Counting rational maps onto surfaces and fundamental groups

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Abstract. We consider the class of quasiprojective varieties admitting a dominant morphism onto a curve with negative Euler characteristic. The existence of such a morphism is a property of the fundamental group. We show that for a variety in this class the number of maps onto a hyperbolic curve or surfaces can be estimated in terms of the numerical invariants of the fundamental group. We use this estimates to find the number of biholomorphic automorphisms of complements to some arrangements of lines.

1.Introduction. In this paper we study quasiprojective varieties X admitting a dominant morphism onto a curve with negative Euler characteristic. A fundamental result of D.Arapura ([1]) shows that such maps can be described in terms of the moduli spaces of rank one local systems with non vanishing cohomology. His work was built on previous extensive studies going back to DeFranchis and, more recently, to A.Beauville ([2]), Catanese, Green-Lazarsfeld and Deligne-Simpson.

The moduli spaces of local systems with non-vanishing H^1 can be described in terms of the invariants of the fundamental group of X alone. These invariants, in fact depending only on the quotient of the fundamental group by its second commutator, are the algebraic subvarieties of $Hom(\pi_1(X), \mathbb{C}^*)$. They were considered also in ([3] in the case when X is a complement to a plane curve and called the

[1]Supported by the Ministry of Absorption (Israel), the Israeli Science Foundation (Israeli Academy of Sciences, Center of Excellence Program), the Minerva Foundation (Emmy Noether Research Institute of Mathematics)

[2]Supported by NSF grant

1991 *Mathematics Subject Classification.* Primary 14FXX, 14R25;.

Key words and phrases. Affine varieties, fundamental group, rational maps, local systems, characteristic variety of fundamental group..

characteristic varieties. A consequence of the result of Arapura is that the number of possible dominant maps with connected fibers from a quasiprojective variety X onto hyperbolic (i.e. having negative Euler characteristic) curves is equal to the number $n(X)$ of those irreducible components of positive dimension in the characteristic variety of $\pi_1(X)$ which contain the identity of $Hom(\pi_1(X),\mathbb{C}^*)$ (the number of such $i-$dimensional components we denote by $n_i(X)$).

Here we use the characteristic varieties to find effective bounds for the number of maps and targets which are surfaces of hyperbolic type. For two quasiprojective varieties of hyperbolic type the number $R(X,Y)$ of dominant morphisms of X into Y is finite ([4], [5], [6], [7]). The number of these maps and the number of targets ([8]) was a subject of numerous works. In the case of automorphisms, for a projective variety X with at most terminal singularities, nef canonical bundle $K(X)$ and having an index r and the dimension n, the number of birational automorphisms has polynomial bound of the form $f(n)(rK(X)^n)^{g(n)}$ ([9]). For a minimal surface one has: $\#Aut(X) \leq CK(X)^2$ ([10],[11]); for a smooth hyperbolic curve X of genus g with k punctures $\#Aut(X) \leq A(g,k)$, where

$$A(g,r) \leq \begin{cases} 5k & \text{if } g=0 \\ 6(2g+k-2) & \text{if } k>0, g\geq 1 \\ 84(g-1) & \text{if } k=0 \end{cases} \tag{1}$$

(see [12]).

In the case when $dimX = 1$, the number of maps of a compact curve of genus g onto other hyperbolic curves (up to isomorphism of the image) is bounded by

$$T_c(g) = (g-1)2^{2g^2-2}(2^{2g^2-1}-1) \tag{2}$$

([13], [14],[15]). This bound cannot be significantly improved ([14]).

Non-effective bounds for $R(X,Y)$ were obtained in [16], [17] in the case when X,Y are projective varieties of general type having at most canonical singularities and $K(X), K(Y)$ are nef. The number of targets for threefolds was found in ([18]) and the bounds for some special classes of pairs of (X,Y) were obtained in ([19],[20]. The case of canonically polarized varieties was considered in ([21]).

An effective estimate for the number of holomorphic maps of a quasiprojective X with the target being an arbitrary hyperbolic curve is available in the case when $dimX = 1$. If X is smooth, has genus g and r punctures then the number of its dominant maps with hyperbolic targets is at most

$$T(g,r) = T_c(g) + T_1(g,r) + T_0(g,r), \tag{3}$$

where $T_c(g)$ is given by (2),

$$T_1(g,r) = \begin{cases} r2^{3g^2-1}(2g+r)^{2g^2-2}[\zeta(2g^2-2)\zeta(2g^2-3)] & \text{if } g>1 \\ r^2 ln(r) & \text{if } g=1 \\ 0 & \text{if } g=0 \end{cases} \tag{4}$$

and

$$T_0(g,r) = [3(2g+r-1)]^{(2g+r)} \tag{5}$$

Here $T_1(g,r)$ (resp. $T_0(g,r)$) is the bound for the maps onto punctured tori ([14]) (resp. the number of possible maps onto punctured plane ([22])); ζ is the Riemann ζ-function).

Previous estimates in the above problems about the maps of X with hyperbolic targets depend on the numerical data of X. The purpose of this paper is to describe the estimates in such kind of problems rather in terms of the *fundamental group* and find explicit bounds for the number of maps onto the curve and surfaces for which the characteristic varieties have components of positive dimension.

Sections 2 contain preliminaries. In section 3 we consider maps of projective varieties (Theorems 1 and 2) and in section 4 we find the bounds for the number of maps between smooth affine surfaces S with $n(S)>0$ (Theorems 3,4,5). Here is a sample of our results:

Theorem 4. *Let S be an affine surface of general type with $n(S)\geq 0$, Then*

$$\#Aut(S) \leq \min_{\phi,\nu_\Gamma\neq 0}\{\nu_\Gamma \cdot L(\Gamma)\cdot L_{\phi}\}.$$

where ν_Γ is the length of an orbit of a component of characteristic variety corresponding to a map $\phi: S\to\Gamma$, $L(\Gamma)=\#Aut(\Gamma)$, and F_ϕ is a generic fiber of ϕ and $L_\phi = \max\{\#Aut(F_\phi)\}$ for all such ϕ. In particular

$$\#Aut(S) \leq \min_{n_i(S)\neq 0}\{252 n_i(S)\cdot e(S)\}.$$

As an application of this theorem, in the appendix we calculate the order of the *biholomorphic* automorphisms to the complements to line arrangements. In particular the order of the automorphism of the complement to the Ceva arrangement is equal to 120 and hence contains non linear automorphism (cf. example 1). On the other hand all biholomorphic maps of the complements to Hesse arrangement are linear.

We have to acknowledge that for projective varieties the estimates are so big, that their main importance is just the information about the invariants of the variety, which are responsible for the existence and the number of these maps. It seems that $n(X)$ is just one of these invariants.

For affine varieties (section 4) the situation is different. Theorem 3 provides not only the effective estimate of the number of maps into curves, but shows that the main invariant in this game is the fundamental group. This theorem is an answer to a long - standing question, formulated originally in terms of the holomorphic functions (see Remarks after Theorem 3). The estimate for the number of automorphisms for affine surfaces (Theorem 4) could be obtained without $n(S)$, but then it would include the additional factor of type $T(g, r)$, which is exponential, instead of polynomial one.

Our estimates include the function $T(g, r)$ though we never use its explicit form. Roughly speaking, in projective case we prove that if T is the best estimate for the number of maps into hyperbolic curves, then T^2 (with some modifications) is the estimate for the number of dominant maps into surfaces with special fundamental groups.

The main idea used in this paper is the following. If $S_1 \to S_2$ is a map between the surfaces and S_2, is equipped with a fibration, then S_1 is equipped by a fibration as well. Any map of a fibration with base Γ_1 and a general fiber F_1 into a fibration with base Γ_2 and a general fiber F_2 provides the maps of Γ_1 into Γ_2 and F_1 into F_2. Since all these curves are hyperbolic, there exist estimates for the number of maps. The number of fibrations is given in terms of the fundamental group.

Finally, we use the following terminology: a variety X is of general (hyperbolic) type if it has a smooth compactification $\overline{X}$ such that :

a) $X = \overline{X} - D$, where D is a normal crossing divisor;

b) linear system $|m(K(\overline{X}) + D)|$ provides a birational map for some m.

2. Preliminaries: characteristic variety of the fundamental group and maps onto the curves.

In this section we shall describe the characteristic varieties of the fundamental group in terms of which we estimate of the order of the automorphisms groups. They depends only on the quotient $\pi_1'(X)/\pi_1''(X)$ of the fundamental group by its second commutator. For additional details cf. ([3],[23]).

Let G be a finitely generated, finitely presented group such that the abelianization $G/G' \neq 1$. Let r be a number of generators of G/G' and let s be a surjection: $\mathbb{Z}^r \to G/G'$. We shall consider the exact sequence:

$$0 \to G'/G'' \to G/G'' \to G/G' \to 0$$

where $G'' = [G', G']$ is the second commutator. Since the left group in this sequence is abelian, a lifting the elements in G/G' to elements in G/G'' yields the action of

G/G' on G'/G'' and the module $M_G = G'/G'' \otimes \mathbb{C}$ over the group ring $\mathbb{C}[G/G']$ of G/G'. The characteristic variety C_G of G is the support of this module. This is the subset of $Spec\mathbb{C}[G/G']$ consisting of prime ideals in $\wp \subset \mathbb{C}[G/G']$ such that $M_G \otimes_{\mathbb{C}[G/G']} (\mathbb{C}[G/G']/\wp) \neq 0$.

The surjection s allows to view $Spec\mathbb{C}[G/G']$ as a subset in the torus $\mathbb{C}^{*r}$. C_G has canonical filtration defined as follows. Let $\Phi : \mathbb{C}[G/G']^m \to \mathbb{C}[G/G']^n \to M_G \to 0$ be a presentation of the module M_G with n generators and m relations. Let $F_i(G)$ be the ideal in $\mathbb{C}[G/G']$ generated by $(n-i+1) \times (n-i+1)$ minors of the matrix of Φ (Fitting ideal of the module M_G). Let C_G^k be the (reduced) zero set of $F_k(G)$. We have the inclusions:

$$... \subseteq C_G^{k+1} \subseteq C_G^k \subseteq ... \subseteq C_G^1 \subseteq Spec\mathbb{C}[G/G']$$

All affine varieties C_G^k and ideals $F_k(G)$ are invariants of the fundamental group depending only on G/G''.

Let X be a connected CW-complex. A local system of rank one on X is a homomorphism $\pi_1(X) \to \mathbb{C}^*$ i.e. a character of the fundamental group $G = \pi_1(X)$. Non trivial rank one local systems exist if and only if $G/G' \neq 0$. Clearly the group of characters of G can be identified with $Spec\mathbb{C}[G/G']$. The above filtration on $Spec\mathbb{C}[G/G']$ has a description in terms of local systems as follows.

The cohomology $H^i(X,\rho)$ of a local system ρ can be defined as the cohomology of a chain complex:

$$... \to C_*(\tilde{X},\mathbb{C}) \otimes_{\pi_1(X)} \mathbb{C}_\rho \to ...$$

where $\tilde{X}$ is the universal cover of X, $C_*(\tilde{X},\mathbb{C})$ is a $\mathbb{C}$-vector space of chains on $\tilde{X}$ considered as a $\pi_1(X)$-module with $\pi_1(X)$-action coming from the action on $\tilde{X}$ and $\mathbb{C}_\rho$ is $\mathbb{C}$ with the structure of $\mathbb{C}[G/G']$-module via ρ. The set C_G^k coincides with the set of characters ρ with the property $dim H^1(X,\rho) \geq k$.

In the case when X is a quasi-projective variety and $G = \pi_1(X)$, the varieties C_G^k have a remarkably simple structure (cf. [1]): each C_G^k is a union of translated subgroups of $Spec\mathbb{C}[G/G']$. Moreover, one has the following. Let us call a map $f : X \to C$ *admissible* if there is an extension $\bar{f} : \bar{X} \to \bar{C}$ with smooth $\bar{X}, \bar{C}$ such that $\bar{f}$ has connected fibers (cf. [1]). For any quasi-projective X, there exist unitary characters ρ'_j, the torsion characters ρ_i, the curves C_i and the admissible maps $f_i : X \to C_i$ such that

$$C^1_{\pi_1(X)} = \bigcup \rho_i f_i^*(H^1(C_i)) \cup \bigcup \rho'_j$$

Vise versa, for a surjective map $f : X \to C$ on a curve with $rkH^1(C) \geq 2$ and a character ρ we have $rkH^1(C,\rho) = rkH^1(X, f^*\rho)$ (cf. [1],). Since on a hyperbolic curve $H^1(C,\rho) \neq 0$, the pullbacks of the local systems from C belong to a component of $C^1_{\pi_1(X)}$ containing the identity and in fact fills it. Moreover, the component $\rho_i f_i^*(H^1(C_i))$ belongs to $C^{-e(C)}_{\pi_1(X)}$ and has the dimension equal to $h^1(C)$. Different, modulo automorphisms of the target, admissible f (and C) correspond to different components. Hence the equivalences classes of admissible maps onto hyperbolic curves are in one to one correspondence with components containing the identity (here two maps are equivalent, if they differ by an automorphism of the target). In particular one has enumeration of the holomorphic maps from X onto all possible hyperbolic curves in terms of $\pi_1(X)/\pi_1(X)''$.

The group $Aut(X)$ of automorphisms of X acts of $H^1(X,\mathbb{C}^*) = Char(\pi_1(X)$ and hence on the components of $C^1_{\pi_1(X)}$. In particular, $\#Aut(X) = \nu_C \times \mu_C$ where ν_C is the number of components of $C^1_{\pi_1}$ containing the identity and belonging to the orbit corresponding to a map $X \to C$ and μ_C is the order of the stabilizer of this component in the action of $Aut(X)$ on components of $C^1_{\pi_1(X)}$.

We shall conclude this section with two lemmas:

Lemma 1.

- **a)** *$\varphi_* : \pi_1(S) \to \pi_1(\Gamma)$ and $\varphi_* : h_1(S) \to h_1(\Gamma)$ are surjective;*
- **b)** *([24] Ch. 3, 11.4), [25], Ch. 4. Th.6): If $\overline{S}$ is of general type and $\overline{\Gamma}$ is hyperbolic, then*

$$h_1(\overline{F}) \leq \frac{e(\overline{S})}{|e(\overline{\Gamma})|} + 2. \tag{6}$$

Proof. a) is well known. For b) see ([24] Ch. 3, 11.4), [25], Ch. 4. Th.6)

□

Note that the estimate (6) is sharp: if $\overline{S}$ is a product of two hyperbolic curves, we have the equality.

Lemma 2. *If the surface S is affine, then*

$$|e(F)| \leq \frac{e(S)}{|e(\Gamma)|}.$$

Proof of Lemma 2.

We may assume that D (in notations of Lemma 1) intersects the general fiber normally. So, outside the singular fibers the map φ is locally trivial fiber bundle as well. Thus, we may use the standard procedure ([24] (Ch. 3, 11.4), [25], Ch. 4. Th.6):

Let F be a general and $F_1 \dots F_s$ – the singular fibers of the map φ. Then

$$e(S) = e(F) \cdot e(\Gamma) + \sum_{i=1}^{s} (e(F_i) - e(F)), \tag{7}$$

Similarly to projective case, since we assume that F is connected, we have

$$e(F_i) \geq e(F). \tag{8}$$

Inequality (8) for the affine surfaces was proven by M. Suzuki ([2]6) and later, with more details, by M. Zaidenberg ([27]).

Since S is of general type, F should be hyperbolic and (8),(7) imply

$$e(F) \cdot e(\Gamma) = e(S) - \sum_{i=1}^{s} (e(\tilde{F}_i) - e(\tilde{F})) \geq 0.$$

Thus,

$$|e(F)| \leq \frac{e(S)}{|e(\Gamma)|},$$

and

$$h_1(F) = |e(F)| + 1 \leq \frac{e(S)}{|e(\Gamma)|} + 1.$$

□

3. Projective case.

In this section we describe a bound for the number of maps onto projective surfaces with $n(S) > 0$ ($n(S)$ is the number of components of characteristic variety containing the identity (cf. sect.1). We shall put $T(h) = T_c(h/2)$ (see (2)).

Definition. *We call two maps $f_1 : X \to Y_1$ and $f_1 : X \to Y_2$ equivalent if there exists an isomorphism $\varphi : Y_1 \to Y_2$ such that the diagram*

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Y_1 \arrow[rr, "\varphi"'] & & Y_2 \end{tikzcd}$$

is commutative.

Theorem 1. *Let S be a smooth projective surface. Then for the number $R(S)$ of possible rational dominant non-equivalent maps from S onto all minimal surfaces S_1 of general type with $n(S_1) > 0$, the following inequality is valid:*

$$R(S) \leq \sum_{i=1}^{n(S)} T(a_i) T\left(\frac{e(S)}{a_i - 1} + 2\right) \leq n(S) T(h_1(S)) T(e(S) + 2),$$

where a_i stands for dimension of $i-th$ component of characteristic variety of $\pi_1(S)$.

Remark. If $n(S) > 0$ for a smooth surface S, then $n(S') > 0$ for any other smooth surface S', which is birational to S.

Lemma 3. *Let $f_1 : S \to S_1$ and $f_2 : S \to S_2$ be two rational dominant maps from a smooth surface S onto smooth projective minimal surfaces of general type S_1, S_2 respectively. Let $\varphi_1 : S_1 \to \Gamma$, $\varphi_2 : S_2 \to \Gamma$ and $\varphi : S \to \Gamma$ be morphisms onto a projective hyperbolic curve Γ such that the diagram*

$$\begin{tikzcd} S_1 \arrow[dr, "\varphi_1"'] & S \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "\varphi"] & S_2 \arrow[dl, "\varphi_2"] \\ & \Gamma & \end{tikzcd}$$

is commutative. Let $G_\gamma = \varphi_1^{-1}(\gamma)$, $H_\gamma = \varphi_2^{-1}(\gamma)$, $F_\gamma = \varphi^{-1}(\gamma)$. Assume that there is a Zariski open set $U \subset \Gamma$ such that for each $\gamma \in U$ there exists an isomorphism $h_\gamma : G_\gamma \to H_\gamma$ and the diagram

$$\begin{tikzcd} & F_\gamma \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ G_\gamma \arrow[rr, "h_\gamma"'] & & H_\gamma \end{tikzcd}$$

is commutative as well. Then $S_1 \cong S_2$ and the maps f_1, f_2 are equivalent.

Proof. Our aim is to obtain an isomorphism g between S_1 and S_2 such that the diagram

$$\begin{tikzcd} & S \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ S_1 \arrow[rr, "g"'] & & S_2, \end{tikzcd} \tag{9}$$

is commutative.

Let $\gamma_1, \ldots, \gamma_n$ be the images in Γ of the indeterminacy points of the maps f_1 and f_2. Resolving the singularities of f_1 and f_2 we obtain a new surface which we also denote S. The new map φ takes each exceptional component into a point which is one of the points $\gamma_1, \ldots, \gamma_n$. In particular the exceptional components would not intersect a general fiber. It follows that we may assume that f_1, f_2 are morphisms (resolving singularities of both of them).

Let W be the image of S in $S_1 \times_\Gamma S_2$. Consider the following diagram:

$$\begin{tikzcd} & S \arrow[dl, "f_1"'] \arrow[d, "\pi"] \arrow[dr, "f_2"] & \\ S_1 \arrow[dr, "\varphi_1"'] & W \arrow[l, "f_1'"'] \arrow[r, "f_2'"] & S_2 \arrow[dl, "\varphi_2"] \\ & \Gamma & \end{tikzcd}$$

where $f_1',\ f_2'$ are induced rational maps. Since in the open set $\varphi^{-1}(U)$ the maps f_1' and f_2' are one-to-one in $\pi(\varphi^{-1}(U))$ (due to the isomorphism on the general fiber), it follows that S_1 is birational to S_2. But both are minimal, i.e. $S_1 \cong S_2$. Moreover, for $g = f_2' \circ (f_1')^{-1}$ diagram (9) is commutative. □

Proof of Theorem 1. Let $S_i,\ i = 1, \dots, N$ be surfaces of general type with $n(S_i) > 0$. Let $\varphi_i : S_i \to \Gamma_i$ be regular maps with a connected general fiber onto a hyperbolic curve and $f_i : S \to S_i$ be a rational dominant map. Consider the commutative diagram:

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{f}_i"] & \\ S \arrow[rr, "f_i"'] \arrow[d, "\sigma_i"'] & & S_i \arrow[d, "\varphi_i"] \\ \Sigma_i \arrow[rr, "\varphi_{f_i}"'] & & \Gamma_i \end{tikzcd} \tag{10}$$

where $\pi : \tilde{S} \to S$ is a resolution of singularities of all the maps f_i (i.e., $\tilde{f}_i$ are regular on $\tilde{S}$), Σ_i stands for the Stein factorization of the map $\varphi_i \circ \tilde{f}_i$. Since Γ_i is hyperbolic, for any exceptional curve E of the map π the dimension $\dim \varphi_i \circ \tilde{f}_i(E)$ is zero, i.e., the map σ_i is everywhere defined on S.

If two maps f_1, f_2 are not equivalent, then (due to Lemma 3)

– either the maps $\varphi_{f_i} \circ \sigma_i,\ i = 1, 2$ are not equivalent;

–or the maps $\varphi_{f_i} \circ \sigma_i,\ i = 1, 2$ are equivalent, but the restrictions of f_1 and f_2 on a general fiber $F_\sigma = \sigma_i^{-1}(s),\ i = 1, 2$ are not equivalent for a general $s \in \Sigma_1$ (in this case $\Sigma_1 = \Sigma_2$, since they are the Stein factorization of the same map).

There may be at most $n(S)$ (up to isomorphism) curves Σ_i, which may appear in diagram (10), at most $T(h_1(\Sigma_i))$ non-equivalent maps φ_{f_i} for each Σ_i, and at most $T(h_1(F_{\sigma_i}))$ non-equivalent maps of the fiber. Thus,

$$R(S) \leq \sum_{\Sigma_i} T(h_1(\Sigma_i)) T(h_1(F_{\sigma_i})).$$

Here $h_1(\Sigma_i) = a_i$ is precisely the dimension of the corresponding component of characteristic variety of $\pi_1(S)$ and estimate for $h_1(F_{\sigma_i})$ is provided by Lemma 1.

Thus

$$R(S) \leq \sum_{i=1}^{n(S)} T(a_i) T\left(\frac{e(S)}{a_i - 1} + 2\right).$$

Due to the same Lemma 1 we have $2 \leq a_i \leq h_1(S)$, i.e.

$$\sum_{i=1}^{n(S)} T(a_i) T\left(\frac{e(S)}{a_i - 1} + 2\right) \leq n(S) T(h_1(S)) T(e(S) + 2),$$

□

Remark. *An alternative way to estimate the number of pairs* (Σ_i, Γ_i) *is the following. Since the surface* S *is of general type,* $e(S) \geq e(S_m)$*, where* S_m *is its minimal model. The linear system* $|5K(S_m)|$ *is base point free and provides the birational map (*[28]*). Hence the number of non-equivalent maps* $S \to \Gamma_i$ *does not exceed the same number for the general divisor* $L \in |5K(S_m)|$.

$$g(L) = \frac{(L, L + K(S_m))}{2} + 1 \leq 15(K(S_m))^2 + 1 \leq 45e(S_m) + 1 \leq 45e(S) + 1.$$

Therefore,

$$R(S) \leq T(45e(S) + 1)T(e(S) + 2).$$

This estimate is obviously worse then one, obtained in Theorem 1. For example, if $n(S) = 0$*, then* $R(S) = 0$*. Moreover,* $h_1(S) \leq h_1(L) \leq 45e(S) + 1$.

Theorem 2. *Let* $X \subset \mathbb{P}^N$ *be a smooth projective variety with nef and big canonical bundle and* $n = dimX$*. Let* $s \geq 1$ *be a rational number such that*

1) $sK(X)$ *is a Cartier divisor on* X*;*

2) linear system $|sK(X)|$ *is base point free.*

Then

$$R(X) \leq n(X)T(h_1(X))T(5s^nK(X)^n + 38),$$

where $R(X)$ *is, as above, the number of dominant maps from* X *onto minimal surfaces* S *of general type with* $n(S) > 0$.

Remark. *Due to Theorems of V.Shokurov and Y. Kawamata (*[29], [30]*) such* s *always exists. J.Kollar (*[31]*) proved that any integral* $s > 2(n+2)!(2+n)$ *will do. If* $K(X)$ *is ample, then any* $s > 1/2(n(n+1))$ *will be good (*[32]*). If* $dimX = 3$ *then* $s = 7$ *(*[33]*). Theorem 2 may be proved without much changes for varieties, having at most canonical singularities (then the estimate would depend on the index of* X*), but then we do not know the good effective estimate of* s.

Proof of Theorem 2. We use induction on $n = dimX$.

Let $dimX = 2$. Then, by Theorem 1

$$R(S) \leq n(S)T(h_1(S))T(e(S) + 2).$$

Since for minimal surfaces of general type

$$e(S) \leq 5K(X)^2 + 36$$

([24], Corollary 3.2, ch. VII), we have:

$$R(S) \leq n(S)T(h_1(S))T(5K(S)^2 + 38),$$

and the claim follows for $s \geq 1$.

Now let $dimX = n > 2$. By Bertini Theorem, the general divisor $L \in |sK(X)|$ is a smooth subvariety of X. It has the following properties:

1) $dimL = n - 1$;
2) $K(L) = (K(X) + L)\mid_L = (s+1)K(X)\mid_L$;
3) $K(L)$ is nef and big;
4) $[K(L)]^{n-1} = ((K(X) + L)^{n-1}, L) = (s+1)^{n-1}s[K(X)]^n$;
5) $\frac{s}{s+1}K(L) = [\frac{s}{s+1}(s+1)K(X)]\mid_L = sK(X)\mid_L$ is a Cartier divisor;
6) $\mid \frac{s}{s+1}K(L) \mid$ is a base point free linear system.

Let $\Phi : X \to \mathbb{P}^M$ be morphism, defined by the linear system $|sK(X)|$ and $\overline{\Phi} : \mathbb{P}^N \to \mathbb{P}^M$ some its rational extension into projective space. By construction, $L = \Phi^{-1}(H \cap \Phi(X)) \cap X = H' \cap X$, where H is a hyperplane in $\mathbb{P}^M$ and $H' = \overline{\Phi}^{-1}(H)$, i.e. H' is a hypersurface (smooth for a general H due to Bertini Theorem).

Since a smooth hypersurface in $\mathbb{P}^N$ is its hyperplane section for some Veronese embedding, the Lefschetz Theorem on hyperplane section ([34]) is valid for it , and

$$\pi_1(L) = \pi_1(H' \cap X) \cong \pi_1(X).$$

Hence

7) $n(L) = n(X)$.

The inductive assumption yields:

$$R(L) \leq n(L)T(h_1(L))T\big(5\big(\frac{s}{s+1}\big)^{n-1}K(L)^{n-1} + 38\big) =$$

$$n(X)T(h_1(X))T(5s^nK(X)^n + 38).$$

Since $R(L) \geq R(X)$ for a general L, the Theorem is proved. □

4. Maps into affine curves and surfaces of general (hyperbolic) type.

Let $T'(h) = max\{T(g,r),\ \text{with}\quad 2g + r - 1 = h\}$ (see(3)).

Theorem 3. *Let X be a quasiprojective variety and $N(X)$ – the number of all regular dominant maps from X into hyperbolic curves. Then*

$$N(X) \leq \sum_{i=1}^{n(X)} T'(a_i) \leq n(X) \cdot T'(h_1(X)),$$

where a_i is the dimension of the $i-th$ component of the characteristic variety of $\pi_1(X)$.

Proof of Theorem 3. Let $X \xrightarrow{\varphi} \Gamma$ be a dominant regular map from X into a hyperbolic curve Γ. Let $\overline{X}, \overline{\Gamma}$ be closures of X and Γ, respectively, such that the map φ may be extended to a regular map $\overline{\varphi} : \overline{X} \to \overline{\Gamma}$.

Consider the Stein factorization of $\overline{\varphi}$, i.e. a smooth projective curve $\tilde{\Gamma}$ and a map $\tilde{\varphi} : \overline{X} \to \tilde{\Gamma}$, such that in the commutative diagram

$$\begin{tikzcd} \overline{X} \arrow[rr, "\overline{\varphi}"] \arrow[dr, "\tilde{\varphi}"'] & & \overline{\Gamma} \\ & \tilde{\Gamma} \arrow[ur, "h"'] & \end{tikzcd}$$

the regular map $\tilde{\varphi}$ has connected fibers, and h is finite. Let $\tilde{\Gamma}' = \tilde{\varphi}(X)$. We have

$$N(X) \leq \sum_{\tilde{\Gamma}'} T'(h_1(\tilde{\Gamma}')) \leq \sum_{i=1}^{n(X)} T'(a_i).$$

Since the fibers of $\tilde{\varphi}$ are connected, $h^1(\tilde{\Gamma}) \leq h^1(X)$ (see Lemma 1), and the number of possible $\tilde{\Gamma}'$ is equal to $n(X)$, we have

$$N(X) \leq n(X) \cdot T'(h^1(X)).$$

□

Remark. *Both numbers depend only on the fundamental group of X.*

Remark. *In particular, this Theorem provides positive answer to an old conjecture of V.Lin and E.Gorin (*[22]*,4.2).*

Conjecture*(E.Gorin, V.Lin) The number $\sigma(X)$ of all holomorphic functions omitting two values on affine variety X may be bounded from above by a function, depending on topology of X only.*

This conjecture is true for curves. In [20] *a bound was found for $\sigma(X)$, depending on $h_1(X)$ and $h_2(X)$, when $dim X = 2$. Theorem 3 provides the following estimate.*

Corollary 1.

$$\sigma(X) \leq \sum_{i=1}^{n(X)} \Big(\sum_{2g+r-1=a_i} T_0(g,r)\Big) \leq N(X)$$

has a bound, depending on the fundamental group of X.

Now we shall estimate the number of maps between two affine surfaces of general type with $n(S) > 0$ and meeting some additional conditions. In particular, we are going to estimate $\#Aut(S)$ of all the automorphisms of this class of affine surface S. We want to use the fact that if $n(S) > 0$, then the surface is fibered.

The idea is the following one. Consider all the maps, preserving a fibering. If the map preserve fibering, it maps a fiber onto itself and provides an automorphism of the fiber. If two maps coincide on the open set, they coincide. That means that two different maps may coincide only on the finite number of fibers. Since the number of maps is finite, in the general fiber all the maps are different. Hence, the number of such automorphisms do not exceed he number of automorphisms of a general fiber. Further on we denote by $L(C)$ the number of automorphisms of a curve C.

Theorem 4. *For smooth affine surface S of general type with $n(S) \geq 0$, $\#Aut(S)$ has an estimate*

$$\#Aut(S) \leq \min_{\nu_\Gamma \neq 0} \{\nu_\Gamma \cdot L(\Gamma) \cdot L_\varphi\}. \tag{11}$$

where ν_Γ is the length of an orbit of a component of characteristic variety corresponding to a map $\varphi : S \to \Gamma$, F_φ is a generic fiber of φ and $L_\varphi = \max L(F_\varphi)$ for all such φ.

This estimate may be simplified as

$$\#Aut(S) \leq \min_{n_i(S) \neq 0} \{252 n_i(S) \cdot e(S)\}. \tag{12}$$

Proof of Theorem 4. Let $\varphi : S \to \Gamma$ be a dominant map of S (with a connected general fiber) into a hyperbolic curve Γ, $f : S \to S$ an automorphism.

We have the following diagram:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\varphi"'] & S \arrow[d, "\varphi"'] \\ \Gamma \arrow[r, dashed, "\varphi_f"] & \Gamma \end{tikzcd} \tag{13}$$

Since f is an isomorphism, the general fibers of φ and $\varphi \circ f$ are both connected. That's why if φ_f is defined, it has to be an isomorphism, i.e. φ and $\varphi \circ f$ define the same irreducible component of characteristic variety of $\pi_1(S)$. Then φ induces an isomorphism of $F_\varphi = \varphi^{-1}(\gamma)$ onto $\varphi^{-1}(\varphi_f(\gamma))$ for a general point $\gamma \in \Gamma$. Thus, there may be at most

$$a_\Gamma = L(\Gamma) \cdot L(F_\varphi)$$

automorphisms f for which φ_f is defined.

Assume that diagram (13) cannot be completed by the map φ_f, that is, φ and $\varphi \circ f$ define different components (of the same dimension) of the characteristic variety of $\pi_1(S)$. Let f_1, f_2 be two automorphisms such that $\varphi \circ f_1$, and $\varphi \circ f_2$ define the same component. Then changing f for $f_2 \circ f_1^{-1}$ in diagram (13), we obtain the new diagram (13), which may be completed by the isomorphism of the curve Γ.

Therefore,

$$\#Aut(S) \leq \nu_\Gamma \cdot L(\Gamma) \cdot L_\varphi,$$

We remind that here ν_Γ stands for the number of the components of characteristic variety of S, which correspond to (non-equivalent) maps $\varphi : S \to \Gamma$ belonging to the same orbit, and $L_\varphi = \max L(F_\varphi)$ for all such φ.

Since this is valid for any Γ, provided that a dominant map $S \to \Gamma$ exists, we have

$$\#Aut(S) \leq \min_{\nu_\Gamma \neq 0} \{\nu_\Gamma \cdot L(\Gamma) \cdot L_\varphi\}.$$

Γ may be *apriori* compact, that is why the best estimate is $L(\Gamma) \leq 42|e(\Gamma)|$. On the other hand, F_φ is an affine curve and (due to Lemma 2)

$$|e(F_\varphi)| \leq \frac{e(S)}{|e(\Gamma)|}.$$

Hence, (see(1)),

$$L(F_\varphi) \leq 6\frac{e(S)}{|e(\Gamma)|}.$$

Inserting these estimates into (11), we get (12). □

Remark. *The second bound in this theorem is not sharp, but the bound (11) is. In the Appendix we demonstrate this by examples.*

Theorem 5. *Let S be an affine surface of general type. Let S' be any other surface of general type with $n(S) > 0$. Then the number $R(S,S')$ of dominant morphisms from S into S'*

$$R(S,S') \leq n(S)T'(e(S)+1).$$

Proof of Theorem 5. Let $f : S \to S'$ be a dominant morphism, φ' a map with connected general fiber from S' onto a hyperbolic curve Γ'. Denote:

– $\overline{S}'$ such a closure of S' that the extension $\overline{\varphi}'$ of φ' onto $\overline{S}'$ is a morphism;

– $\pi : \overline{S} \to \tilde{S}$ such a resolution of $\tilde{S}$, that the extension $\overline{f}$ of f onto $\overline{S}$ is a morphism;

– $\overline{\Gamma}'$ - the closure of Γ';

– $\overline{\Gamma}$ -the Stein factorization of $\overline{\varphi}' \circ \overline{f}$;

– $\Gamma = \varphi(S) \subset \overline{\Gamma}$.

The following diagram is commutative:

$$\begin{array}{ccccc}
\tilde{S} & \underset{\pi}{\longleftarrow} & \overline{S} & \xrightarrow{\overline{f}} & \overline{S}' \\
\cup & & \downarrow{\scriptstyle \overline{\varphi}} & & \downarrow{\scriptstyle \overline{\varphi}'} \\
S & & \overline{\Gamma} & \rightarrow & \overline{\Gamma}' \\
& \varphi\searrow & \cup & & \cup \\
& & \Gamma & \underset{\varphi_f}{\longrightarrow} & \Gamma'
\end{array}$$

There may be at most $n(S)$ ways to choose the map φ and ($\overline{\varphi}$) in this digram. For each choice of φ may be at most $T'(h_1(F))$ maps from a fiber $F = \varphi^{-1}(\gamma)$ over a general point $\gamma \in \Gamma$ into a fiber $(\varphi')^{-1}(\varphi_f(\gamma))$. As it was shown in Corollary 1

$$h_1(F) = |e(F)| + 1 \leq \frac{e(S)}{|e(\Gamma)|} + 1.$$

Hence,

$$R(S, S') \leq n(S)T'(e(S) + 1)$$

□

Appendix: Automorphisms of arrangements.

Let $M = \bigcup_{i=1}^{n} L_i$ be an arrangement of n lines in $\mathbb{P}^2$, $S = \mathbb{P}^2 - M$ and $A_1, \ldots, A_t$ be the intersection points of $d(A_i) \geq 3$ lines from M. If $t > 0$, then projection from A_1 yields a regular map $\varphi_1 : S \to \mathbb{P}^1 - \{z_1, \ldots z_d\}$, where $d = d(A_1)$, of S onto a hyperbolic curve. If S is of general (hyperbolic) type, the fibers should be hyperbolic as well. Thus, we may use the above results to estimate the number of automorphisms of S. According to (1) and (11) (we take minimum here over a smaller set, than in (11))

$$\#Aut(S) \leq \min_{0 \leq k \leq t} \{n_{d_k - 1}(S) A(0, d_k) A(0, n - d_k + 1)\} \leq$$

$$\min_{0 \leq k \leq t} \{25 n_{d_k - 1}(S) d_k (n - d_k + 1)\} \leq \frac{25}{4} n(S)(n + 1)^2.$$

Example 1. Consider the arrangement of 6 lines, which are the sides and medians of a triangle.

Let $\{w_1 : w_2 : w_3\}$ be coordinates in $\mathbb{P}^2$, $L_1 = \{w_1 = 0\}$, $L_2 = \{w_2 = 0\}$, $L_3 = \{w_1 = w_2\}$, $L_4 = \{w_1 + w_2 - w_3 = 0\}$, $L_5 = \{w_1 + 2w_2 - w_3 = 0\}$, $L_6 = \{2w_1 + w_2 - w_3 = 0\}$, $S = \mathbb{P}^2 - \bigcup_{i=1}^{4} L_i$.

The vertices of the triangle: $A_1 = (0:0:1),\ A_2 = (1:0:1),\ A_3 = (0:1:1)$, and the intersection point of medians: $A_4 = (1:1:3)$ have $d_i = 3,\ i = 1, ..., 4$. Linear projection from each A_i yields the map φ_i of S onto $\Gamma = \mathbb{P}^1 - \{0, 1, \infty\}$. There is also the fifth map onto Γ :

$$\varphi_5 = \frac{w_1(2w_1 + w_2 - w_3)}{w_2(w_1 + 2w_2 - w_3)}.$$

A calculation with the fundamental group as in section 2 or the results in [3] yield $n(S) = n_2(S) = 5$.

The first component is defined by the fibering $\varphi_1 : S \to \Gamma$, where $\varphi_1 = (w_1 : w_2)$. The general fiber, being $\mathbb{P}^1$ with four general punctures has four automorphisms, i.e. $L_{\varphi_1} = 4,\ L(\Gamma) = A(0, 3) = 6$. Let us show, that all these automorphisms may be realized as the automorphisms of the surface S.

Let $k = w_1/w_2,\ t = (w_3 - w_1 - w_2)/w_2$. In these coordinates $S = \{(k, t) \in \mathbb{C}^2 : (k, t) \notin \cup L'_i\}$, where $L'_1 = \{k = 0\},\ \ L'_2 = \{k = 1\}$, $L'_3 = \{t = 0\},\ \ L'_4 = \{t = 1\}$, $L'_5 = \{t = k\};\ \ \varphi_1(k, t) = k;\ \ \varphi_1^{-1}(k) = F_{\varphi_1} = \{t \neq 0, 1, k\}$. Four automorphisms of the fiber are induced by the following automorphisms of S :

$$f_1(k, t) = (k, t),\quad f_2(k, t) = (k, k/t),\ f_3(k, t) = (k, \frac{k - t}{1 - t}),\quad f_4(k, t) = (k, \frac{k(t - 1)}{t - k}).$$

Six automorphisms of the base $\Gamma = \mathbb{P}^1 - \{0, 1, \infty\}$ may be realized as $g_1(k, t) = (1/k, 1/t), g_2(k, t) = (1-k, 1-t), g_3(k, t) = (1-1/k, 1-1/t), g_4(k, t) = (\frac{1}{1-1/k}, \frac{1}{1-1/t})$, $g_5(k, t) = (\frac{1}{1-k}, \frac{1}{1-t})$, $g_6(k, t) = (k, t)$.

It follows that we have $L(\Gamma) \cdot L_{\varphi_1} = A(0, 3) \times 4 = 6 \times 4 = 24$ automorphisms of the surface S, corresponding to the map φ_1.

Now we want to show that all five components belong to the same orbit. In other words, we want to "connect" the different components of the characteristic variety by automorphisms, i.e. to find $f_i \in Aut(S)$, such that $\varphi_1 \circ f_i$ is $\varphi_i,\ i = 2, ..., 5$ (see digram (13)).

The map $\alpha : \mathbb{P}^2 \to \mathbb{P}^2,\ \alpha(w_1 : w_2 : w_3) = (w_2 : -(w_1 + w_2 - w_3) : w_3)$, permutes the vertices of the triangle, and leaves the point $A_4 = (1 : 1 : 3)$ fixed. Hence, it preserves the arrangement and "connects" the components, defined by $\varphi_1, \varphi_2, \varphi_3$.

The map $\beta : \mathbb{P}^2 \to \mathbb{P}^2,\ \beta(w_1 : w_2 : w_3) = (2w_1 + w_2 - w_3 : w_1 - w_2 : -w_3 + 4w_1)$ sends A_4 to A_1, preserving the arrangement, i.e. "connects" the fourth and the first components.

The map $\gamma : \mathbb{P}^2 \to \mathbb{P}^2,\ \ \gamma(w_1 : w_2 : w_3) = (w_1(w_1 + 2w_2 - w_3) : w_2(2w_1 + w_2 - w_3) : w_1(w_1 + 2w_2 - w_3) + w_2(2w_1 + w_2 - w_3) + w_1w_2)$ connects the first and the fifth component:

$$\gamma(L_1 \cup L_5) = L_1,\ \gamma(L_2 \cup L_6) = L_2,\ \gamma(L_3 \cup L_4) = L_3,\ \gamma(A_1) = L_4,\ \gamma(A_2) = L_5,\ \gamma(A_3) = L_6.$$

$\gamma(L_1') = A_3,\ \gamma(L_2') = A_2,\ \gamma(L_3') = L_3,\ \gamma(L_4') = A_4,\ \gamma(L_5') = L_1,\ \gamma(L_6') = L_2,\ \gamma(A_1) = L_4,\ \gamma(A_2) = L_5,\ \gamma(A_3) = L_6$.

Altogether, $\#Aut(S) = n_2(S)A(0,3)L_{\varphi_1} = 5 \times 6 \times 4 = 120$, which shows that the estimate (11) is sharp.

Example 2. Consider the following arrangement of 12 lines and 9 points:

$$L_1 = \{w_1 = 0\},\ L_2 = \{w_2 = 0\},\ L_3 = \{w_3 = 0\},$$

$$L_4 = \{w_1+w_2+w_3 = 0\},\ L_5 = \{w_1+ew_2+e^2w_3 = 0\},\ L_6 = \{w_1+e^2w_2+ew_3 = 0\},$$

$$L_7 = \{w_1+w_2+ew_3 = 0\},\ L_8 = \{w_1+ew_2+w_3 = 0\},\ L_9 = \{w_1+e^2w_2+e^2w_3 = 0\},$$

$$L_{10} = \{w_1+w_2+e^2w_3 = 0\},\ L_{11} = \{w_1+ew_2+ew_3 = 0\},\ L_{12} = \{w_1+e^2w_2+w_3 = 0\}.$$

Let $N = \{(1:1),(0:1),(e:1),(e^2:1)\}$, where $e^3 = 1$ be the set of 4 points in $\mathbb{P}^1$. There is one component (see[3]), corresponding to the map φ of all the arrangement to $\mathbb{P}^1 - N$. This map sends the point on the cubic

$$a(w_1^3 + w_2^3 + w_3^3) + bw_1w_2w_3 = 0$$

to $(-3a, b)$. The fibers are general tori with 9 punctures, so

$$\nu_\Gamma = 1,\ L(\Gamma) = 12,\ L_\varphi = 18,\ 12 \times 18 = 216$$

Thus, in this case formula (11) gives at most 216, which is sharp ([35] p. 298)

Example 3. Consider the arrangement dual to the one in example 2: 9 lines $L_i, i = 1, ..., 9$ intersecting in 12 points $A_j, j = 1, ..., 12$. At each point three lines are intersecting.

$$L_1 = \{w_2 - w_3 = 0\},\ L_2 = \{w_1 - w_3 = 0\},\ L_3 = \{w_1 - w_2 = 0\},$$

$$L_4 = \{ew_2 - w_3 = 0\},\ L_5 = \{ew_1 - w_3 = 0\},\ L_6 = \{ew_1 - w_2 = 0\}$$

$$L_7 = \{e^2w_2 - w_3 = 0\},\ L_8 = \{e^2w_1 - w_3 = 0\},\ L_9 = \{e^2w_1 - w_2 = 0\}.$$

$$A_1 = (1:0:0),\ A_2 = (0:1:0),\ A_3 = (0:0:1),\ A_4 = (1:1:1),$$

$$A_5 = (1:1:e),\ A_6 = (1:1:e^2), A_7 = (1:e:e^2),\ A_8 = (1:e:1),$$

$$A_9 = (1:e:e),\ A_{10} = (1:e^2:e),\ A_{11} = (1:e^2:e^2),\ A_{12} = (1:e^2:1).$$

Let $S = \mathbb{P}^2 - \cup L_i$. There are 16 2-dimensional components of characteristic variety of S ([3]): 12 maps φ_i to $\Gamma = \mathbb{P}^1 - \{0, 1, \infty\}$, with the only singular point at each $A_i, i = 1, ..., 12$ and four maps $\psi_i, i = 1, ..., 4$, to $\Gamma' \sim \Gamma$, each for a choice of 3 points which are not connected by lines. The fibers of φ_i are lines, the fibers of ψ_i are elliptic curves, thus the components corresponding to φ_i and ψ_i belong to different orbits. On the other hand, one can write explicitly 12 linear automorphisms of S, permuting the points A_i. Hence, $\nu_\Gamma = 12$.

Consider a fiber $F_a = \{\varphi_1 = a\} = \{(w_1 : w_2 : w_3) : w_2 - w_3 = a(ew_2 - w_3)\}$. It is isomorphic to $C = \mathbb{P}^1 - \{0, 1, t, e, et, e^2, e^2 t\}$, where $e^3 = 1$ and $t = \frac{ea-1}{a-1}$, and $L(C) = 3$. Thus the final estimate is $12 \times 6 \times 3 = 216$, which is an expected estimate, because this configuration of points and lines is dual to one from example 2.

Acknowledgments. The authors are grateful to S. Yuzvinsky for his help and to V. Lin for providing the most useful information and references.

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