HOMOTOPY GROUPS OF COMPLEMENTS AND NON-ISOLATED SINGULARITIES

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1. Introduction

It has been known for some time that the topology of non-isolated singularities, at least in some cases, has something to do with the "position" of the singularities (cf. [St, p.164], [D1], [D2]). The starting point of this work was an attempt to clarify this relationship. We shall consider two situations in which non-isolated singularities arise:

(1) *polynomials in* $\mathbb{C}^{n+1}$ *with a single atypical value*,

(2) *germs of analytic functions with non-isolated singularities.*

It turns out that in the first case, say for a polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ having 0 as the only atypical value, the first non-trivial (in the appropriate sense, cf. section 4) homology group of a fiber $F_t = \{f = t\}$, for generic t, is related to the first non-trivial homotopy group of the complement of $F_0 \cap H$ in H, for a generic linear section H of appropriate dimension. The study of the first non-trivial homotopy groups of complements $\mathbb{C}^{n+1} \setminus V$, was started in [L3]. If V does not have singularities at infinity and the dimension of the singular locus of V is equal to k, the first possibly non-trivial higher homotopy group is the group $\pi_{n-k}(\mathbb{C}^{n+1} \setminus V)$ (since $\pi_1(\mathbb{C}^{n+1} \setminus V) = \mathbb{Z}$ and $\pi_i(\mathbb{C}^{n+1} \setminus V) = 0$ for $i < n-k$; cf. [L3]). Although the information on higher homotopy groups of $\mathbb{C}^{n+1} \setminus V$ depends heavily on the information about homotopy groups of spheres, the first non-trivial homotopy group has algebro-geometric meaning and depends on the local type and the position of singularities of V (cf. [L3], [L4]). This identification allows one to relate in a direct way the homology of a smoothing to the position of singularities of a generic hyperplane section of the atypical fiber.

In the case (2) we obtain a similar relation. More precisely, we relate the first non-trivial homology group of the Milnor fiber of an analytic function f (with non-isolated singularity at the origin) to the corresponding homotopy group of the complement to the zero set $f^{-1}(0)$ in a generic and close to the origin linear section of $f^{-1}(0)$, inside a small ball.

In case (1), we consider the first non-trivial homotopy group for polynomials which may have certain singularities at infinity. This expands the class of polynomials which we can

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handle. It appears to be useful when, in section 4, we prove the aforementioned result for polynomials with one atypical value (say 0): the first non-trivial higher homology group of the generic fiber can be identified with the first non-trivial higher homotopy group of the complement to the hypersurface $H \cap f^{-1}(0)$ in a generic linear subspace H (the codimension of H is the dimension of the singular locus of $f^{-1}(0)$). This identification immediately yields several conditions for vanishing and non-vanishing of the homology of generic fibers of such polynomials by applying vanishing and non-vanishing results for the homotopy groups of the complements (cf. section 4). These results are based on divisibility theorems for the orders of the homotopy groups (cf. [L3]) and on vanishing results for hypersurfaces having mild singularities.

In the case of germs of holomorphic functions, we show that most of the results discussed above in the case of polynomials with one atypical value can be proven in the local case. Besides the already mentioned relation between the homology of the Milnor fiber in certain dimension and the homotopy group of the complements to hypersurfaces in a ball, we prove some divisibility results similar to those in the case of polynomials with one atypical value. We suspect that this analog can be extended further and that the latter homotopy groups can be related to geometry of singularities in a precise way, but because of technical difficulties, we postpone the discussion to later publication.

The contents of this note are the following. In the next section we discuss a way to measure the dimension of *singularities of a polynomial at infinity* (in a certain strong sense) and a method of constructing such polynomials, motivated by [Ti2]. In section 3 we consider the homotopy groups of the complements to hypersurfaces which may have singularities at infinity, expanding the results from [L3]. We consider separately two situations. One is the case when the hypersurface is a generic fiber of a polynomial and another is the case of $f = t$, where t is an atypical value. In the last section we prove results on the homology of smoothings and homotopy groups of the complements and the consequences discussed earlier in this introduction in cases (1), cf. Theorem 4.2, and (2), cf. Theorem 4.12.

2. Classes of hypersurfaces with singularities at infinity

Let V be a hypersurface in $\mathbb{C}^{n+1}$. In [L3], the first author investigates the homotopy of the complement $\mathbb{C}^{n+1} \setminus V$, which depends on the singularities of V and also on *singularities at infinity of* V, defined as follows in *loc.cit.*:

$$\mathrm{Sing}^{\infty}(V) := \mathrm{Sing}(\bar{V} \cap H^{\infty}), \tag{1}$$

where $H^{\infty} \subset \mathbb{P}^{n+1}$ is the hyperplane at infinity and $\bar{V} \subset \mathbb{P}^{n+1}$ is the projective closure of V.

The hypersurface V is a fibre of a polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$. We show that the consideration of singularities of f instead of those of V may refine the study of the homotopy of the complement $\mathbb{C}^{n+1} \setminus V$. Of course, we have to take into account the *singularities of* f *at infinity*; our definition extends Definitions 1.1, 2.3 in [Ti2] and has common flavor with Definition 2.2 in [L2].

Consider an embedding of $\mathbb{C}^{n+1}$ into some complex space $\mathbb{X}$ such that there exists a proper algebraic morphism $f^{\mathbb{X}} : \mathbb{X} \to \mathbb{C}$ extending f. In particular, the space $\mathbb{X}$ compactifies the fibres of f. Let $\mathcal{S}$ denote some Whitney stratification of $\mathbb{X}$ such that $\mathbb{C}^{n+1}$ is contained in a stratum. Let $\mathrm{Sing}\, f^{\mathbb{X}}_{|\mathcal{S}_i}$ denote the singular locus of the restriction of $f^{\mathbb{X}}$ to $\mathcal{S}_i$ (i.e. $\{x \in \mathcal{S}_i \mid \mathrm{grad}\, f^{\mathbb{X}}_{|\mathcal{S}_i}(x) = 0\}$). Then $\mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} := \cup_{\mathcal{S}_i \in \mathcal{S}} \mathrm{Sing}\, f^{\mathbb{X}}_{|\mathcal{S}_i}$ is the *singular locus* of $f^{\mathbb{X}}$ with respect to the stratification $\mathcal{S}$. The Whitney conditions imply that $\mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}}$ is a closed analytic set. Notice that $\mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}}$ depends on the choice of the embedding $\mathbb{C}^{n+1} \subset \mathbb{X}$, whereas its intersection with $\mathbb{C}^{n+1}$ does not, namely it is $\mathrm{Sing}\, f$.

Definition 2.1. We call the germ of $\mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}}$ at the set $\mathbb{X}^\infty := \mathbb{X} \setminus \mathbb{C}^{n+1}$ the *singular set of f at infinity* with respect to the proper extension $f^{\mathbb{X}}$ and to the stratification $\mathcal{S}$. We say that f *has isolated singularities at infinity* if the dimension of $\mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}}$ at any point of $\mathbb{X}^\infty$ is ≤ 0.

Observe that, if f has isolated singularities at infinity for some proper extension $f^{\mathbb{X}}$ and some stratification $\mathcal{S}$, then $\dim \mathrm{Sing}\, f \leq 0$.

We show that we have good control of the topology over a class of hypersurfaces which satisfy Definition 2.1, although they might have a large singular locus at infinity in the sense of (1).

Using the fact that $f^{\mathbb{X}}$ is algebraic and Thom's First Isotopy Lemma, it follows that the image $f^{\mathbb{X}}(\mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}})$ is a finite set $\Lambda \in \mathbb{C}$ (which depends on the embedding $\mathbb{C}^{n+1} \subset \mathbb{X}$) and that $f : \mathbb{C}^{n+1} \setminus f^{-1}(\Lambda) \to \mathbb{C} \setminus \Lambda$ is a locally trivial fibration (see e.g. [Ve]). We assume that $\Lambda := \{b_1, \ldots, b_r\}$ is minimal with this property and we call it the set of *atypical values*. We call $f^{-1}(b_i)$ an *atypical fiber*. It follows from the definition that Λ contains the critical values of f and is contained in $f^{\mathbb{X}}(\mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}})$.

The embedding $\mathbb{C}^{n+1} \subset \mathbb{X}$ and extension $f^{\mathbb{X}} : \mathbb{X} \to \mathbb{C}$ that we mainly use in this paper are the following.

Example 2.2. To each coordinate x_i of $\mathbb{C}^{n+1}$ we associate a positive weight w_i and write $f = f_d + f_{d-k} + \cdots$ where f_j is the degree j weighted-homogeneous part of f and where $f_{d-k} \neq 0$. Let $\tilde{f}$ be the degree d homogenization of f, in the weighted sense, by a new variable z of weight 1, and define:

$$\mathbb{X} := \{\tilde{f}(x, z) - tz^d = 0\} \subset \mathbb{P}(w) \times \mathbb{C}, \tag{2}$$

where $\mathbb{P}(w)$ denotes the weighted projective space $\mathbb{P}(w_0, \ldots, w_n, 1)$. Since $\mathbb{P}(w)$ is the space of orbits by the $\mathbb{C}^*$-action on $\mathbb{C}^{n+2} \setminus \{0\}$ given by $\lambda * x = (\lambda^{w_0} x_0, \cdots \lambda^{w_n} x_n, \lambda z)$, it has a canonical Whitney stratification by the orbit type (see [Fe] or [GWPL, p. 21]), which is moreover the coarsest one. We take on $\mathbb{P}(w) \times \mathbb{C}$ the product stratification. This induces a stratification on the subspace $\mathbb{X} \subset \mathbb{P}(w) \times \mathbb{C}$ and we consider the coarsest Whitney stratification $\mathcal{S}$ on $\mathbb{X}$ containing it. We then define $f^{\mathbb{X}} : \mathbb{X} \to \mathbb{C}$ to be the projection on the second factor. We identify $\mathbb{C}^{n+1}$ with $\mathbb{X} \setminus \mathbb{X}^\infty$, where $\mathbb{X}^\infty := \mathbb{X} \cap \{z = 0\}$ denotes the "divisor at infinity" of $\mathbb{X}$. Notice that the top-dimensional stratum of $\mathcal{S}$ is $\mathbb{X} \setminus \mathrm{Sing}\, \mathbb{X}$ and it contains $\mathbb{C}^{n+1}$, under the above identification.

In Example 2.2, if all the weights are 1, the singularities at infinity of f can be estimated by the following practical criterion (the case $s = 0$ has been proved in [Ti4] for any weights). We state the result only in case of weights 1 but this holds in fact for any weights, with a slightly more extended proof.

Proposition 2.3. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial and consider the extension $f^{\mathbb{X}} : \mathbb{X} \to \mathbb{C}$ as in Example 2.2. Let $\Sigma := \{\mathrm{grad}\, f_d = 0, f_{d-k} = 0\} \subset \mathbb{P}^{n+1} \cap \{z = 0\}$. If the singular locus of f in $\mathbb{C}^{n+1}$ is of dimension $\leq s$ and if $\dim \Sigma \leq s$, then $\dim \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \leq s$.*

Proof. Since the singularities of $f^{\mathbb{X}}$ on $\mathbb{C}^{n+1} = \mathbb{X} \setminus \mathbb{X}^\infty$ are of dimension $\leq s$ by hypothesis, we only have to look at singularities of $f^{\mathbb{X}}$ within $\mathbb{X}^\infty = \{f_d = 0\} \times \mathbb{C} \subset (\mathbb{P}^{n+1} \cap \{z = 0\}) \times \mathbb{C}$. So we need to know the stratified structure of $\mathbb{X}$ in the neighbourhood of $\mathbb{X}^\infty$. We prove in the following that $\mathbb{X}^\infty \cap \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \subset \Sigma \times \mathbb{C}$.

Consider the hypersurface $\tilde{\mathbb{X}}$ within $(\mathbb{C}^{n+2} \setminus \{0\}) \times \mathbb{C}$, defined by the same equation as in (2). Notice first that the subset $\tilde{\mathbb{X}} \cap \{z = 0\} \cap (\{f_d = 0\} \setminus \{\mathrm{grad}\, f_d = 0\})$ is contained in the regular part of $\tilde{\mathbb{X}}$. Next, at some point $\xi = (q, 0, t_0) \in \tilde{\mathbb{X}} \cap \{z = 0\} \cap (\{\mathrm{grad}\, f_d = 0\} \setminus \{f_{d-k} = 0\})$, we claim that there exists, locally at that point, a stratification of $\tilde{\mathbb{X}}$ with the property that its strata are product-spaces by the t-coordinate (i.e., if (p, t) belongs to a stratum, then (p, t') belongs to the same stratum for all $t' \in \mathbb{C}$). Indeed, the local equation of $\tilde{\mathbb{X}}$ is $\tilde{f}(x, z) - tz^d = 0$ and can be written as $f_d(x) + z^k g = 0$, where $g(x, z, t) = f_{d-k}(x) + zh(x, z, t)$. Since $f_{d-k}(q) \neq 0$, one can define, locally at ξ, a new coordinate $z' = z \sqrt[k]{g}$, by choosing a k^{th} root of g. It follows that, locally, our hypersurface is equivalent, via an analytic change of coordinates at ξ, to the product of $\{f_d(x) + (z')^k = 0\}$ by the t-coordinate. Consequently, there exists a local Whitney stratification at ξ which is a product by the t-coordinate. Notice that $\{z' = 0\}$ corresponds to $\{z = 0\}$ at ξ, hence the complement of $\{z' = 0\}$ is nonsingular too.

It follows that $\tilde{\mathbb{X}}$ may be endowed with a global Whitney stratification $\mathcal{W}$, such that $\tilde{\mathbb{X}}^\infty$ is a union of strata and that, locally at each point $\xi \in \tilde{\mathbb{X}}^\infty \setminus (\tilde{\Sigma} \times \mathbb{C})$, the strata contained in $\tilde{\mathbb{X}}^\infty$ are products by the t-coordinate. In particular, the projection to $\mathbb{C}$ is a stratified submersion at all points of $\tilde{\mathbb{X}}^\infty \setminus (\tilde{\Sigma} \times \mathbb{C})$. Moreover, since in charts $\{x_i = 1\}$ the $\mathbb{C}^*$ action on $\tilde{\mathbb{X}}$ is the identity, it also follows that the projection $\mathbb{X} \to \mathbb{C}$, which is just our $f^{\mathbb{X}}$, is a stratified submersion at every point $\xi \in \mathbb{X}^\infty \setminus (\Sigma \times \mathbb{C})$. Taking the coarsest stratification on $\mathbb{X}$ such that $\mathbb{X}^\infty$ is a union of strata, the map $f^{\mathbb{X}}$ clearly remains a stratified submersion at all points of $\mathbb{X}^\infty \setminus (\Sigma \times \mathbb{C})$.

Our claim that $\mathbb{X}^\infty \cap \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \subset \Sigma \times \mathbb{C}$, is now completely proved. To conclude the proof of 2.3, we notice that the stratified singularities of the restriction of $f^{\mathbb{X}}$ to $\Sigma \times \mathbb{C}$ can only occur on a finite number of fibres of $f^{\mathbb{X}}$. Since $\dim \Sigma \leq s$, it follows that $\dim \mathbb{X}^\infty \cap \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \leq s$. □

Corollary 2.4. *If $\dim \Sigma \leq s$ then $\dim \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \leq s + 1$ and, in particular, $\dim \mathrm{Sing}\, f \leq s + 1$.*

Proof. The proof of 2.3 shows that, if $\dim \Sigma \leq s$ then $\dim \mathbb{X}^\infty \cap \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \leq s$. This implies that $\dim \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \leq s + 1$. In turn, this yields $\dim \mathrm{Sing}\, f \leq s + 1$, since $\mathbb{C}^{n+1}$ is a stratum of the stratification $\mathcal{S}$ of $\mathbb{X}$. □

3. Higher homotopy groups

Let $V \subset \mathbb{C}^{n+1}$ be a hypersurface. It may be a general fiber of a polynomial f, or an atypical one. In case V is a general fiber, we have the following result on the homotopy type of the complement:

Proposition 3.1. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be any polynomial and let V be a general fibre of f. Then $\mathbb{C}^{n+1} \setminus V \overset{\mathrm{ht}}{\simeq} S^1 \vee S(V)$, where $S(V)$ denotes the suspension over V.*

In particular, the cup-product in the cohomology ring of $\mathbb{C}^{n+1} \setminus V$ is trivial.

Proof. Let $V = f^{-1}(\beta)$ and take a small enough closed disc $D \subset \mathbb{C}$ centered at β. Take a path γ_i from some point α on ∂D to a small enough disc D_i centered at an atypical value b_i of f. Now, f is a trivial fibration over D, hence f is trivial over ∂D too. Since $\mathbb{C}^{n+1} \setminus V \overset{\mathrm{ht}}{\simeq} f^{-1}(\cup_i(\gamma_i \cup D_i)) \cup f^{-1}(\partial D)$, it follows that $\mathbb{C}^{n+1} \setminus V$ is obtained from $f^{-1}(\partial D) \overset{\mathrm{ht}}{\simeq} \partial D \times V$ by attaching the space $f^{-1}(\cup_i(\gamma_i \cup D_i)) \overset{\mathrm{ht}}{\simeq} \mathbb{C}^{n+1}$ over $f^{-1}(\beta) \overset{\mathrm{ht}}{\simeq} V$. This is the attaching of a cone over $\{\beta\} \times V$ to $\partial D \times V$, so we get the claimed result. $\square$

We may derive the following consequence; this has been proved in [Ti4, 4.5] for a particular extension.

Corollary 3.2. *If V is a general fibre of f and f has isolated singularities at infinity for some extension $f^{\mathbb{X}}$, then $\mathbb{C}^{n+1} \setminus V \overset{\mathrm{ht}}{\simeq} S^1 \vee \bigvee_\lambda S^{n+1}$.*

Proof. Since f has isolated singularities at infinity, we may use a result of the second author [Ti2, Theorem 4.6, Corollary 4.7] which works in our general setting. This bouquet theorem says that the general fibre of our f is homotopy equivalent to a bouquet of spheres $\bigvee_\lambda S^n$, where λ is the sum of the local Milnor numbers at singular points of f and at singularities at infinity of $f^{\mathbb{X}}$. $\square$

Corollary 3.2 holds in particular if f has isolated singularities in $\mathbb{C}^{n+1}$ and $\dim \Sigma \leq 0$, by Proposition 2.3.

Example 3.3. $f : \mathbb{C}^3 \to \mathbb{C}$, $f = x + x^2 y + z^2$.
This polynomial has no singularities in $\mathbb{C}^3$. If we consider the extension $f^{\mathbb{X}}$ as in Example 2.2, with all weights equal to 1, then f has isolated singularities at infinity since it satisfies the assumptions of Proposition 2.3 for $s = 0$. According to Proposition 3.1, we have the homotopy equivalence $\mathbb{C}^3 \setminus V \overset{\mathrm{ht}}{\simeq} S^1 \vee S^3$; in particular $\pi_2(\mathbb{C}^3 \setminus V) = 0$. This works for general fibres, i.e. for $V = f^{-1}(t)$, $\forall t \neq 0$, since the only atypical value of f is 0, as one can easily check.

Example 3.4. $f : \mathbb{C}^4 \to \mathbb{C}$, $f = x_1^4 x_2^4 + (x_1 + x_2)^6 + x_3^5 + x_4^4 + x_1^2$.
Note that, according to definition (1), V is not transversal at infinity along a 2-dimensional set. Nevertheless, we may observe that $f = g(x_1, x_2) + h(x_3, x_4)$ is a sum of two polynomials in separate variables. For g, we have that $\dim \operatorname{Sing} g \leq 0$ and that there are no singularities at infinity, since $\Sigma = \emptyset$ (use Proposition 2.3). We get that the general fibre

of g is, homotopically, a bouquet $\bigvee S^1$. On the other hand, the polynomial h is weighted homogeneous with a unique singularity at the origin, hence its general fibre is $\overset{\mathrm{ht}}{\simeq} \bigvee S^3$.

By a Thom-Sebastiani result, the general fibre of $f = g + h$ has the homotopy type of a bouquet $\bigvee S^3$. Now, by Proposition 3.1, for a general fibre V of f, the complement $\mathbb{C}^4 \setminus V$ is homotopy equivalent to $S^1 \vee \bigvee S^4$.

For the complement of a hypersurface V which is an atypical fibre of a polynomial f, we have the following result:

Proposition 3.5. *Let $V = f^{-1}(0)$. If the general fibre of the polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is s-connected, $s \geq 2$, then $\pi_i(\mathbb{C}^{n+1} \setminus V) = 0$, for $1 < i \leq s$, and $\pi_1(\mathbb{C}^{n+1} \setminus V) = \mathbb{Z}$. In particular, if f has isolated singularities at infinity in some extension $f^{\mathbb{X}}$, then $\pi_i(\mathbb{C}^{n+1} \setminus V) = 0$ for $1 < i \leq n-1$.*

Proof. We use the notations D_i, γ_i as in the proof of Proposition 3.1 and take $b_1 := 0$, so that $V = f^{-1}(b_1)$. We first claim that the space $T_1 := f^{-1}(\cup_{i \neq 1} \gamma_i \cup D_i)$ is homotopy equivalent to a general fibre $F := f^{-1}(\alpha)$ to which one attaches cells of dimension $\geq s+2$, in other words that the pair (T_1, F) is $(s+1)$-connected.

Since F is s-connected by hypothesis, we have that $(\mathbb{C}^{n+1}, F)$ is $(s+1)$-connected. Then, by excision in homology, we have that $H_j(\mathbb{C}^{n+1}, F) = H_j(T_1, F) \oplus H_j(f^{-1}(D_1), F)$, for any j. Hence $H_j(T_1, F) = 0$ for $j \leq s+1$. By Blakers-Massey theorem [BM], the excision works in homotopy within a certain range. Namely, since F is s-connected, we get that the inclusion:

$$\pi_j(T_1, F) \oplus \pi_j(f^{-1}(D_1), F) \to \pi_j(\mathbb{C}^{n+1}, F)$$

is an isomorphism for $j \leq s-1$ and an epimorphism for $j = s$. This shows that (T_1, F) is $(s-1)$-connected and in particular simply connected, since $s \geq 2$. We may furthermore apply the relative Hurewicz isomorphism theorem and get that $\pi_j(T_1, F)$ is trivial for $j \leq s+1$. (We also get the isomorphism $\pi_{s+2}(T_1, F) \simeq H_{s+2}(T_1, F)$.) By Switzer's result [Sw, Proposition 6.13], it follows that T_1 is homotopy equivalent to the space F to which one attaches cells of dimension $\geq s+2$. The claimed property is proved.

Next, we have that $\pi_j(f^{-1}(\partial D_1)) = 0$ for $1 < j \leq s$ and that $\pi_1(f^{-1}(\partial D_1) = \mathbb{Z}$, due to the homotopy exact sequence of the fibration $f_| : f^{-1}(\partial D_1) \to \partial D_1$ and the s-connectivity of the fibre. (Note that this holds even for $s = 1$.)

Finally, $\mathbb{C}^{n+1} \setminus V$ is obtained from $f^{-1}(\partial D_1)$ by attaching the space T_1 over a general fibre F, which, we have proved above, means attaching only cells of dimension $\geq s+2$. It follows that $\pi_j(\mathbb{C}^{n+1} \setminus V) = 0$ for $1 < j \leq s$ and that $\pi_1(\mathbb{C}^{n+1} \setminus V) = \mathbb{Z}$.

The second statement follows from the first one. Indeed, if f has isolated singularities at infinity then, as mentioned in the proof of Corollary 3.2, the general fibre of f is $(n-1)$-connected. $\square$

Theorem 3.6. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial function and let's consider the embedding $\mathbb{C}^{n+1} \subset \mathbb{P}(w_0, \ldots, w_n, 1)$, for some system of weights w, as in Example 2.2. Suppose that* $\dim \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \leq k$. *Then:*

(a) *for a general fibre F of f, $\pi_i(\mathbb{C}^{n+1} \setminus F) = 0$, for $2 \leq i \leq n-k$.*

(b) *for an atypical fibre* V *of* f, $\pi_i(\mathbb{C}^{n+1} \setminus V) = 0$, *for* $2 \le i \le n-k-1$.

Proof. If we prove that the general fibre F is $(n-k-1)$-connected, then (a) follows by Proposition 3.1 and (b) follows by Proposition 3.5.

So let us show that the condition $\mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \le k$ indeed implies that the general fibre F is $(n-k-1)$-connected. This is true in the particular case when all the weights are 1, by [Ti2, Theorem 5.5]. The proof in *loc.cit.* goes by induction and uses generic hyperplane sections in $\mathbb{P}^{n+1}$, which do not exist in the case of weighted projective space.

Nevertheless, the proof could work in a similar spirit, provided that we can use, instead of generic hyperplane sections, a class of hypersurface sections with good enough properties. Let us start defining that.

Consider the finite map $\Psi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, given by $(x_0, \dots, x_n) \mapsto (x_0^{m_0}, \dots, x_n^{m_n})$, where $m_i = N/w_i$ and N is a common multiple of all w_i, $i = \overline{0,n}$. This induces $\bar{\Psi}$: $\mathbb{P}(w_0, \dots, w_n) \to \mathbb{P}^n$ and so a finite map $\mathbb{X} \subset \mathbb{P}(w) \times \mathbb{C} \xrightarrow{\hat{\Psi}} \mathbb{P}^{n+1} \times \mathbb{C}$. The class of "generic" hypersurfaces will be an open subset of $\{a_0 x_0^{m_0} + \cdots + a_n x_n^{m_n} = s \mid a_i, s \in \mathbb{C}\}$. Actually, we prove our statement by reduction to the space $\mathbb{P}^{n+1}$, via the finite map $\hat{\Psi}$.

We denote by H_s the affine hyperplane $\{l_H = s\} \subset \mathbb{C}^{n+1}$, where $H \in \check{\mathbb{P}}^n$ is a hyperplane defined by a linear form $l_H : \mathbb{C}^{n+1} \to \mathbb{C}$. We consider the restriction of f to $\mathcal{H}_s := \Psi^{-1}(H_s)$. We denote by $\mathcal{S}_{\mathcal{H}_s}$ the coarsest Whitney stratification of the space $\mathbb{H}_s := \mathbb{X} \cap (\bar{\mathcal{H}}_s \times \mathbb{C})$, where $\bar{\mathcal{H}}_s$ denotes the degree N weighted projective hypersurface $\{l_H \circ \Psi(x) - s z^N = 0\}$.

By eventually refining the stratification $\mathcal{S}$, we may assume without loss of generality that the restriction of $\hat{\Psi}$ to each stratum is an unramified covering. To the images by $\hat{\Psi}$ of the strata and of the levels of $f^{\mathbb{X}}$, we may apply the method of slicing by generic hyperplanes, as described in [Ti2, §5]. Then we may transfer back the transversality results via $\hat{\Psi}^{-1}$.

In this way, by using [Ti2, Lemma 5.4], it follows that there exists a Zariski-open set $\Omega \subset \check{\mathbb{P}}^n$ and a finite set $A \subset \mathbb{C}$ such that, if $H \in \Omega$ and $s \in \mathbb{C} \setminus A$ and if $\dim \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \ge 1$, then $\dim \mathrm{Sing}_{\mathcal{S}_{\mathcal{H}_s}} (f_{|\mathcal{H}_s})^{\mathbb{H}_s} \le \dim \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} - 1$.

Slicing a general fibre F of f by a "generic" hypersurface $\mathcal{H}_s$ gives a general fibre of the restriction $f_{|\mathcal{H}_s}$. Moreover, by the Lefschetz type theorem, the pair $(F, F \cap \mathcal{H}_s)$ is $\dim F - 1$ connected.

The general setting of [Ti2] and the use of the pull-back by $\hat{\Psi}$ allow one to continue this slicing procedure until the singularities of the restriction of f become zero-dimensional. Namely, there exist k generic hyperplanes $H^i \in \check{\mathbb{P}}^n$ and generic $s_1, \dots, s_k \in \mathbb{C}$ such that $A := \mathcal{H}^1_{s_1} \cap \cdots \cap \mathcal{H}^k_{s_k}$ is the global Milnor fibre of a weighted homogeneous affine complete intersection with isolated singularity at the origin and that the restriction $f_| : A \to \mathbb{C}$ has isolated singularities in the affine and at infinity, in the sense of Definition 2.1.

In this situation, we may apply to $f_| : A \to \mathbb{C}$ the results concerning isolated singularities at infinity [Ti2, Theorem 4.6 and Corollary 4.7], namely: A is obtained from a general fibre of $f_{|A}$ by attaching a finite number of cells of dimension $n+1-k$. Since A is homotopy equivalent to a bouquet of spheres of dimension $n+1-k$, it follows that this general fibre is homotopy equivalent to a bouquet of $n-k$ spheres, since it is a Stein space of dimension $n-k$ and $(n-k-1)$-connected. By tracing back the vanishing of homotopy in the slicing sequence, we get that the general fibre F is at least $(n-k-1)$-connected. □

The assumption of Theorem 3.6 holds in particular if $\dim(\Sigma \cup \operatorname{Sing} f) \le k$, by Proposition 2.3. About π_1, we can say the following:

Remark 3.7. (a) In case of a generic fibre V, it follows from the proof of Theorem 3.6(a) via Proposition 3.1, that $\pi_1(\mathbb{C}^{n+1} \setminus V)$ is trivial as long as $n-k \ge 1$, where $k \ge 0$.

(b) In case of an atypical fibre V, one may prove that $\pi_1(\mathbb{C}^{n+1} \setminus V)$ is trivial, as long as $n-k \ge 2$, where $k \ge 0$. We have to modify the arguments in the proofs of the above results, as follows. We slice as in the proof of Theorem 3.6, but instead of following a generic fibre F, we work with the atypical V. We get that the restriction $f_| : A \to \mathbb{C}$ has isolated singularities in the affine and at infinity and that $V \cap A$ is an atypical fibre of $f_{|A}$. Next, revisiting the proof of Proposition 3.5, we notice that one can prove by some different arguments that, for our restriction $f_{|A}$, the pair (T_1, F) is $(n-k)$-connected. Namely, we may use here again [Ti2, Theorem 4.6 and Corollary 4.7], as in the proof of Theorem 3.6, to show that T_1 is obtained from a general fibre of $f_{|A}$ by attaching a number of cells of dimension $n+1-k$. The last part of the proof of Proposition 3.5 still works in case $s=1$.

Note 3.8. The proof of Theorem 3.6 yields, in particular, that the general fibre of f is $(n-k-1)$-connected and that any atypical fibre of f is at least $(n-k)$-connected. This bound for the connectivity of the fibres appears to be sharp.

4. Monodromy of non-isolated singularities

4.1. **The global case.** We show how the monodromy of certain non-isolated singularities is related to the "first" non-vanishing homotopy group of the complement of the hypersurface singularity. Though most of the material below can be carried out in the framework of definition 2.1 we shall start by working with the weighted projective embedding $\mathbb{C}^{n+1} \subset \mathbb{X} \subset \mathbb{P}^{n+1}(w) \times \mathbb{C}$, as defined at 2.2. Let us assume that $\dim \operatorname{Sing}_{\mathcal{S}} f^{\mathbb{X}} \le k$ and denote by $V = f^{-1}(0)$ an atypical fibre. The generic fiber F of f is $(n-k-1)$-connected, by Theorem 3.6. Our goal is to calculate the monodromy acting on *the first possibly non-trivial group* $H_{n-k}(F)$.

Let $L_{k-1} = \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_{k-1}$ denote the intersection of $k-1$ generic hypersurfaces, as defined in the proof of Theorem 3.6. Then the Lefschetz theorem yields:

$$H_{n-k}(F) = H_{n-k}(F \cap L_{k-1}), \tag{3}$$

where $F \cap L_{k-1}$ can be viewed as a generic fiber of the polynomial $f_{|L_{k-1}}$. Moreover, $\dim \operatorname{Sing}_{\mathcal{S}'}(f_{|L_{k-1}})^{\mathbb{L}_{k+1}} \le 1$, after [Ti2], where $\mathbb{L}_{k+1} := \mathbb{X} \cap (\bar{L}_{k-1} \times \mathbb{C})$ (see the proof of 3.6 for the definition of the stratification $\mathcal{S}'$ on L_{k-1}).

On the other hand if $L_k = L_{k-1} \cap \mathcal{H}_k$ is the cut by another generic hypersurface, then by the Zariski-Lefschetz theorem of Hamm and Lê [HL] we have:

$$\pi_j(L_k \setminus L_k \cap V) \simeq \pi_j(\mathbb{C}^{n+1} \setminus V), \tag{4}$$

for $j \leq n-k$. This is an isomorphism of $\mathbb{C}[\mathbb{Z}]$-modules. Moreover, for $i < n-k$, we have $\pi_1(L_k \setminus L_k \cap V) = \mathbb{Z}$ and $\pi_i(L_k \setminus L_k \cap V) = 0$, by 3.5. The polynomial $f_{|L_k}$ has only isolated singularities at infinity and $\pi_{n-k}(L_k \setminus L_k \cap V)$ is the *first possibly non-trivial homotopy group* (cf. Theorem 3.6).

Definition 4.1. Let

$$\pi_{n-k}(\mathbb{C}^{n+1} \setminus V) \otimes \mathbb{Q} = \bigoplus_i \mathbb{Q}[t, t^{-1}]/(\lambda_i) \oplus \mathbb{Q}[t, t^{-1}]^{\kappa} \tag{5}$$

be the decomposition as $\mathbb{Q}[t, t^{-1}]$ module, where λ_i are some polynomials, defined up to units in $\mathbb{Q}[t, t^{-1}]$.

When $\kappa = 0$ in the decomposition (5), one calls the product $\prod_i \lambda_i$ the *order of* $\pi_{n-k}(\mathbb{C}^{n+1} \setminus V) \otimes \mathbb{Q}$. We denote it by $\Delta(\mathbb{C}^{n+1} \setminus V)$ or by $\Delta(L_k \setminus V \cap L_k)$. In case $n-k = 1$, this is nothing else than the Alexander polynomial of the curve $L_k \cap V$ in L_k. When $\kappa \neq 0$, one says the order is 0.

Theorem 4.2. *Let $V = f^{-1}(0)$ be the only atypical fiber*[1] *of a polynomial f with $\mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \leq k$. Let L_k denote the intersection of k generic hyperplanes in $\mathbb{C}^{n+1}$. Then*

$$\pi_{n-k}(L_k \setminus V \cap L_k) = H_{n-k}(F, \mathbb{C})$$

as $\mathbb{C}[\mathbb{Z}]$-modules. In particular, for the orders of these modules we have:

$$\Delta(L_k \setminus V \cap L_k) = \det[h_0 - t \cdot \mathrm{id} : H_{n-k}(F, \mathbb{C}) \to H_{n-k}(F, \mathbb{C})],$$

where h_0 is the monodromy of the general fibre F of f around the value 0.

Proof. Since f has at most one atypical value, the infinite cyclic cover $\widetilde{\mathbb{C}^{n+1} \setminus f^{-1}(0)}$ is homotopy equivalent to the general fiber F. The monodromy h_0 is the deck transform of the infinite cyclic cover $\widetilde{\mathbb{C}^{n+1} \setminus f^{-1}(0)} \to \mathbb{C}^{n+1} \setminus f^{-1}(0)$. This, together with (4) yields the first equality above. Moreover, we have the following:

$$\begin{aligned} \Delta(L_k \setminus L_k \cap V) &= \text{order of } \pi_{n-k}(L_k \setminus L_k \cap V) \otimes \mathbb{C} = \\ &= \text{order of } \pi_{n-k}(\mathbb{C}^{n+1} \setminus V) \otimes \mathbb{C} = \\ &= \text{characteristic polynomial of the deck transform on } H_{n-k}(\widetilde{\mathbb{C}^{n+1} \setminus V}) = \\ &= \text{characteristic polynomial of the monodromy on } H_{n-k}(F, \mathbb{C}). \end{aligned}$$

□

The above theorem can be used to obtain results for the homology of Milnor fibers of polynomials with non-isolated singularities, as follows.

Corollary 4.3. *Under the assumptions of Theorem 4.2, let $k = 1$ and $n > 2$. Then the number of cyclic factors corresponding to $t-1$ in the cyclic decomposition of $H_{n-1}(F, \mathbb{Q})$ is equal to the dimension of $H_{n-1}(\mathbb{C}^{n+1} \setminus V, \mathbb{Q})$. In particular $\mathrm{rk}\, H_{n-1}(F, \mathbb{Q}) \geq \mathrm{rk}\, H_{n-1}(\mathbb{C}^{n+1} \setminus V, \mathbb{Q})$ and this equality takes place if and only if $\Delta(\mathbb{C}^{n+1} \setminus V)$ has no other roots except 1 and $k_i = 1$ for any i.*

If $n = 2$ and V is irreducible, then $\mathrm{rk}\, H_{n-1}(F, \mathbb{Q}) = 0$ if and only if $\Delta(\mathbb{C}^{n+1} \setminus V) = 1$.

[1]For a partial classification of polynomials in 2 variables with one atypical value, we refer to [Ti3, 4.4].

Proof. Let

$$H_{n-1}(F,\mathbb{Q}) = H_{n-1}(\widetilde{\mathbb{C}^{n+1}\setminus V},\mathbb{Q}) = \bigoplus_{i=1}^{l'} \mathbb{Q}[t,t^{-1}]/(t-1)^{k_i} \bigoplus \oplus_{j=1}^{l''}\mathbb{Q}[t,t^{-1}]/(\lambda_j),$$

where λ_j does not have 1 as root.

From the spectral sequence $H_p(\mathbb{Z}, H_q(\widetilde{\mathbb{C}^{n+1}\setminus V}),\mathbb{Q}) \Rightarrow H_{p+q}(\mathbb{C}^{n+1}\setminus V,\mathbb{Q})$ and since $H_{n-2}(\mathbb{C}^{n+1}\setminus V,\mathbb{Q}) = 0$, we derive that $H_{n-1}(\widetilde{\mathbb{C}^{n+1}\setminus V},\mathbb{Q})^{Inv} = H_{n-1}(\mathbb{C}^{n+1}\setminus V,\mathbb{Q})$. Since $H_{n-1}(\widetilde{\mathbb{C}^{n+1}\setminus V},\mathbb{Q})^{Inv}$ is the kernel of the multiplication by $t-1$ on $H_{n-1}(\widetilde{\mathbb{C}^{n+1}\setminus V},\mathbb{Q})$, the result follows for $n > 2$. If $n = 2$, then the same spectral sequence shows that the number of cyclic summands is equal to $\operatorname{rk} H_1(\mathbb{C}^3\setminus V,\mathbb{Z}) - 1$ and the rank of the latter homology group can be identified with the number of irreducible components of V.

□

Example 4.4. Let us consider a polynomial of the form

$$f = P_a(x,y,z)^b + P_b(x,y,z)^a, \tag{6}$$

where P_b and P_a are generic and homogeneous of degrees b, resp. a. The homogeneity of f yields that 0 is the single atypical value; the singularities of f form the union of lines corresponding to the points $\{P_a = P_b = 0\} \subset \mathbb{P}^2$ and $\operatorname{Sing}_{\mathcal{S}} f^{\mathbb{X}}$ is just the closure of $\operatorname{Sing} f$ in $\mathbb{X}$. The standard identification of the Milnor fiber of f at 0, denoted M_f, with the the cover of degree ab of the complement of the projective curve given by $f = 0$ and the calculation of the Alexander polynomial for such curves yields that $H_1(M_f,\mathbb{Z}) = \mathbb{Z}^{(a-1)(b-1)}$ (cf. [L1]). On the other hand, by 4.2, the characteristic polynomial of this singularity is the Alexander polynomial of the affine curve $P_a(x,y,1) + P_b(x,y,1) = 0$ (since the plane $z = 1$ is generic), which is $\frac{(t^{a+b}-1)(t-1)}{(t^a-1)(t^b-1)}$.

More generally, we can consider the polynomial:

$$f = \sum P_{a_1\cdots\hat{a}_i\cdots a_n}(x_1,...,x_{n+1})^{a_i} \tag{7}$$

where P_k denotes a generic homogeneous polynomial of degree k. Again it satisfies the conditions of Theorem 4.2 with $k = 1$. We have $H_{n-1}(M_f,\mathbb{C}) \neq 0$, according to [L4].

Example 4.5. One can obtain new examples of polynomials by applying automorphisms of $\mathbb{C}^{n+1}$. For example, let $f_{i,j}(x_j,..,x_{n+1}), 1 \le i < j \le n+1$ be arbitrary polynomials. The automorphism

$$x_i \to x_i + \sum_{j>i} f_{i,j}(x_j,...,x_{n+1})x_j$$

applied to the polynomial (7) yields non-homogeneous examples of polynomials with non-isolated singularities and non-trivial monodromy on the first non-vanishing homology of the Milnor fiber.

In the remainder we refer to the embedding of $\mathbb{C}^{n+1}$ into $\mathbb{X} \subset \mathbb{P}^{n+1}\times\mathbb{C}$, where $\mathbb{P}^{n+1}$ is the usual projective space. Recall that $\operatorname{Sing} V \subset \operatorname{Sing} f \subset \operatorname{Sing}_{\mathcal{S}} f^{\mathbb{X}}$, where $V = f^{-1}(0)$.

We consider the restriction $f_{|H}$ of the polynomial f to a generic hyperplane in $\mathbb{C}^{n+1}$. Let $\mathcal{S}'$ denote the Whitney stratification induced by $\mathcal{S}$ and the cut by H, as defined in [Ti2, 5.2] and recalled in 3.6 above. Therefore $\mathrm{Sing}_{\mathcal{S}'}(f_{|H})^{\mathbb{H}}$ is well defined. According to [Ti2, §5], there exist, and we shall use in the following, generic hyperplanes H such that $\bar{V} \cap \mathrm{Sing}_{\mathcal{S}'}(f_{|H})^{\mathbb{H}} = \bar{H} \cap \bar{V} \cap \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}}$.

Suppose $\dim \mathrm{Sing}\, V = 1$. For each irreducible component Σ_i of $\mathrm{Sing}\, V$ let Δ_i be the characteristic polynomial of the monodromy of the isolated singularity one obtains as the transversal intersection of a small disk within H with the one-dimensional stratum Σ_i. We may call it *the horizontal* monodromy corresponding to Σ_i, in analogy to the case of germs, cf. [St].

Suppose also that $\dim(\mathbb{X}^\infty \cap \bar{V} \cap \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}}) = 1$ and denote by Σ_i^∞ some one-dimensional irreducible component of $\mathbb{X}^\infty \cap \bar{V} \cap \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}}$. Then Δ_j denotes the characteristic polynomial of the monodromy of the isolated singularity at infinity of the polynomial $f_{|H}$, for generic H. Remark that the monodromy around an isolated singularity at infinity is well defined. Indeed, if $g : \mathbb{C}^n \to \mathbb{C}$ is a polynomial and if $p_j \in \mathbb{X}^\infty$ is an isolated singularity of $g^{\mathbb{X}}$, there is a locally trivial fibration

$$B_j \cap \mathbb{X}_{D^*} \setminus \mathbb{X}^\infty \to D^*,$$

where D and B_j are Milnor data, i.e. B_j is a small Milnor ball centered at p_j and D is a small enough disk centered at $g^{\mathbb{X}}(p_j)$.

Following [L3, 4.5-4.7], let us denote by Δ^∞ the order of $\pi_{n-1}(S^\infty \cap H \setminus (S^\infty \cap V \cap H))$, for a sphere $S^\infty \subset \mathbb{C}^{n+1}$ of sufficiently large radius.

With these preliminaries, we can prove the following.

Corollary 4.6. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial such that $V = f^{-1}(0)$ is the only atypical fibre and such that $\dim \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} = 1$ and let F denotes a generic fiber of f. If none of the roots of Δ_i's and of Δ_j's distinct from 1 is a root of Δ^∞ and $k_i = 1$ for the cyclic summands in $H_{n-1}(F, \mathbb{Q})$ corresponding to 1 then $H_{n-1}(F, \mathbb{Q}) = H_{n-1}(\mathbb{C}^{n+1} \setminus V, \mathbb{Q})$.*

Proof. Note that $f_{|H}$ does not have a single atypical value, in general, even if f has only one. Due to the genericity of H, the polynomial $f_{|H}$ has isolated singularities at infinity and for such a polynomial the divisibility theorem [L3, 4.3] works. More precisely, the cited result can be extended in case of isolated singularities at infinity in the sense of this paper. The conclusion from *loc.cit.* is that $\Delta(\mathbb{C}^{n+1} \setminus V) = \Delta(H \setminus V \cap H)$ divides the the product of $\prod_i \Delta_i \cdot (1-t)^\kappa$, for some non-negative integer κ.

To complete the proof we have to use also [L3, Theorem 4.5]. This result applies to $f_{|H}$, which has isolated singularities at infinity. Actually, it can be extended to our new definition of singularities at infinity and it yields that $\Delta(H \setminus V \cap H)$ divides Δ^∞ of $f_{|H}$. Now our claim follows from Corollary 4.3. □

Still in case $\dim \mathrm{Sing}\, V = 1$, our statement 4.6 may become more precise, provided that we have a closer control on singularities at infinity.

Corollary 4.7. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial with one atypical fibre $V = f^{-1}(0)$ and such that $\dim \mathrm{Sing}\, f = 1$ and that $\dim \mathrm{Sing}(\bar{V} \cap H^\infty) = 0$ (where H^∞ denotes the hyperplane at infinity in $\mathbb{P}^{n+1}$).*

If none of the roots of Δ_i's and Δ_j's distinct from 1 *is a root of unity of degree* $d = \deg f$*, and* $k_i = 1$ *for the cyclic summands corresponding to* 1 *then* $H_{n-1}(F, \mathbb{Q}) = H_{n-1}(\mathbb{C}^{n+1} \setminus V, \mathbb{Q})$.

Proof. Due to the condition on singularities, the cut by a generic H gives that $V \cap H$ is transversal at infinity H^∞. We may apply [L3, Theorem 4.8], which amounts to saying that Δ^∞ has only roots of unity of degree d. The conclusion then follows by Corollary 4.6. □

Remark 4.8. Corollary 4.7 remains true when replacing the condition $\dim \mathrm{Sing}(\bar{V} \cap H^\infty) = 0$ by the more general one $\dim \mathbb{X}^\infty \cap \mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} = 0$. See the preliminaries before Corollary 4.6.

Example 4.9. Let $f : \mathbb{C}^3 \to \mathbb{C}$ be as in Corollary 4.7 (e.g. homogeneous or obtained via an automorphism applied to a homogeneous polynomial) and having transversal A_1 or A_2-singularities along each of the strata of its singular locus. If the degree of f is not divisible by 6, then the rank of $H_1(F, \mathbb{C})$ is equal to the number of irreducible components of the atypical fiber minus one. Indeed, the roots of the characteristic polynomial of the monodromy for an A_2 singularity are roots of unity of degree 6. Hence if d is not divisible by 6 then the only root of the characteristic polynomial is 1. Therefore, by 4.3, the rank of $H_1(F, \mathbb{Z})$ is the equal to the rank of $H_1(\mathbb{C}^3 \setminus f^{-1}(0))$ minus one and the rank of the last homology group is equal to the number of irreducible components of $f^{-1}(0)$.

Other vanishing, resp. non-vanishing results (cf. [L3]) combined with 4.2 yield corresponding vanishing, resp. non-vanishing results for the homology of the Milnor fibers of non-isolated singularities. Let us quote one of the results along these lines.

Corollary 4.10. *Let* $f : \mathbb{C}^3 \to \mathbb{C}$ *be a polynomial having* 0 *as single atypical value, such that* $f^{-1}(0)$ *is irreducible, that* $\dim \mathrm{Sing} f^{-1}(0) = 1$ *and that* $\dim(\mathrm{Sing}_{\mathcal{S}} f^{\mathbb{X}} \cap \mathbb{X}^\infty) = 0$. *Let the transversal type of singularities along the strata of* $\mathrm{Sing} f^{-1}(0)$ *be either* A_1 *or* A_2. *Assume that the degree* d *of* f *is divisible by* 6. *Let* $\Sigma = \mathrm{Sing} f^{-1}(0) \cap H$*, for some generic hyperplane* H *and let the superabundance of the curves in* H *having the degree equal to* $d - 3 - \frac{d}{6}$ *and passing through* Σ *be* s. *Then the characteristic polynomial of the monodromy of* f *acting on* $H_1(F, \mathbb{C})$ *is equal to* $(t^2 - t + 1)^s$. □

4.2. **The local case.** The above global case, with a single atypical value, is somehow close to a semi-local situation. So, let us formulate the problem in the local case.

Let $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ denote a germ of a holomorphic function, with $\dim \mathrm{Sing}\, g \leq 1$. Then g has a $(n-2)$-connected Milnor fibre M_g, by Kato and Matsumoto's result [KM]. Take a closed Milnor ball B at 0, with boundary $S := \partial B$ and take a general hyperplane H passing close to the origin. Denote $B_H = B \cap H$, $S_H = S \cap H$ and $V = g^{-1}(0)$. Then $B_H \cap V$ has at most isolated singularities, since $\dim \mathrm{Sing}\, g \leq 1$. Let $\{\Delta_i\}_i$ be the collection of characteristic polynomials of the monodromies of the singularities of $B_H \cap V$. With these notations, we have the following:

Theorem 4.11. (a) $\pi_i(S_H \setminus S_H \cap V) = \pi_i(B_H \setminus B_H \cap V) = 0$, *for* $1 < i \leq n-2$ *and* $\pi_1(S_H \setminus S_H \cap V) = \pi_1(B_H \setminus B_H \cap V) = \mathbb{Z}$.

(b) $\pi_{n-1}(S_H \setminus S_H \cap V)$ *is a torsion* π_1*-module.*
(c) *The module* $\pi_{n-1}(B_H \setminus B_H \cap V) \otimes \mathbb{C}$ *is a* $\mathbb{C}[\mathbb{Z}]$*-torsion module and the order of* $\pi_{n-1}(B_H \setminus B_H \cap V) \otimes \mathbb{C}$ *divides the order of* $\pi_{n-1}(S_H \setminus S_H \cap V)$*.*
(d) *The order of* $\pi_{n-1}(B_H \setminus B_H \cap V) \otimes \mathbb{C}$ *divides* $\prod_i \Delta_i$*.*

Proof. We may assume that H is a member of a linear pencil $h : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ which scans the space B and is in general position with respect to V. Denote $H_t = \{h = t\}$.
(a). The space $S_{H_t} \setminus S_{H_t} \cap V$ is diffeomorphic to $S_{H_0} \setminus S_{H_0} \cap V$, due to the transversality of the sphere to $H_t \cap V$, for any t within a small neighbourhood of 0. This comes from the fact that the restriction $g_{|H_0}$ is an isolated singularity.

Now $S_{H_0} \setminus S_{H_0} \cap V$ is the total space of the Milnor fibration over the circle S^1, defined by the restriction of $g/\|g\|$. The claim follows from the homotopy exact sequence and from the fact that the Milnor fibre of the restriction $g_{|H_0}$ is homotopically a bouquet of spheres $\bigvee S^{n-1}$.

(c). We have the following commuting diagram:

$$\begin{array}{ccc} S_{H_0} \setminus S_{H_0} \cap V & \stackrel{\sim}{\hookrightarrow} & B_{H_0} \setminus B_{H_0} \cap V \\ \updownarrow & & \downarrow \\ S_{H_t} \setminus S_{H_t} \cap V & \hookrightarrow & B_{H_t} \setminus B_{H_t} \cap V \end{array} \tag{8}$$

The embedding on the first line is a homotopy equivalence, which comes from the local cone structure. The arrow at the left is a diffeomorphism, as shown above. The arrow to the right is also an embedding. Moreover, we claim that the space $B_{H_t} \setminus B_{H_t} \cap V$ is homotopy equivalent to the space $B_{H_0} \setminus B_{H_0} \cap V$ to which one attaches cells of dimension n. For example one can argue along the following lines using the map $\phi = (h, g) : (\mathbb{C}^{n+1}, 0) \to \mathbb{C}$ (cf. e.g. [Le] and [Ti1]). Let $\Gamma(h, g)$ be the polar curve of g with respect to h and let Δ denote the image of $\Gamma(h, g)$ by ϕ, see Figure 1.

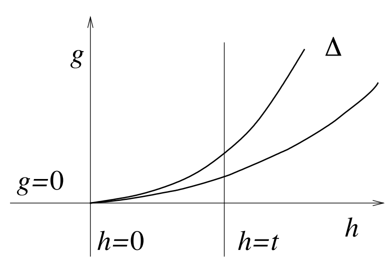

FIGURE 1. The image by the map $(h, g) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^2, 0)$.

For suitable ε and t, one has the following homotopy equivalences: $B_{H_0} \setminus B_{H_0} \cap V \stackrel{\text{ht}}{\simeq} \{|g| = \varepsilon\} \cap B_{H_0}$, by retraction, and $\{|g| = \varepsilon\} \cap B_{H_0} \stackrel{\text{ht}}{\simeq} \{|g| = \varepsilon\} \cap B_{H_t}$, by isotopy. The function g has isolated singularities on $B_{H_t} \setminus B_{H_t} \cap V$, which are, by definition, exactly the points where the polar curve $\Gamma(h, g)$ cuts this space. It then follows that, up to homotopy equivalence, the space $B_{H_t} \setminus B_{H_t} \cap V$ can be constructed by attaching to

$\{|g| = \varepsilon\} \cap B_{H_t}$ a certain number of n-cells for each singular point of the function g on $g^{-1}(D_\varepsilon) \cap B_{H_t} \setminus B_{H_t} \cap V$.

Now, we apply π_{n-1} to the diagram (8). The arrow to the right becomes a surjection in π_{n-1} and this implies that the arrow on the bottom is also a surjection in π_{n-1}. By the multiplicativity of the order in exact sequences, it follows that $\Delta(B_H \setminus B_H \cap V)$ divides $\Delta(S_H \setminus S_H \cap V)$. □

The local analogue of Theorem 4.2 looks as follows:

Theorem 4.12. *Let g be a germ of a holomorphic function with* $\dim \mathrm{Sing}\, g \leq 1$. *Then the characteristic polynomial of the monodromy of g acting on $H_{n-1}(M_g, \mathbb{C})$ coincides with the order of $\pi_{n-1}(B_H \setminus B_H \cap V)$.*

Proof. The proof follows from the Zariski-Lefschetz theorem of Hamm and Lê [HL] and the equivariant identification of the universal cover of $B \setminus g^{-1}(0)$ with the Milnor fibre $M_g = g^{-1}(t)$, similar to the one used in the proof of 4.2. □

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