Elliptic genera of singular varieties, orbifold elliptic genus and chiral de Rham complex.

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Abstract

This paper surveys the authors' recent work on the two-variable elliptic genus of singular varieties. The last section calculates a generating function for the elliptic genera of symmetric products. This generalizes the classical results of Macdonald and Zagier.

1 Introduction

Elliptic genera appeared in the mid-1980's in several diverse problems both in topology, e.g., circle actions on manifolds, construction of generalized cohomology theories, genera satisfying multiplicative properties, and in physics, as part of the study of Dirac-like operators on loop spaces (cf.[35]). Elliptic genera are certain modular functions attached to manifolds which interpolate many known genera of manifolds e.g., Todd, L and $\hat{A}$-genera. Following a suggestion of E.Witten (cf. [50]), a two-variable elliptic genus was formulated as an invariant of superconformal field theory, and was systematically studied as a tool for comparison of $N = 2$ minimal models and Landau-Ginzburg models in the work of T.Kawai,Y.Yamada and S-K. Yang (cf. [32]). From a mathematical point of view, the two-variable elliptic genus was studied in the work of Krichever, G.Hohn, B.Totaro and V.Gritsenko (cf. also [28]). While various generalizations were proposed (for example to complex manifolds, cf. section 2), the two-variable elliptic genus appears to be the most general elliptic genus in the sense that almost all versions of elliptic genera are its specializations.

The aim of these notes is to discuss generalizations of the two-variable elliptic genus to singular varieties from the mathematical point of view proposed in [9] and

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[10], in particular without reference to superconformal field theories (it is curious to note, however, the resemblance of the definition of elliptic genus in terms of the cohomology of the chiral de Rham complex and the definition of the elliptic genus of SCFT). First, we shall discuss the definition in terms of the cohomology of the chiral de Rham complex. Such a cohomology can be defined for hypersurfaces in Fano toric varieties in terms of the combinatorics of the toric variety, which allows one to define the elliptic genus in this case. Secondly, we shall discuss the definition of elliptic genus of singular algebraic varieties in terms of their resolutions and for singular spaces which are orbifolds X/G in terms of the action of G on X. These definitions can be used to give mathematical proofs for results which were previously obtained from the point of view of string theory, notably the Dijkgraaf-Moore-Verlinde-Verlinde formula for the generating function of the orbifold elliptic genera of symmetric groups acting on products of a fixed manifold X (cf. section 4). We shall finish with a derivation of generating functions for elliptic genera of symmetric products and containing as special cases old calculations of generating functions for Euler characteristics (I.Macdonald) and signatures (D.Zagier).

This subject is extremely vast and no claim to completeness is made. An excellent book by F.Hirzebruch, T.Berger and R.Jung ([29]) is particularly recommended for everybody interested in this subject.

2 Elliptic genera of manifolds.

Let Ω_*^{SO} (resp. Ω_*^U) be the cobordism ring of oriented (resp. almost complex) manifolds. Recall that cobordism ring is defined as the quotient of the free abelian group generated by manifolds (C^∞, almost complex, Spin, etc.) by the subgroup generated by manifolds which are boundaries (of manifolds with the same structure); the product is given by the product of manifolds. An R-valued genus is a ring homomorphism $E : \Omega_*^{SO} \otimes \mathbf{Q} \to R$. Similarly, a complex genus is a ring homomorphism $E : \Omega_*^U \otimes \mathbf{Q} \to R$. The class of an almost complex manifold in $\Omega_*^U \otimes \mathbf{Q}$ is completely specified by Chern numbers (cf. [25]), i.e. products of Chern classes evaluated on the fundamental class of the manifold. In particular, for complex cobordism a genus can be written as $E(M) = \int_M \mathcal{E}_{\mathrm{dim} M}(c_1, ...c_k, ..)$ for some polynomial $\mathcal{E}_{\mathrm{dim} M}$ having coefficients in the ring R. Similarly, in the oriented case, a class of $\Omega_*^{SO} \otimes \mathbf{Q}$ is determined by Pontryagin numbers and the genus is the integral of a polynomial in the Pontryagin classes.

The collection of polynomials $\mathcal{E}_i$ can be specified by a characteristic series: $Q(x) = 1 + b_1 x + b_2 x^2 + ...$ ($b_i \in R$) such that for the factorized total Chern class $c(T_M) = 1 + c_1(M) + ... + c_{\mathrm{dim} M}(M) = (1 + x_1) \cdots (1 + x_r)$ one has $\mathcal{E}(c_1, ...) = \prod Q(x_i)$ (cf. [25]). This condition determines the polynomials $\mathcal{E}_i$ from $Q(x)$ completely. For example (cf. [25]), the holomorphic Euler characteristic of a trivial bundle on a complex manifold extends to the complex genus and equals the Todd genus, with the corresponding characteristic series being $\frac{x}{1-e^{-x}}$ (Hirzebruch's Riemann-Roch theorem). The corresponding polynomials in Chern classes are $\frac{c_1}{2}, \frac{c_1^2+c_2}{12}, \frac{c_1 c_2}{24}$, etc. In the case of oriented manifolds, the same methods work after replacing Chern classes by

Pontryagin classes. The integer-valued genera which attracted the most attention, besides the Todd genus, are the L-genus (corresponding to the series $\frac{x}{\tanh(x)}$; L-genus is equal to the signature of the intersection form on the middle dimensional cohomology cf. [25]) and the $\hat{A}$-genus (corresponding to the series $\frac{x/2}{\sinh(x/2)}$ and equal to the index of the Dirac operator cf. [2]).

In the simplest version of the elliptic genus, the ring R is the ring of modular forms $Mod^*(\Gamma)$ for a certain subgroup Γ of $SL_2(\mathbf{Z}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbf{Z}$, i.e. the functions on the upper half-plane satisfying $\phi(\gamma \cdot \tau) = (c\tau + d)^k \phi(\tau), \gamma \in \Gamma$; k is an integer called the *weight* of ϕ and which provides the grading of the ring of modular forms; such functions often are written in terms of the variable $q = e^{2\pi i \tau}$.

Landweber-Stong (cf [35]) and S.Ochanine ([45]), while studying the circle actions on manifolds and the ideals in the cobordism ring generated by the projectivizations of vector bundles, considered the genus $\Omega^* \to Mod^*(\Gamma_0(2)) \subset \mathbf{Q}[[q]]$, where $\Gamma_0(2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) | c$ even). Its characteristic series is given by

$$Q_{LSO}(x) = \frac{x/2}{\sinh(x/2)} \prod_{n=1}^{\infty} [\frac{(1-q^n)^2}{(1-q^n e^x)(1-q^n e^{-x})}]^{(-1)^n}. \tag{1}$$

E.Witten ([51]) proposed the following expression for this genus:

$$\hat{A}(X) ch\{\frac{R(T_X)}{R(1)^{\dim X}}\}[X]$$

where

$$R(T_X) = \otimes_{l>0, l \equiv 0(2)} S_{q^l}(T_X) \otimes_{l>0, l \equiv 1(2)} \Lambda_{q^l}(T_X) \tag{2}$$

and the cohomology class $ch(E) = \sum e^{x_i}$ for a bundle E for which $c(E) = \prod(1 + x_i)$ is the Chern character of E. In the same paper he gave an interpretation of the elliptic genus as the index of a Dirac-like (or a signature-like) operator on the loop space $\mathcal{L}M$.

Elliptic genera of complex manifolds were defined by F.Hirzebruch ([27]) and E.Witten([51]). Such an elliptic genus takes values in the ring of modular forms for the group

$$\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) | c \equiv 0(N), a \equiv d \equiv 1(N)\} \tag{3}$$

provided the first Chern class of the manifold satisfies $c_1 \equiv 0(N)$.

The characteristic series depends on a choice of a point of order N on an elliptic curve with periods $2\pi i(1, \tau)$, say $\alpha = 2\pi i(\frac{k}{N}\tau + \frac{l}{N}) \neq 0$, and is given in terms of

$$\Phi(x, \tau) = (1 - e^{-x}) \prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})}{(1 - q^n)^2}. \tag{4}$$

It is equal to:

$$Q_{HW}(x, \tau) = x e^{-\frac{k}{N}x} \frac{\Phi(x - \alpha)}{\Phi(x)\Phi(-\alpha)}. \tag{5}$$

I.Krichever ([34]) considered the complex genus with characteristic series

$$Q_K(x, z, \omega_1, \omega_2, \kappa) = xe^{-\kappa x} \frac{\sigma_{\omega_1,\omega_2}(x - z)}{\sigma_{\omega_1,\omega_2}(x)\sigma_{\omega_1,\omega_2}(-z)} e^{\zeta_{\omega_1,\omega_2}(z)x} \tag{6}$$

where $z, \kappa \in \mathbf{C}^*$, $\sigma_{\omega_1,\omega_2}(z)$ and $\zeta_{\omega_1,\omega_2}(z)$ are Weierstrass ζ ($\zeta' = -\wp$) and σ-functions ($\zeta = \frac{\sigma'}{\sigma}$) corresponding to the same lattice in $\mathbf{C}$. It was further studied by G.Höhn (cf. [30]) and B.Totaro (cf.[49]). In this paper B.Totaro gives an important characterization of the genus introduced by Krichever as the universal genus of Ω^*_{SU} invariant under classical flops.

Note that the series Q_K specializes to Q_{HW} for $z = \alpha$ and $\kappa = -\frac{2k}{N}\zeta(\pi i\tau) - \frac{2l}{N}\zeta(\pi i) + \zeta(z)$. In addition, the Hirzebruch-Witten genus for $N = 2$ can be expressed in terms of Pontrjagin classes, so that it is an invariant of SO-cobordism which up to a factor coincides with the genus of Ochanine, Landweber and Stong.

One should mention that much of the interest in elliptic genera first come from a conjecture by Witten later proven by Bott and Taubes (cf. [11]), Hirzebruch (cf. [27]), Krichever (cf. [34]) and Liu (cf. [37], [39]) concerning the *rigidity* property which claims the following. Suppose a compact group G acts on M and a bundle V so that an operator P acting on V commutes with the the action of G. Let us consider the character $L_{M,V,P}(g) = \mathrm{Tr}_g\mathrm{Ker}P - \mathrm{Tr}_g\mathrm{Im}P$. The operator is rigid if this character is independent of g. The above mentioned results (generalizing [1]) state that the bundles which are the coefficients of the q-expansion of (2) support operators which are rigid. This is the case for other genera, including (5) and (6). Another important issue in which the elliptic genus was essential is known under the title *anomaly cancellation*, which yields a series of nontrivial identities and congruences among various classical (i.e. $L, \hat{A}$ etc.) genera (cf. [38] and survey [40]).

In the physics literature a two-variable elliptic genus was associated with an $N = (2, 2)$ superconformal field theory (cf. Eguchi-Ooguri-Taormina-Yang [18], E.Witten [50] and Kawai-Yamada-Yang cf.[32]). It is given by:

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F y^{J_0} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \tag{7}$$

where $\mathcal{H}$ is the Hilbert space of the SCFT, L_0 (resp. $\bar{L}_0$) is the Virasoro generator of left (resp. right)-movers and J_0 (resp. $\bar{J}_0$) is the $U(1)$ charge operator of left (resp. right) movers, the trace is taken over Ramond sector and $F = F_L - F_R$ with F_L (resp. F_R) the fermion number of left (resp. right) movers. In the case when the field theory comes from a smooth Calabi-Yau manifold M, one has the following mathematical expression for the genus (cf. [32],[17], [9])

$$Ell(M) = \int_M ch(\mathcal{E}ll_{q,y})td(M) \tag{8}$$

where

$$\mathcal{E}ll_{q,y} = y^{-\frac{\dim M}{2}} \otimes_{n\geq 1} (\Lambda_{-yq^{n-1}}\bar{T}_M \otimes \Lambda_{-y^{-1}q^n}T_M \otimes S_{q^n}\bar{T}_M \otimes S_{q^n}T_M). \tag{9}$$

The characteristic series for the genus (8) can be written in terms of the theta-function as follows. Let

$$\theta(z,\tau)=q^{\frac{1}{8}}(2\sin\pi z)\prod_{l=1}^{l=\infty}(1-q^l)\prod_{l=1}^{l=\infty}(1-q^le^{2\pi iz})(1-q^le^{-2\pi iz}) \tag{10}$$

where $q=e^{2\pi i\tau}$ (the Jacobi theta-function [13] or $\theta_{1,1}$, the theta-function with theta-characteristic, cf. [43]). Then the elliptic genus (8) corresponds to the characteristic series (with $y=e^{2\pi iz}$):

$$x\cdot\frac{\theta(\frac{x}{2\pi i}-z,\tau)}{\theta(\frac{x}{2\pi i},\tau)} \tag{11}$$

(cf. [32] and [9]). Note that the use of theta-functions in connection with elliptic genera goes back to D.Zagier (cf. [52]) and J.L. Brylinski ([12]).

The elliptic genus $K(M,\omega_1,\omega_2,z,\kappa)$ introduced by I.Krichever for a Calabi-Yau manifold M differs from the elliptic genus (8) only by a factor which depends only on dimension (and is independent of κ cf. [9] Sect.2):

$$K(2\pi\mathrm{i}z,2\pi\mathrm{i},2\pi\mathrm{i}\tau,\kappa)(X)=Ell(z,\tau)(X)\cdot(-\frac{\theta'(0,\tau)}{2\pi\mathrm{i}\,\theta(z,\tau)})^d. \tag{12}$$

The automorphic property of the elliptic genus is central for understanding this invariant. Recall that a weak Jacobi form of weight k and index r ($k\in\mathbf{Z},r\in\frac{1}{2}\mathbf{Z}$: we consider forms of half-integral index) is a holomorphic function on $H\times\mathbf{C}$ satisfying:

$$\phi(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d})=(c\tau+d)^ke^{2\pi i\frac{rcz^2}{c\tau+d}}\phi(\tau,z) \tag{13}$$

$$\phi(\tau,z+m\tau+n)=(-1)^{2r(\lambda+\mu)}e^{-2\pi ir(m^2\tau+2mz)}\phi(\tau,z) \tag{14}$$

In addition, a weak Jacobi form must have a Fourier expansion with non-negative powers of $q=e^{2\pi i\tau}$. This is weaker than the usual condition on Fourier modes, which explains the name (cf. [19]).

Using the expression via θ-functions for the characteristic series of the elliptic genus (11), one can show that the elliptic genus of an (almost) complex manifold of dimension d is a weak Jacobi form of weight 0, index $\frac{d}{2}$ (cf. [9]). A description of the space of weak Jacobi forms in [19] yields that elliptic genera of Calabi-Yau manifolds span the space of Jacobi forms of weight 0 and index $\frac{d}{2}$ (cf. [9], theorem 2.6). Gritsenko ([23]) has calculated the $\mathbf{Z}$-span of elliptic genera.

Such calculations in particular allow one to decide to what extent the elliptic genus depends on the χ_y-genus. Note that the elliptic genus is a combination of Chern numbers and there are non-trivial relations among Chern and Hodge numbers (e.g. $\sum_{p=2}^{d}(-1)^p\binom{p}{2}\chi^p=\frac{1}{12}\{\frac{1}{2}d(3d-5)c_d+c_{d-1}c_1\}[X]$ cf. [36]). More precisely,

Theorem 2.1 *If the dimension of a Calabi-Yau manifold is less than 12 or is equal to 13, then the numbers χ_p determine its elliptic genus uniquely. In all other dimensions there exist Calabi-Yau manifolds with the same $\{\chi_p\}$ but distinct elliptic genera.*

For example, if $e(X)$ (resp. $\chi(X)$) denotes the topological (resp. holomorphic) Euler characteristic then (cf. [44], [32]) the elliptic genus in the case of threefolds is

$$\frac{e(X)}{2}(y^{-\frac{1}{2}}+y^{\frac{1}{2}})\prod_{n=1}^{n=\infty}\frac{(1-q^ny^2)(1-q^ny^{-2})}{(1-q^ny)(1-q^ny^{-1})} \tag{15}$$

and for fourfolds is

$$\chi(X)E_4A^2+\frac{e(X)}{144}(B^2-E_4A^2). \tag{16}$$

Here, $A=\frac{\phi_{10,1}(\tau,z)}{\eta^{24}(\tau)}$, $B=\frac{\phi_{10,1}(\tau,z)}{\eta^{24}(\tau)}$, where $\phi_{10,1}$ and $\phi_{12,1}$ are the unique cusp forms of index 1 and weights 10 and 12, resp. (cf.[19]), $\eta(\tau)$ is the Dedekind η-function and $E_4(\tau)$ is the normalized Eisenstein series of weight 4.
However, as follows from the above theorem, for manifolds of high dimension the elliptic genus contains information not available from the χ_y-genus. It is interesting, therefore, to know what are the values of this invariant for concrete manifolds. For example, the χ_y characteristic of toric varieties is well known (cf. [15], [46] or [21]). For elliptic genera of smooth toric varieties we have the following:

Theorem 2.2 *Let* $\mathbf{P}$ *be a smooth toric variety corresponding to a fan* Σ *in* $N\otimes\mathbf{R}$ *for some lattice of rank* d*. Let* M *be the lattice dual to* N*. For the cone* C^* *of* Σ *(which is simplicial due to the smoothness of* $\mathbf{P}$*) let* $n_i (i=1,...,d)$ *be a system of its generators. Then:*

$$Ell(\mathbf{P},y,q)=y^{-d/2}\sum_{m\in M}\sum_{C^*\in\Sigma}(-1)^{\mathrm{codim}C^*}\left(\prod_{i=1,\dots,\mathrm{dim}C^*}\frac{1}{1-yq^{m\cdot n_i}}\right)G(y,q)^d \tag{17}$$

where

$$G(y,q)=\prod_{k\geq 1}\frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2}.$$

We shall sketch the proof, which uses the calculation of the cohomology via a split of the Čech complex according to characters.

First, let us consider the Leray spectral sequence for the cover of the toric variety by open sets $\mathbf{A}_C=\mathrm{Spec}\mathbf{C}[C]$ defined by the cones $C^*\in\Sigma$ and apply this spectral sequence to the bundle $\mathcal{E}ll_{q,y}(\mathbf{P})$ (cf.(9)). By abuse of language, the bundle here actually is a bigraded bundle whose components are the coefficients of y^aq^b in $\mathcal{E}ll_{q,y}(\mathbf{P})$; these coefficients are bundles having finite rank. Since the cohomology of positive dimension of the bundle $\mathcal{E}ll_{q,y}(\mathbf{P})$ vanishes over affine sets, it yields

$$Ell(\mathbf{P};y,q)=y^{-d/2}\sum_{m\in M}(\sum_{C_0^*,\dots,C_k^*}(-1)^k\mathrm{dim}_mH^0(\mathbf{A}_{C_0}\cap\dots\cap\mathbf{A}_{C_k},\mathcal{E}ll_{q,y}(\mathbf{P})).$$

Second, over each such open set $\mathbf{A}_C$ of maximal dimension, since $\mathbf{A}_C$ is just an affine space, a direct calculation shows

$$\sum_{m\in M}t^m\mathrm{dim}_mH^0(\mathbf{A}_C,\mathcal{E}ll_{q,y}(\mathbf{P}))=\prod_{i=1,\dots,d}\prod_{k\geq 1}\frac{(1-t^{m_i}yq^{k-1})(1-t^{-m_i}y^{-1}q^k)}{(1-t^{m_i}q^{k-1})(1-t^{-m_i}q^k).} \tag{18}$$

where m_i are generators of the cone C forming a basis in the lattice of the space containing the cone. Third, one notices that the latter can be rewritten as

$$\prod_{i=1,\ldots,d}\prod_{k\geq 1}\frac{(1-t^{m_i}yq^{k-1})(1-t^{-m_i}y^{-1}q^k)}{(1-t^{m_i}q^{k-1})(1-t^{-m_i}q^k)}=\sum_{m\in M}t^m\prod_{i=1,\ldots,d}\left(\frac{1}{1-yq^{m\cdot n_i}}\right)G(y,q)^d \tag{19}$$

where

$$G(y,q)=\prod_{k\geq 1}\frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2}$$

and n_i are generators of C^*.

One checks that the combined result of (18) and (19) is true for cones of arbitrary (i.e. possibly nonmaximal) dimension since $\mathbf{A}_C=\mathbf{C}^{\dim C^*}\times(\mathbf{C}-0)^{d-\dim C^*}$.

Finally, a combinatorial argument shows that the total contribution of each cone in the terms of the Čech complex, i.e. $\sum_{C_0\cap\ldots\cap C_k=C}(-1)^k$, is equal to $(-1)^{\mathrm{codim} C^*}$. This yields the theorem.

Since compact toric varieties are never Calabi-Yau, the expression (17) is not expected to have automorphic properties. However, its specialization to one-variable genera must satisfy modular relations. For example, for the Landweber-Stong-Ochanine elliptic genus

$$\widehat{Ell}(X;q)=(-1)^{d/2}Ell(X;-1,q)G(-1,q)^{-d}$$

we obtain

Theorem 2.3 *If* $\mathbf{P}$ *is a smooth complete toric variety, then*

$$\widehat{Ell}(\mathbf{P};q)=\sum_{m\in M}\left(\sum_{C^*\in\Sigma}(-1)^{\mathrm{codim}\ C^*}\prod_{i=1,\ldots,\dim\ C^*}\frac{1}{1+q^{m\cdot n_i}}\right).$$

In particular, the series in the right hand side is a modular form.

It is interesting that neither the modular property nor the relation to previous calculations of elliptic genera are obvious but, rather, lead to interesting new identities. For example, since

$$\widehat{Ell}(\mathbf{P}^2)=\delta=-\frac{1}{8}-3\sum_{n\geq 1}(\sum_{d|n,d\ \mathrm{odd}}d)q^d$$

we have

$$\sum_{m\geq 1,n\geq 1}\frac{q^{m+n}}{(1+q^m)(1+q^n)(1+q^{m+n})}=\sum_{r\geq 1}q^{2r}\sum_{k|r}k.$$

(cf. [9] for a direct proof of this identity, rather than as a consequence of two different calculations of elliptic genera).

The next problem is how to calculate the elliptic genus of hypersurfaces in toric varieties. To describe this, one needs a description of the elliptic genus via the chiral de Rham complex.

3 Elliptic genera in the singular case and the chiral de Rham complex

The two-variable elliptic genus is closely related to the chiral de Rham complex constructed by Malikov, Schechtman and Vaintrob in [41] for algebraic (analytic, C^∞ etc.) manifolds. This is a sheaf of vector spaces which has the structure of sheaf of vertex operator algebras. In particular, it supports the action of the Virasoro algebra, whose role in the theory of elliptic genera was anticipated from the very beginning (cf. [47]; for another attempt to clarify the role of the Virasoro algebra cf. [48]).

For convenience, let us recall the definition of a vertex operator algebra and conformal vertex operator algebra (cf. for example [31]).

Definition 3.1 *A vertex operator algebra is a vector space V, endowed with*
1. a decomposition

$$V = V_0 \oplus V_1 \tag{20}$$

2. a vector denoted $|0> \in V_0$ and called the vacuum vector
3. a linear map $V \to End(V)[z, z^{-1}]$ called the states to fields correspondence; the image of $a \in V$ is denoted $Y(a, z) = \sum_{n \in \mathbf{Z}} a_{(n)} z^{-n-1}, a_{(n)} \in End(V)$. One requires that for fixed a and b there holds $a_{(n)} b = 0$ for $n \gg 0$.
4. a linear map $T : V \to V$ called the infinitesimal translation operator.

This data are required to satisfy the following axioms:
a)Translation covariance: $\{T, Y(a, z)\}_- = \partial Y(a, z)$.
b)Vacuum: $|0>$ satisfies: $Y(|0>, z) = \mathrm{Id}_V$, $Y(a, z)|0> |_{z=0} = a$, $T|0>= 0$
c)Locality: $(z - w)^N Y(a, z)Y(b, z) = (-1)^{p(a)p(b)}(z - w)^N Y(b, z)Y(a, z)$ for $N \gg 0$

Definition 3.2 *A conformal vertex algebra is a pair (V, L), where V is a vertex algebra and L is a field that corresponds to an even element with the following properties:*
1. The components of $L(z) = \sum_n L_n z^{-n-2}$ satisfy the Virasoro commutation relations:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{n^3 - n}{12} \cdot c \cdot \delta^n_{-m}$$

2. $L_{-1} = T$ is an infinitesimal translation operator.
3. L_0 is diagonalizable.

In [41] the authors prove the following:

Theorem 3.3 *Let X be a nonsingular compact complex manifold. There exists a sheaf Ω^{ch}_X of vector spaces on X with the properties:*

a) For each Zariski open set U, $\Gamma(U, \Omega^{ch}_X)$ has a structure of conformal vertex algebra, with the restriction maps being morphisms of vertex algebras.

b) Ω^{ch}_X has two gradings with degrees called fermionic charge and conformal weight.

c) Ω^{ch}_X *has de Rham differential* d^{ch}_{DR} *of (fermionic) degree 1,* $(d^{ch}_{DR})^2 = 0$.

d) The usual de Rham complex $\Omega^{\bullet}_X$ *is isomorphic to the conformal weight zero component of* Ω^{ch}_{DR}.

e)The complex $(\Omega^{ch}_X, d^{ch}_{DR})$ *is quasi-isomorphic to* $(\Omega^{\bullet}_X, d_{DR})$.

f) Each component of fixed conformal weight has a canonical filtration with gr_F *isomorphic to the tensor product of exterior powers of the tangent and cotangent bundles, so that the corresponding generating function is*

$$\otimes_{n\geq 1}(\Lambda_{yq^{n-1}}\bar{T}_X \otimes \Lambda_{y^{-1}q^n}T_X \otimes S_{q^n}\bar{T}_X \otimes S_{q^n}T_X)$$

Recall that the supertrace of an operator S acting on a space (20) is $\mathrm{tr}S|_{V_0} - \mathrm{tr}S|_{V_1}$. By the Riemann-Roch theorem, the integral in (8) is just $\sum_i(-1)^i \mathrm{dim}H^i(\mathcal{E}ll_{q,y}(M))$. If one considers the bigraded sheaf with components being the coefficients of (8), the parity given by the parity of the exponent of y and endowed with the operators A and B acting on the coefficient of of $y^a q^b$ as multiplication by a and b respectively, then we see that elliptic genus can be written as $y^{\frac{-\mathrm{dim}M}{2}}\mathrm{Supertrace}_{H^*(\mathcal{E}ll_{q,y}(M))}y^A q^B$. Since the Euler characteristics of a filtered sheaf and its associated graded sheaf are the same, this suggests the following:

Definition 3.4 *Let* X *be a variety for which one can define a chiral de Rham complex* $\Omega^{ch}_X = \mathcal{MSV}(X)$ *with properties a)-f) as above. The elliptic genus of* X *is then defined as*

$$y^{-\frac{\mathrm{dim}X}{2}}\mathrm{SuperTrace}_{H^*(\mathcal{MSV}(X))}y^{J[0]}q^{L[0]}.$$

The usefulness of this definition stems from the following: the first-named author did construct such a complex $\mathcal{MSV}(X)$ in the case when X is a hypersurface in a toric varieties with Gorenstein singularities (cf. [7]) or for toric varieties themselves. In [7], a purely combinatorial construction of the cohomology of $\mathcal{MSV}(X)$ is given in these cases. It contains a description of the latter as the BRST cohomology of Fock spaces with an explicit description of those in terms of combinatorics. This yields the following explicit formulas for elliptic genera.

Theorem 3.5 Let X be a generic hypersurface in a Gorenstein toric Fano variety corresponding to a reflexive polytope Δ in a lattice M_1, $\mathrm{rk}M_1 = d+1$. Let $M = M_1 \oplus \mathbf{Z}$, N_1 and N be the lattices dual to M_1 and M respectively and Δ^* be the polytope dual to Δ. Denote the elements $(0,1) \in M, (0,1) \in N$ as deg and deg* respectively. Let K (resp. K^*) be the cone in M (resp. N) over $(\Delta, 1)$ (resp. $(\Delta^*, 1)$) with the vertex at $(0,0)_M$ (resp. $(0,0)_N$). Then

$$Ell(X,y,q) = y^{-\frac{d}{2}}\sum_{m\in M}\left(\sum_{n\in K^*} y^{n\cdot \mathrm{deg} - m\cdot \mathrm{deg}^*}q^{m\cdot n + m\cdot \mathrm{deg}^*}G(y,q)^{d+2}\right)$$

where

$$G(y,q) = \prod_{k\geq 1}\frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2}.$$

On the other hand, in the toric case one obtains:

Theorem 3.6 For a toric Gorenstein variety $\mathbf{P}$

$$Ell(\mathbf{P},y,q)=y^{-d/2}\sum_{m\in M}\sum_{C^*\in\Sigma}(-1)^{\mathrm{codim}C^*}(\sum_{n\in C^*}q^{m\cdot n}y^{\deg\cdot n})G(y,q)^d.$$

The Gorenstein property is needed since $\mathbf{P}$ has Gorenstein singularities if and only if the function of n given by $n\cdot\deg$ takes integer values. Inspection of the formulas in these theorems yields the following:

Corollary 3.7 If X admits a crepant toric desingularization $\hat{X}$, then

$$Ell(X,y,q)=Ell(\hat{X},y,q).$$

Similarly to the non singular case we have:

Theorem 3.8 The elliptic genus of a generic Calabi-Yau hypersurface in a any toric Gorenstein Fano variety is a weak Jacobi form of weight 0 and index $\frac{d}{2}$.

The proof uses an extension of the elliptic genus to a three-variable function and expression of the latter via theta functions, which reduces to the Bott formula in the smooth case. (cf. lemma 5.3 in [9])
An explicit description of the Fock spaces for which BRST cohomology yields the cohomology of the chiral de Rham complex $\mathcal{MSV}(X)$ in the case of theorem 3.5 and use of the Jacobi property of their elliptic genus provides the following relation:

Theorem 3.9 *Let* X, X^* be Calabi-Yau hypersurfaces in toric Gorenstein Fano varieties corresponding to dual reflexive polytopes Δ and Δ^*. Then:

$$Ell(X;y,q)=(-1)^dEll(X^*;y,q).$$

Such a result certainly is expected from physics considerations and assuming that Calabi-Yau hypersurfaces corresponding to dual polytopes form a mirror pair in the strong sense of correspondence between CFT's. Also, one can check it in small dimensions when the elliptic genus is a combination of Hodge numbers (cf. 2.1 and [44] for explicit formulas). But in higher dimensions the relation in theorem 3.9 can be viewed as a test for deciding if two Calabi-Yau manifolds form a mirror pair.

4 Elliptic genus of singular varieties via resolution of singularities and orbifold elliptic genera.

The definition of elliptic genus for special singular varieties in the last section suggests the following problem: find an expression for the elliptic genus of singular varieties in terms of a resolution and define the elliptic genus for varieties more general than hypersurfaces in singular toric spaces. These problems were addressed in [10], where the following approach was proposed.

Definition 4.1 Let Z be a complex space with $\mathbf{Q}$-Gorenstein singularities and let $Y \to Z$ be a resolution of singularities. Let $\alpha_k \in \mathbf{Q}$ be the discrepancies, i.e. rational numbers defined from the relation: $K_Y = \pi^* K_Z + \sum \alpha_k E_k$. Then

$$Ell_{sing}(Z; z, \tau) := \int_Y \Big(\prod_l \frac{(\frac{y_l}{2\pi \mathrm{i}})\theta(\frac{y_l}{2\pi \mathrm{i}} - z, \tau)\theta'(0, \tau)}{\theta(-z, \tau)\theta(\frac{y_l}{2\pi \mathrm{i}}, \tau)}\Big) \times \Big(\prod_k \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} - (\alpha_k + 1)z, \tau)\theta(-z, \tau)}{\theta(\frac{e_k}{2\pi \mathrm{i}} - z, \tau)\theta(-(\alpha_k + 1)z, \tau)}\Big)$$

(This definition can be generalized to define the elliptic genus of log-terminal pairs; cf. [10] for details).

It turns out that $Ell_{sing}(Z; z, \tau)$ is independent of Y and hence defines an invariant of Z. Several results make this invariant interesting.

1. It does specialize to the normalized version of the elliptic genus discussed earlier in the case when Z is non-singular, i.e.

$$Ell_{sing}(Z, z, \tau) = Ell(Z, z, \tau)\big(-\frac{\theta'(0, \tau)}{2\pi \mathrm{i}\, \theta(z, \tau)}\big)^d \tag{21}$$

2. If Z admits a crepant resolution, i.e. such that all discrepancies are zero, the singular elliptic genus coincides with the elliptic genus of a crepant resolution (up to the same factor as in (21)).

3. In the case when Z is a Calabi-Yau, the singular elliptic genus has the transformation properties of a Jacobi form.

4. For Calabi-Yau hypersurfaces in Fano Gorenstein toric varieties, the elliptic genus in 4.1 coincides with the elliptic genus considered in the last section (again up to the factor in (21)).

5. If $q \to 0$ then the singular elliptic genus specializes (up to a factor) to the χ_y genus that is a specialization of the E-function studied by Batyrev ([6]).

Finally, in many situations Ell_{sing} is related to the elliptic genus of orbifolds, also introduced in [10]. Let X be a complex manifold on which a finite group G is acting via holomorphic transformations. Let X^h will be the fixed point set of $h \in G$ and $X^{g,h} = X^g \cap X^h (g, h \in G)$. Let

$$TX|_{X^h} = \oplus_{\lambda(h) \in \mathbf{Q} \cap [0,1)} V_\lambda. \tag{22}$$

where the bundle V_λ on X^h is determined by the requirement that h acts on V_λ via multiplication by $e^{2\pi i \lambda(h)}$. For a connected component of X^h (which by abuse of notation we also will denote X^h), the fermionic shift is defined as $F(h, X^h \subseteq X) = \sum_\lambda \lambda(h)$ (cf. [53], [5]). Let us consider the bundle:

$$\begin{aligned} V_{h, X^h \subseteq X} := \otimes_{k \geq 1} \Big[&(\Lambda^\bullet V_0^* y q^{k-1}) \otimes (\Lambda^\bullet V_0 y^{-1} q^k) \otimes (Sym^\bullet V_0^* q^k) \otimes (Sym^\bullet V_0 q^k) \otimes \\ \otimes \big[\otimes_{\lambda \neq 0} &(\Lambda^\bullet V_\lambda^* y q^{k-1+\lambda(h)}) \otimes (\Lambda^\bullet V_\lambda y^{-1} q^{k-\lambda(h)}) \otimes (Sym^\bullet V_\lambda^* q^{k-1+\lambda(h)}) \otimes (Sym^\bullet V_\lambda q^{k-\lambda(h)}) \big] \Big] \end{aligned} \tag{23}$$

Definition 4.2 The orbifold elliptic genus of a G-manifold X is the function on $H \times \mathbf{C}$ given by:

$$Ell_{orb}(X,G;y,q) := y^{-\dim/X/2} \sum_{\{h\},X^h} y^{F(h,X^h \subseteq X)} \frac{1}{|C(h)|} \sum_{g \in C(h)} L(g, V_{h,X^h \subseteq X})$$

where the summation in the first sum is over all conjugacy classes in G and connected components X^h of an element $h \in \{h\}$, $C(h)$ is the centralizer of $h \in G$ and $L(g, V_{h,X^h \subseteq X}) = \sum_i (-1)^i \mathrm{tr}(g, H^i(V_{h,X^h \subseteq X}))$ is the holomorphic Lefschetz number.

Using the holomorphic Lefschetz formula ([2]) one can rewrite this definition as follows.

Theorem 4.3 Let $TX|_{X^{g,f}} = \oplus W_\lambda$ and let x_λ be the collection of Chern roots of W_λ. Let

$$\Phi(g,h,\lambda,z,\tau,x) = \frac{\theta(\frac{x}{2\pi i} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x}{2\pi i} + \lambda(g) - \tau\lambda(h))} e^{2\pi i z \lambda(h)}.$$

Then:

$$E_{orb}(X,G,z,\tau) = \frac{1}{|G|} \sum_{gh=hg} \prod_{\lambda(g)=\lambda(h)=0} x_\lambda \prod_\lambda \Phi(g,h,\lambda,z,\tau,x_\lambda)[X^{g,h}].$$

An orbifold elliptic genus so defined specializes for $q = 0, y = -1$ to the orbifold Euler characteristic: $e_{orb}(X,G) = \frac{1}{|G|} \sum_{fg=gf} e(X^{f,g})$ (cf. [26] and [3] where such an orbifold Euler characteristic is interpreted as the Euler characteristic of equivariant K-theory: $\mathrm{rk} K^0_G(X) - \mathrm{rk} K^1_G(X)$). Such an orbifold elliptic genus also can be specialized to the orbifold E-function studied by Batyrev-Dais ([5]). Moreover, one can show that $Ell_{orb}(X,G)$ is an invariant of cobordisms of G-actions.

One of the consequences of 4.3 is the Jacobi property of $Ell_{orb}(X,G)$ in the case when X is Calabi-Yau and the action of G preserves a holomorphic volume form (for more general actions, Ell_{orb} still has the Jacobi property but only for a subgroup of the Jacobi group described in terms of the order of the image of G in$\mathrm{Aut} H^0(X, \Omega^d(X))$).

In the case when $X \to X/G$ does not have ramification we have the following conjecture.

Conjecture 4.4 Let X be a complex manifold equipped with an effective action of a finite group G. Then

$$Ell_{orb}(X,G;y,q) = \left(\frac{2\pi \mathrm{i}\theta(-z,\tau)}{\theta'(0,\tau)}\right)^{\dim X} \widehat{Ell}(X/G,;y,q)$$

(for a more general statement, which allows ramification, cf. [10]). This conjecture is proven in [10] in the case when X is a smooth toric variety and G is a subgroup of the big torus and also for arbitrary X in the case when $G = \mathbf{Z}/2\mathbf{Z}$ (using the description of generators of the cobordisms of $\mathbf{Z}/2\mathbf{Z}$-actions given in [33]). Assuming this conjecture in the case when X/G admits a crepant resolution $\widetilde{X/G}$, the orbifold ellptic genus is just the elliptic genus of such a resolution. So it is natural to think about $Ell_{orb}(X,G)$ as a substitute for the elliptic genus of crepant resolution in the cases when it does not exist.

The most interesting property of $E_{orb}(X,G)$ is that it yields the remarkable formula due to R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde (cf. [17], also cf. [16]) obtained as part of the identification of the elliptic genus of the supersymmetric sigma model of the N-symmetric product of a manifold X and the partition function of a second quantized string theory on $X \times S^1$. Namely, in [10] a mathematical proof is given for the following.

Theorem 4.5 *Let X be a smooth variety X with elliptic genus $\sum_{m,l} c(m,l) y^l q^m$. Then*

$$\sum_{n \geq 0} p^n Ell_{orb}(X^n, \Sigma_n; y, q) = \prod_{i=1}^{\infty} \frac{1}{(1 - p^i y^l q^m)^{c(mi,l)}}.$$

Note that since the elliptic genus can be specialized to χ_y-genus and the Hilbert schemes for surfaces give a crepant resolution of the symmetric product, the results of [20] and [22] can be viewed as special cases of this theorem (cf. also [55]).

5 Generating functions for elliptic genera of symmetric products.

Another interesting question is about a generating function similar to 4.5 but constructed for ordinary elliptic genus of the quotient which we define as

$$\frac{1}{|G|} \sum_g L(g, \mathcal{E}ll_{q,y}(X)) \tag{24}$$

where $\mathcal{E}ll_{q,y}(X)$ is the bundle (9). We remark that this represents a "naive" version of an elliptic genus of the quotient, and is *different* from the orbifold genus considered in the last section. In particular, one cannot expect it to satisfy the formula of [17]. On the other hand, such an elliptic genus of the quotient specializes to the χ_y-genus of the quotient (cf. [52]) and in particular determines the Euler characteristic and the signature of the quotient. Generating functions for these classical invariants of symmetric products of manifolds were obtained earlier: for the Euler characteristic (cf. (5.4) and [42]) and for the signature (cf. [52], [54],[55], and 5.5). The analog of 4.5 is the following:

Theorem 5.1 *Let $Ell(X) = \Sigma c(m,l) q^m y^l$. Then*

$$\sum_n Ell(X^n/\Sigma_n) t^n = \prod_{m,l} \frac{1}{(1 - tq^m y^l)^{c(m,l)}}.$$

The proof is based on the following expression of holomorphic Lefschetz numbers of (9) via theta functions.

Lemma 5.2

$$L(g, y^{-d/2}\Lambda_{-yq^{k-1}}T^* \otimes \Lambda_{-y^{-1}q^k}T \otimes S_{q^k}(T^*) \otimes S_{q^k}(T))$$

$$= \frac{\prod_{i,r,s} y_i\theta(\frac{y_i}{2\pi i} - z, \tau)\theta(\frac{x_{r,s}+\theta_r}{2\pi i} - z, \tau)}{\prod_{r,s,i} \theta(\frac{y_i}{2\pi i}, \tau)\theta(\frac{x_{r,s}+\theta_r}{2\pi i}, \tau)}$$

Proof. We shall use the Atiyah-Singer holomorphic Lefschetz formula:

$$L(g, V) = \frac{[ch\ V|_{X^g}](g)td(T_{X^g})}{ch\ \lambda_{-1}(N^g)^*(g)}[X^g]$$

If $N^g = \oplus N^g(\theta_r)$ has Chern roots $x_{r,s}$ then $ch\ \lambda_{-1}((N^g)^*)(g) = \prod_{r,s}(1 - e^{-x_{r,s}-\theta_r})$
Let y_i be the Chern roots of T_{X^g}. Then we have:

$$\frac{ch\mathcal{E}ll_{q,y}(X)|_{X^g}td(X^g)}{ch\lambda_{-1}((N^g)^*)(g)} =$$

$$y^{-d/2}\frac{\prod_{i,r,s} y_i(1 - yq^{k-1}e^{-y_i})(1 - yq^{k-1}e^{-x_{r,s}-\theta_r})(1 - y^{-1}q^k e^{y_i})(1 - y^{-1}q^k e^{x_{r,s}+\theta_r})}{\prod_{i,r,s}(1 - q^k e^{-y_i})(1 - q^k e^{-x_{r,s}-\theta_r})(1 - q^k e^{y_i})(1 - q^k e^{x_{r,s}+\theta_{r,s}})(1 - e^{-y_i})\prod_{r,s}(1 - e^{-x_{r,s}-\theta_r})}$$

The latter can be written as

$$y^{-d/2}\frac{\prod_{i,r,s} y_i(1 - yq^k e^{-y_i})(1 - yq^k e^{-x_{r,s}-\theta_r})(1 - y^{-1}q^k e^{y_i})(1 - y^{-1}q^k e^{x_{r,s}+\theta_r})(1 - ye^{-y_i})(1 - ye^{-x_{r,s}-\theta_r})}{\prod_{i,r,s}(1 - q^k e^{-y_i})(1 - q^k e^{-x_{r,s}-\theta_r})(1 - q^k e^{y_i})(1 - q^k e^{x_{r,s}+\theta_{r,s}})(1 - e^{-y_i})\prod_{r,s}(1 - e^{-x_{r,s}-\theta_r})}$$

Since $\sin\pi(a-z) = e^{\pi i(a-z)}\frac{(1-e^{-2\pi i(a-z)})}{2i} = y^{-\frac{1}{2}}e^{\pi i a}(1-ye^{-2\pi i a})(\frac{1}{2i})$ this can be written as:

$$\prod_{i,r,s}\frac{2\sin\pi(\frac{y_i}{2\pi i} - z)(1 - e^{2\pi i z}q^k e^{-y_i})(1 - e^{-2\pi i z}q^k e^{y_i})2\sin\pi(\frac{x_{r,s}+\theta_r}{2\pi i} - z)(1 - e^{2\pi i z}q^k e^{-x_{r,s}+\theta_r+2\pi i z})}{2\sin\pi y_i(1 - q^k e^{y_i})(1 - q^k e^{-y_i})2\sin\pi(x_{r,s} + \theta_r)(1 - q^k e^{x_{r,s}+\theta_r})(1 - q^k e^{-x_{r,s}-\theta_{r,s}})}$$

$$\frac{(1 - e^{2\pi i z}q^k e^{-x_{r,s}+\theta_r})}{} = \frac{\prod_{i,r,s} y_i\theta(\frac{y_i}{2\pi i} - z, \tau)\theta(\frac{x_{r,s}+\theta_r}{2\pi i} - z, \tau)}{\prod_{r,s,i}\theta(\frac{y_i}{2\pi i}, \tau)\theta(\frac{x_{r,s}+\theta_r}{2\pi i}, \tau)}.$$

□

We also shall use the following two identities:

$$\prod_{k=0}^{k=r-1}\sin\pi(x + \frac{k}{r}) = \frac{1}{2^{r-1}}\sin\pi r x$$

and

$$\prod_{k=0}^{k=r-1}(1 - q^l e^{2\pi i z + 2\pi i\frac{k}{r}}) = (1 - q^{rl}e^{2\pi i z r})$$

which follow from $(1 - t^r) = \prod(1 - t\zeta_r^k)$.

They yield:

$$\prod_k \theta(x+\frac{r}{r}-z) = \prod_k q^{\frac{1}{8}} 2\mathrm{sin}\pi(x+\frac{k}{r}-z)\prod_l(1-q^l)\prod_l(1-q^l e^{2\pi i(x+\frac{k}{r}-z)})(1-q^l e^{2\pi i-(x+\frac{k}{r}-z)})$$

$$= q^{\frac{r}{8}} 2^r \frac{1}{2^{r-1}} \mathrm{sin}\pi r(x-z)(\prod_l(1-q^l))^r \prod_l(1-q^{rl}e^{2\pi i r(x-z)})(1-q^l e^{2\pi i r(x-z)})$$

$$= \frac{\prod_l(1-q^l))^r}{\prod_l(1-q^{lr})}\theta(r\tau, r(x-z)).$$

If σ_r is a cyclic permutation of X^r then the fixed point set is the diagonal, the representation of σ_r in the normal bundle is the quotient of the regular representation by the trivial representation and each isotrivial component isomorphic to the tangent bundle of X. Therefore:

$$L(\sigma_r, X^r) = \prod_i \prod_{k=0}^{r-1} y_i \frac{\theta(\frac{y_i}{2\pi i}+\frac{k}{r}-z)}{\theta(\frac{y_i}{2\pi i}+\frac{k}{r})}[X] = \prod_i y_i \frac{\theta(r\tau, ry_i - rz))}{\theta(r\tau, ry_i)}[X] =$$

$$\frac{1}{r^d}\prod_i ry_i \frac{\theta(r\tau, ry_i - rz))}{\theta(r\tau, ry_i)}[X] = Ell(r\tau, y^r)$$

(the latter equality follows since replacing $y_i \to ry_i$ multiplies the degree d component of the cohomology class evaluated on $[X]$ by r^d).

We can use arguments similar to those used in [42],[52] and [26] to conclude the proof of 5.1. We have

$$\sum Ell_n(X^n/\Sigma_n)t^n = \sum_n [\frac{1}{|\Sigma_n|}\sum_{g\in\Sigma_n} L(g, X^n)]t^n$$

where $L(g, X^n)$ is the holomorphic Lefschetz number of g acting on the bundle $\mathcal{E}ll(X)$ As usual, one can replace the summation with the summation over the set of conjugacy classes since conjugate g have isomorphic fixed point sets. The number of elements in a conjugacy class is $\frac{|G|}{|C(g)|}$, where $C(g)$ is the centralizer of g. Hence the latter sum can be replaced by $\sum_n \sum_{\{g\}\in\Sigma_n} \frac{L(g,X^n)}{|C(g)|}t^n$. Each conjugacy class is specified by a partition of n which has a_i cycles of length i, so that $\sum ia_i = n$. Let $g_{a_1,\dots,a_r}$ be an element in such a conjugacy class. Change of the order of summation yields

$$\sum_{a_1,\dots,a_n,\dots} L(g_{a_1,..,a_n}, X^n)\frac{1}{(a_1)!\cdots a_n!\cdot 2^{a_2}\cdots n^{a_n}} t^{a_1+2a_2+\dots+na_n}$$

since the number of elements in the conjugacy class corresponding to $(a_1,..,a_n)$ is $\frac{n!}{a_1!\dots a_n! 2^{a_2}\dots n^{a_n}}$. Next, the fixed point set of $g_{a_1,..,a_n}$ is $X^{a_1}\times\dots\times X^{a_n}$. Using the multiplicativity of Lefschetz numbers we obtain

$$\sum_{a_1,\dots a_n,\dots} \frac{\prod_i L(\sigma_i, X^i)^{a_i} t^{a_1+2a_2+\dots na_n}}{a_1!\dots a_n! 2^{a_2}\dots n^{a_n}} = \prod_i \sum_k \frac{L(\sigma_i, X^i)^k t^{ki}}{k! i^k}.$$

The latter can be simplified to

$$\prod_k \exp(\frac{L(\sigma_i, X)t^i}{i}) = \exp(\sum_{i,m,l} \frac{c(m,l)q^{im}y^{il}t^i}{i}) =$$

$$\prod_{m,l} \exp(-c(m,l)log(1-tq^m y^l)) = \prod_{m,n} \frac{1}{(1-tq^m y^l)^{c(m,l)}}.$$

□

We shall mention the following special cases of 5.1:

Corollary 5.3 *Let* $\chi_y(X) = \sum_p \chi^p y^p$. *Then*

$$\sum_n \chi_y(X^n/\Sigma_n)t^n = \prod_p \frac{1}{(1-t(-y)^p)^{(-1)^p \chi^p}}.$$

This follows from 5.1 since $\chi_y(X) = Ell(X, q=0, -y)(-y)^{\frac{d}{2}}$ and in particular if $l + \frac{d}{2} = p$ then $c(0,l) = (-1)^p \chi^p$. Generating series for χ_y were also considered in [54], [55].

Corollary 5.4 *(Macdonald, [42]) Let* e *denote the topological Euler characteristic. Then*

$$\sum_n e(X^n/\Sigma_n)t^n = \frac{1}{(1-t)^{e(X)}}.$$

Corollary 5.5 *(D.Zagier, [52]) Let* σ *denote the signature of the intersection form in the middle dimension. Then*

$$\sum_n \sigma(X^n/\Sigma_n)t^n = \frac{(1+t)^{\frac{\sigma(X)-e(X)}{2}}}{(1-t)^{\frac{\sigma(X)+e(X)}{2}}}.$$

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