This in fact is sufficient for obtaining the collection of subgroups we seek due to the following theorem of M.Sakuma:

Theorem 1. ([S]) Let $C = \bigcup_{i=1}^{i=n} C_i$ be an algebraic curve in $\mathbf{C}^2$ with n components which is transversal to the line at infinity and $V_{m_1,\cdots,m_n}$ (resp. $\widetilde{V_{m_1,..,m_n}}$) be the branched covering of $\mathbf{CP}^2$ (respectively its resolution) defined above. For a torsion point $\omega = (\omega_1 \cdots \omega_n)$ where $\omega_i^{m_i} = 1$ of $\mathbf{T}$ let C_ω be the curve formed by the components of C such that $\omega_i \neq 1$. Then

$$b_1(\widetilde{V_{m_1,\cdots,m_n}}) = \sum_{\omega \in \mu_{m_1} \times \cdots \times \mu_{m_n}} \max(i | \omega \in Char_i(C_\omega)) \tag{6}$$

(Contribution of $\omega = (1, \cdots, 1)$ should be taken equal to a 1, here μ_n is the group of roots of unity of degree n).

Note (cf. [L3]) that the first Betti number of the corresponding *unbranched* cover of $\mathbf{C}^2 - C$ is equal to

$$n + \sum_{\omega \in \mu_{m_1} \times \cdots \mu_{m_n}} \max(i | \omega \in Char_i C) \tag{7}$$

To describe the algebro-geometric calculation of $b_1(\widetilde{V_{m_1,\cdots,m_n}(C)})$ we need to define certain ideals in the local ring of a singular point of C. Let p be such a point and $f_1, \cdots, f_k$ be the local equations of those irreducible components of C in a small neighborhood of p which contain p. Let $I = (i_1, \cdots i_k | l_1, \cdots l_k)$ $(i_s, l_t \in \mathbf{Z}$ be the array of integers such that $0 \leq i_k \leq l_k$.

Definition 2. The ideal of quasiadjunction (corresponding to the array I) is the ideal $\mathcal{J}_p(I) \subset \mathcal{O}_p$ formed by germs $\phi \in \mathcal{O}_p$ such that $z_1^{i_1} \cdots z_k^{i_k} \phi$ belongs to the adjoint ideal of the surface $V(f_1, \cdots, f_k | l_1, \cdots, l_k)$ in $\mathbf{C}^{k+2}$ given by

$$z_1^{l_1} = f_1(x, y), \cdots, z_k^{l_k} = f_k(x, y) \tag{8}$$

We have the following:

Proposition 3. (1) Let $\mathcal{J}(f_1 \cdots f_k)$ be the ideal in $\mathcal{O}_p$ generated by

$$a)\frac{\Pi_{i=1}^{i=k} f_i}{f_i f_j} \cdot Jac(\frac{f_i, f_j}{x, y}), \qquad b)\frac{f'_{i\,x}}{f_i}\Pi_{i=1}^{i=k} f_i$$

$$c)\frac{{f_i}'_y \cdot \Pi_{i=1}^{i=k} f_i}{f_i}, \qquad d)\Pi_{i=1}^{i=k} f_i \tag{9}$$

Then $\mathcal{O}_p/\mathcal{J}(f_1, \cdots, f_k)$ is an Artinian algebra and for any I the ideal quasiadjunction corresponding to I contains $\mathcal{J}(f_1, ..., f_k)$,

(2) Let $\mathcal{J}_p(I)$ be the ideal of quasiadjunction corresponding to an array I. Then there exists a polytope $\Delta(\mathcal{J}_p(I))$ in the unit cube $\mathcal{U} = \{0 \le x_i \le 1 | i = 1, ..., k\}$ (which we shall call the *local* polytope of quasiadjunction) such that for any array $I' = (i_1, \cdots, i_k | l_1, \cdots, l_k)$ the ideal of quasiadjunction corresponding to I' is $\mathcal{J}_p(I)$ if and only if $(\frac{i_1}{l_1}, \cdots, \frac{i_k}{l_k}) \in \Delta$. Polytopes $\Delta(\mathcal{J}(I))$ define the partition of the unit cube into a finite union of non-intersecting polytopes.

In the case $k = 1$ the algebra $\mathcal{O}_p/\mathcal{J}_p(f)$ is just the Milnor algebra of the germ f ([M]), the ideals $\mathcal{J}_p(I)$ are the ideals of quasiadjunction $\mathcal{J}_\kappa$ and $O < \kappa_1 < \cdots \kappa_{\mu/2} < 1 (\mu = \dim_{\mathbf{C}} \mathcal{O}_p/\mathcal{J}_p(f))$ are the constants of quasiadjunction mentioned earlier. We have $\mathcal{J}_{\kappa_s} = \mathcal{J}((i|l))$ iff i, l satisfy $\kappa_{s-1} < \frac{i}{l} \le \kappa_s$ and the polytope of quasiadjunction corresponding to $\mathcal{J}_{\kappa_s}$ is the interval $\kappa_{s-1} < x \le \kappa_s$.

In the case $k = 2$ generators of the ideal $\mathcal{J}(f, g)$ are $f \cdot g, fg'_x, fg'_y, gf'_x, gf'_y, f'_x g'_y - g'_x f'_y$. For example if the local equation of C is $y(y - x^2)$ (tacnode), then $\mathcal{J}(f, g)$ is the maximal ideal of the local ring. If the array $I = (i_1, i_2 | l_1, l_2)$ is such that $\frac{i_1}{l_1} + \frac{i_2}{l_2} \le \frac{1}{2}$ then the corresponding ideal $\mathcal{J}(I)$ of quasiadjunction is the maximal ideal. The polytopes of quasiadjunction are $\{(x, y) | x + y \le \frac{1}{2}, 0 \le x \le 1, 0 \le y \le 1$ and $\{(x, y) | x + y \ge \frac{1}{2}, 0 \le x \le 1, 0 \le y \le 1\}$.

For the ordinary triple point $xy(x - y) = 0$ the local equations of the branches are: $f(x, y) = x, g(x, y) = y, h(x, y) = x - y$. The ideal $\mathcal{J}(f, g, h)$ has as generators $f \cdot g \cdot h, f_x gh, f_y gh, g_x fh, g_y fh, h_x fg, h_y fg, Jac(\frac{f,g}{x,y}) \cdot h, Jac(\frac{g,h}{x,y}) \cdot f, Jac(\frac{f,h}{x,y}) \cdot g$. The polytopes of quasiadjunction are given in the unit cube by $x + y + z > 1$ and $x + y + z \le 1$.

The proof of Proposition 3 is a result of making explicit the conditions for the pull back of a form to be holomorphic on the resolution of (8) which is the normalization of $V(f_1, \cdots, f_k | l_1, \cdots, l_k) \times_{\mathbf{C}^2} \widetilde{\mathbf{C}^2(f_1, \cdots}, f_k)$ where $\widetilde{\mathbf{C}^2(f_1, \cdots}, f_k)$ is an embeded resolution of the plane curve singularity $\Pi f_i = 0$ (cf. [L4]).

Next let us consider the following partition of the unit cube $\mathcal{U}_{\mathbf{R}^n}$ in the space $\mathbf{R}^n$ (n is the number of irreducible components of C) in which the coordinates of $\mathbf{R}^n$ are labeled by the irreducible component of C. For each local polytope of quasiadjunction $\Delta_{p_i,j}$ corresponding to the singular point p_i near which the components of C labeled $C_{i_1}, \cdots C_{i_k}$ have equations $f_1 = \cdots = f_k = 0$ consider the polytope $\bar{\Delta}_{p_i,j} \subset \mathcal{U}_{\mathbf{R}^n}$ which is the preimage of $\Delta_{p_i,j}$ for projection of $\mathcal{U}_{\mathbf{R}^n}$ on the subspace with coordinates labeled $i_1, \cdots i_k$. Let us call two points in $\mathcal{U}_{\mathbf{R}^n}$ equivalent if collections of polytopes $\bar{\Delta}_{p_i,j}$ which contain each of these points coincide. Each equivalence class is a polytope which we call a *global* polytope of quasiadjunction. For example for $n = 1$ (i.e. when C is an irreducible curve) such a polytope is an interval between two consecutive elements in the set of rational numbers which is the union of constants of quasiadjunction corresponding to all singularities of the curve.

To each global polytope of quasiadjunction Δ corresponds the sheaf of ideals $\mathcal{J}_\Delta \subset \mathcal{O}_{\mathbf{CP}^2}$ defined by the conditions:

a) $Supp \mathcal{O}_{\mathbf{CP}^2}/\mathcal{J}_\Delta$ is the set of singular points of C.

b) The stalk $\mathcal{J}_{\Delta,p}$ of $\mathcal{J}_\Delta$ at p is the local ideal of quasiadjunction corresponding to unique local polytope of quasiadjunction of the singularity p containing Δ.

Definition 4. A global polytope of quasiadjunction Δ is called contributing if the intersection of the hyperplane $d_1 \cdot x_1 + \cdots + d_n \cdot x_n = \ell$ for some $\ell \in \mathbf{Z}$ is a face δ (of any dimension) of Δ.

We call δ *the contributing face* of the polytope Δ, ℓ will be called *the level* of Δ, the mentioned hyperplane contaings δ will be called *the supporting hyperplane* of the contributing face δ (or of the global polytope of quasiadjunction Δ) and $d(\Delta) = d_1 + ... + d_{i_n}$ will be called *the total weight* of the supporting hyperplane of the global polytope of quasiadjunction Δ.

For example in the case of irreducible C a contributing hyperplane is a point coinciding with one of constants of quasiadjunction κ for one of singularities of C. Its level is $d \cdot \kappa$ where d is the degree of C (in particular a singularity of C affects G'/G'' only if for one of its constants of quasiadjunction κ the product $d \cdot \kappa$ is an integer e.g. if the only singularities are cusps, having $\frac{1}{6}$ as the only constant of quasiadjunction, $G'/G'' \otimes \mathbf{Q} = 0$ unless $6|d$ which is a well known result).

Theorem 5. Let $\widetilde{V_{m_1,\cdots m_n}}(C)$ be a desingularitation of the abelian cover of $\mathbf{CP}^2$ branched over $C = \bigcup_{i=1}^{i=n} C_i$ (C_i irreducible) and possibly the line at infinity which corresponds to the homomorphism $H_1(\mathbf{C}^2 - C) \to \mathbf{Z}/m_1, \oplus \cdots \oplus \mathbf{Z}/m_n$. Let $C_{i_1,\ldots,i_k} = C_{i_1} \cup \cdots \cup C_{i_k}$ be a curve formed by components $i_1, \cdots, i_k$ of C. Then the first Betti number of $\widetilde{V_{m_1,\cdots m_n}}(C)$ is equal to

$$2 \sum_{i_1,\cdots i_k} \sum_{\Delta(C_{i_1,\cdots,i_k})} N(m_{i_1}, \cdots m_{i_k}, \delta(\Delta(C_{i_1,\cdots,i_k}))) \cdot$$

$$\dim H^1(\mathbf{CP}^2, \mathcal{J}_{\Delta(C_{i_1,\cdots i_k})}(d(\Delta(C_{i_1,\cdots i_k}))) - 3 - \ell(\Delta(C_{i_1,\cdots,\subset_k}))) \qquad (10)$$

where $N(m_1, \cdots, m_k, \delta)$ is the number of points of the form $\left(\frac{i_i+1}{m_1}, \cdots \frac{i_k+1}{m_k}\right)$ in a contributing face $\delta(\Delta)$ of a polytope of quasiadjunction Δ, $\ell(\delta)$ is the level of δ and $\mathcal{J}_\Delta$ is the sheaf of quasiadjunction corresponding to a polytope of quasiadjunction Δ. The summation is over all global polytopes of quasiadjunction Δ (only those admitting a contributing face make a contribution).

We have the following:

Corollary 6. Translated subgroups affording the points of finite order of i-th characteristic variety $Char_i(C)$ are given by:

$$t_{i_1}^{\alpha_1}(\delta)\cdots t_{i_k}^{\alpha_k(\delta)} = e^{2\pi i\beta(\delta)}, t_j = 1(j \neq i_1, \cdots, i_k) \qquad (11)$$

where $\alpha_{i_1}(\delta)x_1 + \cdots \alpha_{i_k}(\delta)x_k = \beta(\delta)$ is the equation of the supporting hyperplane of a contributing face δ for which $\alpha_i(\delta) \in \mathbf{Z}^{>0}, g.c.d(\alpha_1(\delta), .., \alpha_k(\delta)) = 1, \beta(\delta) \in \mathbf{Q}$ and δ is the contributing face of a global polytope of quasiadjunction of a curve formed by the components $C_{i_1}, \cdots, C_{i_k}$ of C for which $dimH^1(\mathbf{C}P^2, \mathcal{J}_\Delta(d(\Delta)) - 3 - \ell(\delta(\Delta))) = i$.

In particular the dimension of the cohomology group $H^1(\mathbf{CP}^2, \mathcal{J}_\Delta(d(\delta) - 3 - \ell(\delta))$ where Δ is a contributing polytope is a *topological* invariant of the complement (since it depends only on the fundamental group)

Example 1. Let us calculate the irregularity of the abelian cover of $\mathbf{C}P^2$ branched over the line arrangement L: $xy(x-y)z = 0$ and corresponding to the homomorphism $H_1(\mathbf{C}P^2 - L) = \mathbf{Z}^3 \to (\mathbf{Z}/n\mathbf{Z})^3$ (the first identification is given by the linking numbers with lines $x = 0, y = 0, x = y$ respectively, $z = 0$ is the line at infinity. The only nontrivial ideal of quasiadjunction is the maximal ideal of the local ring of the point $P : x = y = 0$. The only global ideal of quasiadjunction is the local one corresponding to the polytope cut by $x + y + z \leq 1$. Hence the irregularity of the abelian corer is Card $\{(i,j)|0 < i < n, 0 < j < n, 0 < k < n, \frac{i}{n} + \frac{j}{n} + \frac{k}{n} = 1\} \cdot dimH^1(\mathcal{J}(3-3-1))$ where $\mathcal{J} = ker\mathcal{O}_{\mathbf{C}P^2} \to \mathcal{O}_P$. The sheaf $\mathcal{J}$ has the Koszul resolution $O \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{J} \to O$ which yields $H^1(\mathcal{J}(-1)) = H^2(\mathcal{O}_{\mathbf{CP}^2}(-3)) = \mathbf{C}$. Now the counting points on $x + y + z = 1$ shows that the irregularity of this abelian cover is equal to $\frac{n^2-3n+2}{2}$.

Example 2. Let us consider the arrangement of lines $x_i, i = 1, \cdots, 6$ formed by the sides of an equilateral triangle (x_1, x_2, x_3) and its medians (x_4, x_5, x_6) arranged so that their vertices are the intersection points of $(x_1, x_2, x_4), (x_2, x_3, x_5)$ and (x_3, x_5, x_6) respectively (cf. [I]) . It has 4 triple and 3 double points. The polytopes of quasiadjunction for (the full) arrangement are the connected components of the partition of $\mathcal{U} = \{(x_1, \cdots, x_6)|0 \leq x_1 \leq 1, i = 1, \cdots 6\}$ by the hyperplanes:

$$x_1 + x_2 + x_4 = 1, x_2 + x_3 + x_5 = 1, x_3 + x_1 + x_6 = 1, x_4 + x_5 + x_6 = 1 \qquad (12)$$

(abusing notation we use x_i as the coordinate corresponding to the line x_i). The only hyperplane of the form $H_k : x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = k, k \in \mathbf{Z}$ which contains a face of a global polytope of quasiadjunction (of the full) arrangement is H_2 (which contains the set of solutions of the system of equations formed by all equations in (12)). Moreover the subarrangemets formed by the components of this arrangement which have polytope of quasiadjunction admitting contributing faces are triples of lines passing through common point. There are 4 subarrangements of such type for which the supporting hyperplane of a polytope of quasiadjunction is H_1. The contribution into irregularity from the polytope of the first type is equal to $N \cdot \dim H^1(\mathcal{J}(6-3-2))$ where N is the number of solutions to (12) of the form $x_i = \frac{j+1}{n}$ and $\mathcal{J}$ is ideal of the subvariety of $\mathbf{CP}^2$ form by triple points of the arrangement. To calculate $\dim H^1(\mathcal{J}(6-3-2))$ notice that 4 triple points form a complete intersection of two quadrics and hence $\mathcal{J}$ admits a resolution:

$$0 \to O(-4) \to O(-2) \oplus O(-2) \to \mathcal{J} \to 0 \tag{13}$$

This yields $H^1(\mathcal{J}(1)) = H^2(\mathcal{O}(-3)) = \mathbf{C}$. The contributions from the subarrangents of the second type were considered in example 1. Note that (as follows from this calculation) the translated planes forming the characteristic varieties are

$$t_1t_2t_4 = 1, t_2t_3t_5 = 1, t_1t_3t_6 = 1 = t_4t_5t_6 = 1 \tag{14}$$

(corresponding to the global polytope of quasiadjunction of first type) and

$$t_1t_2t_4 = t_3 = t_5 = t_6 = 1, t_2t_3t_5 = t_1 = t_4 = t_6 = 1,$$

$$t_2t_3t_5 = t_1 = t_4 = t_6 = 1, t_2t_3t_5 = t_1 = t_4 = t_6 = 1 \tag{15}$$

corresponding to 4 polytopes of the second type. The number of the points in $\mu_n \times \cdots \times \mu_n$ which belong to the first of four for subgroups (15) and for which $t_1 \neq 1, t_2 \neq 1, t_4 \neq 1$, or, alternatively, the number of solutions to the first equation in (12) of the form

$$\frac{i}{n}, 0 < \frac{i}{n} < 1 \tag{16}$$

is equal to $\frac{(n-1)(n-2)}{2}$. The number of solutions to the system (12) satisfying (16) is equal to $\frac{(n-1)(n-2)}{2}$ as well since the system (12) is equivalent to $x_1 + x_2 + x_4 = 1, x_3 = x_4, x_5 = x_1, x_6 = x_2$. Hence for this arrangement the irregularity of $\widetilde{V_{n,n,n}(C)}$ is equal to $5\frac{(n-1)(n-2)}{2}$ (in particular it is 30 for $n = 5$ cf. [I]).

Other two arrangements considered in [I] can be treated similarly. Note that the theorem above also shows that the irregularity of the branched abelian cover $\widetilde{V_{m_1,..,m_n}(C)}$ is a polynomial periodic function of $m_1, .., m_n$ (cf. [AS],[H]).

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