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McKay correspondence for elliptic genera

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Abstract

We establish a correspondence between orbifold and singular elliptic genera of a global quotient. While the former is defined in terms of the fixed point set of the action, the latter is defined in terms of the resolution of singularities. As a byproduct, the second quantization formula of Dijkgraaf, Moore, Verlinde and Verlinde is extended to arbitrary Kawamata log-terminal pairs.

1. Introduction

One of the fundamental problems suggested by the intersection homology theory is to determine which characteristic numbers can be defined for singular varieties. Elliptic genus appears to be a key tool for a solution to this problem. In [30] it was shown that the Chern numbers invariant in small resolutions are determined by the elliptic genus of such a resolution. In [7] the elliptic genus was defined for singular varieties with $\mathbb{Q}$-Gorenstein, Kawamata-logterminal singularities and its behavior in resolutions of singularities was studied. Among other things, [7] shows that the elliptic genus is invariant in crepant, and in particular small, resolutions, whenever they exist. Hence, the elliptic genus for such class of singular varieties provides the complete class of Chern numbers which is possible to define in such singular setting.

In present work, we study the elliptic genus of singular varieties which are global quotients. We obtain generalizations for several relations between the numerical invariants of actions of finite groups acting on algebraic varieties and invariant of resolutions. Much of the interest in such relations comes from works in physics and the work on Hilbert schemes (cf. [12], [18], [11], [16]) but starts with the work of McKay [28].

The McKay correspondence was originally proposed in [28] as a relation between minimal resolutions of quotient singularities $\mathbb{C}^2/G$, where G is a finite

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subgroup of $\mathrm{SL}_2(\mathbb{C})$, and the representations of G. Shortly after that, L. Dixon, J. Harvey, C. Vafa and E. Witten (cf. [12]) discovered a formula for the Euler characteristic of certain resolutions of quotients:

$$e(\widetilde{X/G}) = \frac{1}{|G|} \sum_{gh=hg} e(X^{g,h}) \tag{1}$$

where X is a complex manifold, $\pi^* : \widetilde{X/G} \to X/G$ is a resolution of singularities such that $\pi^* K_{X/G} = K_{\widetilde{X/G}}$ and $X^{g,f}$ is the submanifold of X of points fixed by both f and g. The right-hand side in (1) can be written as the sum over the conjugacy classes: $\sum_{\{g\}} e(X^g/C(g))$, where $C(g)$ is the centralizer of g, which for $X = \mathbf{C}^2$ is the number of irreducible representations of G. At the same time, the other side in (1) is the number of exceptional curves in a minimal resolution plus 1 and one obtains the McKay correspondence on the numerical level (cf. [18]). The McKay correspondence became the subject of intense study and the term is now primarily used to indicate a relationship between the various invariants of the actions of finite automorphism groups on quasiprojective varieties and resolutions of the corresponding quotients by such actions generalizing (1). We refer to the report [29] for a survey of the evolution of ideas since original empirical observation of McKay.

One of the main results to date on the relationship between the invariants of actions and resolutions of quotients is the description of the E-function of a crepant resolution in terms of the invariants of the action (cf. [5], [10]). We recall that for a quasiprojective variety M its E-function is defined as

$$E(M; u, v) \colon = \sum_{p,q} u^p v^q \sum_n (-1)^n h^{p,q}(H^n_c(M))$$

where $h^{p,q}(H^n_c(M))$ are the Hodge numbers of Deligne's mixed Hodge structure on the compactly supported cohomology of M. The E-function incorporates many classical numerical invariants of manifolds. For example, if M is a projective manifold and $(u, v) = (y, 1)$ one obtains Hirzebruch's χ_y-genus which in turn has the topological and holomorphic Euler characteristics and the signature as its special values.

In [5], Batyrev extended the definition of the E-function to the case of a global quotient of a smooth variety M by a finite group G. He defined the orbifold E-function, $E_{\mathrm{orb}}(M, G; u, v)$ in terms of the action of a finite group G. Moreover, he extended this definition to G-normal pairs (M, D) composed of a smooth variety M and a simple normal crossing G-equivariant divisor D on it. Batyrev showed that the E-function of the pair $(\widetilde{M/G}, D)$ consisting of a resolution $\mu\colon \widetilde{M/G} \to M/G$ and the divisor defined via the discrepancy $D = K_{\widetilde{M/G}} - \mu^*(K_{M/G})$ (with trivial group action) coincides with the orbifold E-function. The fact that the E-function of the pair does not change under birational morphisms, as well as an alternative proof of the McKay correspondence for E-functions are based on Kontsevich's idea of motivic integration (cf. [23], [5], [10], [26]).

Another generalization of Hirzebruch's χ_y-genus is the (two-variable) elliptic genus, and this paper grew from an attempt to prove the relationship between elliptic genera of resolutions of the quotients M/G and the elliptic genera associated with the actions of G on M. These two versions of the elliptic genus of a global quotient were introduced in our previous paper [7] where the McKay correspondence was stated as a conjecture. The proof given below, similarly to Batyrev's approach, requires a generalization of the elliptic genera considered in [7] to the elliptic genus associated with triples consisting of a manifold, the group acting on it and the divisor with simple normal crossings.

The elliptic genus was extensively studied in recent years (cf. [25], [24], [19], [17], [30], [6], [8] and further references in the latter). For an almost complex compact manifold X with Chern roots x_i (i.e. the total Chern class is $\prod(1+x_i)$) the elliptic genus can be defined as

$$\mathrm{Ell}(X;z,\tau)=\int_X\prod_i x_i\frac{\theta(\frac{x_i}{2\pi\mathrm{i}}-z,\tau)}{\theta(\frac{x_i}{2\pi\mathrm{i}},\tau)} \tag{2}$$

where

$$\theta(z,\tau)=q^{\frac{1}{8}}(2\sin\pi z)\prod_{l=1}^{l=\infty}(1-q^l)\prod_{l=1}^{l=\infty}(1-q^ly)(1-q^ly^{-1})$$

is the classical theta function (cf. [9]) where $y=\mathrm{e}^{2\pi\mathrm{i}z}$, $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$.

Alternatively, the elliptic genus can be written as

$$\mathrm{Ell}(X;z,\tau)=\int_X ch(\mathcal{ELL}_{z,\tau})\mathrm{td}(X) \tag{3}$$

where

$$\mathcal{ELL}_{z,\tau}:=y^{-\frac{\dim X}{2}}\otimes_{n\geq1}\Big(\Lambda_{-yq^{n-1}}T_X^*\otimes\Lambda_{-y^{-1}q^n}T_X\otimes S_{q^n}T_X^*\otimes S_{q^n}T_X\Big).$$

Here T_X (resp. T_X^*) is the complex tangent (resp. cotangent) bundle and as usual for a bundle V, $\Lambda_t(V)=\sum_i\Lambda^i(V)t^i$ and $S_t(V)=\sum_i\mathrm{Sym}^i(V)t^i$ denote generating functions for the exterior and symmetric powers of V (by Riemann-Roch this is also the holomorphic Euler characteristic of $\mathcal{ELL}_{z,\tau}$). The elliptic genus of a projective manifold is a holomorphic function of $(z,\tau)\in\mathbb{C}\times\mathcal{H}$. Moreover, if $c_1(X)=0$ then it is a weak Jacobi form (of weight 0 and index $\frac{\dim X}{2}$, see [6] or earlier references in [8]).

Since $y^{-\frac{\dim X}{2}}\chi_{-y}(X)=\lim_{q\to0}\mathrm{Ell}(X;z,\tau)$, Hirzebruch's χ_y-genus is a specialization of the elliptic genus (and so are various one-variable versions of the elliptic genus due to Landweber-Stong, Ochanine, Witten and Hirzebruch). On the other hand, elliptic genus is a combination of the Chern numbers of X, as is apparent from (2), but it cannot be expressed via the Hodge numbers of X (cf. [19], [6]). Therefore the information about elliptic genera of resolutions of X/G cannot be derived from corresponding information about the E-function, though it can be done for the specialization $q \to 0$ of the elliptic genus. Since the elliptic genus depends only on the Chern numbers, it is a cobordism invariant. Totaro [30] found a characterization of the elliptic genus (2) of SU-manifolds from the point of view of cobordisms as the universal genus invariant under classical flops.

A major difference between the elliptic genus and the E-function is that the latter is defined for quasiprojective varieties. Unfortunately, we do not know if a useful definition of the elliptic genus can be given for arbitrary quasiprojective manifolds. Moreover, while the E-function enjoys strong additivity properties there appears to be no analog of them in the case of the elliptic genus. Additivity allows one to work with E-functions not just in the category of manifolds but in the category of of arbitrary quasiprojective varieties. Nevertheless, in [7] (extending [6]) a definition of the elliptic genus for some singular spaces was proposed as follows. Let X be a $\mathbb{Q}$-Gorenstein complex projective variety and $\pi\colon Y \to X$ be a resolution of singularities with the simply normal crossing divisor $\cup E_k, k = 1, \ldots, r$ as its exceptional locus. If the canonical classes of X and Y are related via

$$K_Y = \pi^* K_X + \sum \alpha_k E_k, \tag{4}$$

then

(5)
$$\widehat{\mathrm{Ell}}_Y(X; z, \tau) := \int_Y \Big(\prod_l \frac{(\frac{y_l}{2\pi \mathrm{i}})\theta(\frac{y_l}{2\pi \mathrm{i}} - z)\theta'(0)}{\theta(-z)\theta(\frac{y_l}{2\pi \mathrm{i}})} \Big) \times \Big(\prod_k \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} - (\alpha_k + 1)z)\theta(-z)}{\theta(\frac{e_k}{2\pi \mathrm{i}} - z)\theta(-(\alpha_k+1)z)} \Big)$$

is independent of the resolution π (here e_k are the cohomology classes of the components E_k of the exceptional divisor and y_l are the Chern roots of Y) and depends only on X. $\widehat{\mathrm{Ell}}_Y(X; z, \tau)$ was called *the singular elliptic genus* of X. When $q \to 0$, the singular elliptic genus specializes to the singular χ_y-genus calculated from Batyrev's E-function. We refer the reader to [7] for further discussion of this invariant.

On the other hand, for a finite group G of automorphisms of a manifold X, an *orbifold elliptic genus* was defined in [7] in terms of the action of G on X as follows. For a pair of commuting elements $g, h \in G$, let $X^{g,h}$ be a connected component of the fixed point set of both g and h. Let $TX|_{X^{g,h}} = \oplus V_\lambda, \lambda(g), \lambda(h) \in \mathbb{Q} \cap [0, 1)$, be the decomposition into a direct sum, such that g (resp. h) acts on V_λ as multiplication by $\mathrm{e}^{2\pi \mathrm{i}\lambda(g)}$ (resp. $\mathrm{e}^{2\pi \mathrm{i}\lambda(h)}$). Then

$$\begin{aligned}(6)\quad & E_{\mathrm{orb}}(X, G; z, \tau) \\ &= \frac{1}{|G|} \sum_{gh=hg} \Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda \Big) \prod_\lambda \frac{\theta(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h))} \mathrm{e}^{2\pi \mathrm{i} z \lambda(h) z}[X^{g,h}].\end{aligned}$$

In [7] it was conjectured that these two notions of elliptic genus coincide. More precisely (cf. Conjecture 5.1, *ibid.*), let X be a nonsingular projective variety on which a group G acts effectively by biholomorphic transformations. Let $\mu\colon X \to X/G$ be the quotient map, $D = \sum(\nu_i - 1)D_i$ be the ramification divisor, and let

$$\Delta_{X/G} := \sum_j \left(\frac{\nu_j - 1}{\nu_j}\right)\mu(D_j).$$

Then

$$\mathrm{Ell}_{\mathrm{orb}}(X,G;z,\tau) = \left(\frac{2\pi\mathrm{i}\,\theta(-z,\tau)}{\theta'(0,\tau)}\right)^{\dim X} \widehat{\mathrm{Ell}}(X/G,\Delta_{X/G};z,\tau) \tag{7}$$

where the elliptic genus of the pair $\widehat{\mathrm{Ell}}(X/G,\Delta_{X/G};z,\tau)$ is defined by (5) but with discrepancies α_k obtained from the relation

$$K_Y = \pi^*(K_{X/G} + \Delta_{X/G}) + \sum \alpha_k E_k$$

rather than the relation (4).

The main goal of this paper is to prove the identity (7), which we accomplish in Theorem 5.3. One of the ingredients of the proof is the systematic use of the "hybrid" orbifold elliptic genus of pairs generalizing both the singular and orbifold elliptic genera. It is defined as follows. Let (X,E) be a resolution of singularities of a Kawamata log-terminal pair (cf. [22] and §2) with $E = -\sum_k \delta_k E_k$. Let X support an action of a finite group G such that (X,E) is a G-normal pair (cf. [5] and Section 3). In addition to notation used in the above definition (6) of the orbifold elliptic genus, let $\varepsilon_k(g), \varepsilon_k(h) \in \mathbb{Q}\cap[0,1)$ be defined as follows. If E_k does not contain $X^{g,h}$ then they are zero and if $X^{g,h} \subseteq E_k$ then g (resp. h) acts on $\mathcal{O}(E_k)$ as multiplication by $\mathrm{e}^{2\pi i\varepsilon(g)}$ (resp. $\mathrm{e}^{2\pi i\varepsilon(g)}$). Then we define (cf. Definition 3.2):

$$\begin{aligned}
(8)\quad & \mathcal{ELL}_{\mathrm{orb}}(X,E,G;z,\tau)\\
&:= \frac{1}{|G|}\sum_{g,h,gh=hg}\sum_{X^{g,h}}[X^{g,h}]\Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda\Big)\\
&\quad\times\prod_\lambda \frac{\theta(\frac{x_\lambda}{2\pi\mathrm{i}} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x_\lambda}{2\pi\mathrm{i}} + \lambda(g) - \tau\lambda(h))}\mathrm{e}^{2\pi\mathrm{i}\lambda(h)z}\\
&\quad\times\prod_k \frac{\theta(\frac{e_k}{2\pi\mathrm{i}} + \varepsilon_k(g) - \varepsilon_k(h)\tau - (\delta_k+1)z)}{\theta(\frac{e_k}{2\pi\mathrm{i}} + \varepsilon_k(g) - \varepsilon_k(h)\tau - z)}\frac{\theta(-z)}{\theta(-(\delta_k+1)z)}\mathrm{e}^{2\pi\mathrm{i}\delta_k\varepsilon_k(h)z}.
\end{aligned}$$

If G is trivial, then this expression yields the elliptic genus (5) if $E = \emptyset$ and the version of (5) for pairs as described earlier for arbitrary E. On the other hand, if G is nontrivial but $E = \emptyset$, then one obtains (6). Moreover $\mathrm{Ell}_{\mathrm{orb}}(X,E,G)$ for $q \to 0$ specializes into Batyrev's $E_{\mathrm{orb}}(X,E,G;y,1)$ (cf. [5]). Thus the defined orbifold elliptic genus of pairs is birationally invariant (cf. §3). In fact, we show that the contribution of each pair of commuting elements in the above definition is invariant under the blowups with normal crossing nonsingular G-invariant centers, which allows us to show that the contribution of each pair (g, h) is a birational invariant.

The second main ingredient of the proof is the pushforward formula for the class in (8) for toroidal morphisms. Finally, we use the results of [3] to show that $X \to X/G$ can be lifted to a toroidal map $\hat{Z} \to Z$ so that in the diagram

$$\begin{array}{ccc} \hat{Z} & \to & Z \\ \downarrow & & \downarrow \\ X & \to & X/G \end{array}$$

the vertical arrows are resolutions of singularities.

As was already pointed out, the singular (resp. orbifold) elliptic genus specializes into some known invariants of singular varieties (resp. orbifolds). The simplest corollary of our main theorem is obtained in the limit $q = 0$, $y = 1$. We see that if X/G admits a crepant resolution of singularities (i.e. such that in (4), one has $\alpha_k = 0$ for any k) then the topological Euler characteristic of a crepant resolution is given by the Dixon, Harvey, Vafa and Witten formula (1). While previous proofs of this relation were based on motivic integration (cf. [5], [10]) the proof presented here uses only birational geometry (but depends on [1] and [3]). Moreover, in projective case, the results in [5], [10] for $E(u, 1)$ also get an alternative proof, independent of motivic integration.

Another corollary is the further clarification of a remarkable formula due to Dijkgraaf, Moore, Verlinde and Verlinde. It was shown in [7] that

$$\sum_{n \geq 0} p^n \mathrm{Ell}_{\mathrm{orb}}(X^n/\Sigma_n; z, \tau) = \prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1 - p^i y^l q^m)^{c(mi,l)}} \tag{9}$$

where Σ_n is the symmetric group acting on the product of n copies of a manifold X such that $\mathrm{Ell}(X) = \sum_{m,l} c(m, l) y^l q^m$. A formula of such type was first proposed in [11]. The main theorem of this paper shows that the orbifold elliptic genus in (9) can be replaced by the singular elliptic genus. While for general X it is not clear how to construct a crepant resolution of the symmetric product (or other kind of resolution leading to a calculation of the singular elliptic genus) in the case $\dim X = 2$ it is well-known that the Hilbert scheme $X^{(n)}$ of subschemes of length n in X yields a crepant resolution. A corollary of the main theorem is the the following:

Corollary 6.7. *Let X be a complex projective surface and $X^{(n)}$ be its n^{th} Hilbert scheme. Let $\sum_{m,l} c(m, l) y^l q^m$ be the elliptic genus of X. Then*

$$\sum_{n \geq 0} p^n \mathrm{Ell}(X^{(n)}; z, \tau) = \prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1 - p^i y^l q^m)^{c(mi,l)}}.$$

This is a generalization of results due to Göttsche on the generating series of χ_y-genera of Hilbert schemes (cf. [16]) which one obtains for $q = 0$. In fact in this paper a substantial generalization of (9) is proposed. We are able to extend the DMVV formula to symmetric powers of log-terminal varieties and, more generally, to symmetric powers of Kawamata log-terminal pairs.

The paper is organized as follows. In Section 2 we recall the concept of Kawamata log-terminal pairs, to the extent necessary for our purposes. Section 3 contains our main definition of the orbifold elliptic genus of a Kawamata log-terminal pair. We prove that it is well-defined, for which we use the full force of the machinery of [1]. In Section 4 we introduce toroidal morphisms between pairs that consist of varieties and simple normal crossing divisors on them. Our main result is the description of the pushforward and pullback in the Chow rings in terms of the combinatorics of the conical polyhedral complexes. In the process we use some combinatorial results related to toric varieties, which are collected in the Appendix 8. In Section 5 we apply these calculations to prove our main Theorem 5.3. In Section 6 we generalize the second quantization formula of [11] to the case of Kawamata log-terminal pairs. Various open questions related to our arguments are collected in Section 7.

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2. Kawamata log-terminal pairs

In this section we present the background material for Kawamata log-terminal pairs, which are a standard tool in the minimal model program. Our main reference is [22].

PROPOSITION 2.1 ([22, Def. 2.25, Notation 2.26]). *Let (X, D) be a pair where X is a normal variety and $D = \sum_i a_i D_i$ is a sum of distinct prime divisors on X. We allow a_i to be arbitrary rational numbers. Assume that $m(K_X + D)$ is a Cartier divisor for some $m > 0$. Suppose $f : Y \to X$ is a birational morphism from a normal variety Y. Denote by E_i the irreducible exceptional divisors and the proper preimages of the components of D. Then there are naturally defined rational numbers $a(E_i, X, D)$ such that*

$$K_Y = f^*(K_X + D) + \sum_{E_i} a(E_i, X, D) E_i.$$

Here the equality holds in the sense that a nonzero multiple of the difference is a divisor of a rational function. The number $a(E_i, X, D)$ is called the discrepancy *of E_i with respect to (X,D) and it depends only on E_i, but not on f. By definition $a(D_i,X,D) = -a_i$ and $a(F,X,D) = 0$ for any divisor $F \subset X$ different from all D_i.*

Remark 2.2. In the notation of the above proposition, we will often call the pair $(Y, -\sum_{E_i} a(E_i,X,D)E_i)$ the pair on Y that corresponds to (X,D) or the pullback of (X,D) by f. It is easy to see that for any birational morphism $g\colon Z \to Y$ from a normal variety Z the pullback by g of the pullback of (X,D) by f is equal to the pullback of (X,D) by $f \circ g$.

Definition 2.3. We call a morphism $f\colon Y \to X$ from a nonsingular variety Y to a normal variety X a resolution of singularities of the pair (X,D) if the exceptional locus of f is a divisor with simple normal crossings, which is additionally simple normal crossing with the proper preimage of D. Every pair admits a resolution; see [22, Theorem 0.2].

Definition 2.4. A pair (X,D) is called *Kawamata log-terminal* if there is a resolution of singularities $f\colon Y \to X$ of (X,D) such that the pullback $(Y, -\sum_i \alpha_i E_i)$ satisfies $\alpha_i > -1$ for all i.

Remark 2.5. It is easy to see that our definition of Kawamata log-terminal pair coincides with [22, Definition 2.34] in view of [22, Corollary 2.31]. This corollary also implies that *any* resolution of singularities of a Kawamata log-terminal pair satisfies the condition $\alpha_i > -1$ for all i.

We will also need to describe the behavior of Kawamata log-terminal pairs under finite morphisms, in particular under quotient morphisms. We will use the following result.

PROPOSITION 2.6 ([22, Prop. 5.20]). *Let $g\colon X' \to X$ be a finite morphism between normal varieties. Let D' and D be $\mathbb{Q}$-Weil divisors on X' and X respectively such that*

$$K'_X + D' = g^*(K_X + D).$$

Then $K'_X + D'$ is $\mathbb{Q}$-Cartier if and only if $K_X + D$ is. Moreover, (X',D') is Kawamata log-terminal if and only if (X,D) is.

Definition 2.7. Let G be a finite group which acts effectively on a normal variety X and preserves a $\mathbb{Q}$-Weil divisor D. Let $g\colon X \to X/G$ be the quotient morphism. Then there is a unique divisor D/G on X/G such that

$$g^*(K_{X/G} + D/G) = K_X + D.$$

The components of D/G are the images of the components of D and the images of the ramification divisors of f. We call the pair $(X/G, D/G)$ the *quotient* *of* (X, D) *by* G. By the above proposition, the quotient pair is Kawamata log-terminal if and only if (X, D) is Kawamata log-terminal.

We remark that this definition is contained in [5] in the particular case of a smooth variety X and trivial divisor D. It allows us to generalize the definition of the pullback of a pair to the case of G-equivariant morphisms as follows.

Definition 2.8. Let $g\colon X' \to X$ be a generically finite morphism from a normal G-variety X' to a normal variety X which is birationally equivalent to the quotient morphism $f\colon X' \to X'/G$. We say that a pair (X', D') is a pullback of a pair (X, D) if the pullback of (X, D) to X'/G coincides with the quotient of (X', D') by G. Just as in the birational case, this pullback preserves Kawamata log-terminality.

3. Orbifold elliptic genera of pairs

Definition 3.1 ([5]). Let X be a smooth manifold with the action of a finite group G. Let E be a G-invariant divisor on X. The pair (X, E) is called *G-normal* if $\mathrm{Supp}(E)$ has simple normal crossings and for every point $x \in X$ the action of the isotropy subgroup of x on the set of irreducible components of $\mathrm{Supp}(E)$ that pass through x is trivial.

We will extensively use the theta function $\theta(z, \tau)$ of [9]. By default, the second argument will be τ. We will suppress it from the notation, unless it is different from τ. We will implicitly assume the standard properties of θ, namely its zeroes and transformation properties under the Jacobi group.

Definition 3.2. Let (X, E) be a Kawamata log-terminal G-normal pair (in particular, X is smooth and E has simple normal crossings) with $E = -\sum_k \delta_k E_k$. We define the *orbifold elliptic class* of the triple (X, E, G) as an element of the Chow group $A_*(X)$ by the formula

$$\begin{aligned}
&\mathcal{ELL}_{\mathrm{orb}}(X, E, G; z, \tau) \\
&\quad := \frac{1}{|G|} \sum_{g,h,gh=hg} \sum_{X^{g,h}} (i_{X^{g,h}})_* \Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda \Big) \\
&\qquad \times \prod_{\lambda} \frac{\theta(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h))} \mathrm{e}^{2\pi \mathrm{i}\lambda(h) z} \\
&\qquad \times \prod_{k} \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} + \varepsilon_k(g) - \varepsilon_k(h)\tau - (\delta_k + 1)z)}{\theta(\frac{e_k}{2\pi \mathrm{i}} + \varepsilon_k(g) - \varepsilon_k(h)\tau - z)} \frac{\theta(-z)}{\theta(-(\delta_k + 1)z)} \mathrm{e}^{2\pi \mathrm{i}\delta_k \varepsilon_k(h) z}.
\end{aligned}$$

Here $X^{g,h}$ denotes an irreducible component of the fixed set of the commuting elements g and h and $i_{X^{g,h}} : X_{g,h} \to X$ is the corresponding embedding. The restriction of TX to $X^{g,h}$ has the splitting $\oplus V_\lambda, \lambda(g), \lambda(h) \in \mathbb{Q} \cap [0,1)$, where g (resp. h) acts on V_λ as multiplication by $\mathrm{e}^{2\pi \mathrm{i}\lambda(g)}$ (resp. $\mathrm{e}^{2\pi \mathrm{i}\lambda(h)}$) and x_λ are the Chern roots of V_λ; see [7]. In addition, $e_k = c_1(E_k)$ and $\varepsilon_k \in \mathbb{Q} \cap [0,1)$ is the character of $\mathcal{O}(E_k)$ restricted to $X^{g,h}$ if E_k contains $X^{g,h}$ and is zero otherwise.

We define the *orbifold elliptic genus* $\mathrm{Ell}_{\mathrm{orb}}(X, E, G)$ of (X, E, G) as the degree of the top component of the orbifold elliptic class $\mathcal{ELL}_{\mathrm{orb}}(X, E, G)$.

Remark 3.3. Throughout this section and elsewhere in the paper the Chow groups A_* and A^* will always be thought of as Chow groups with complex coefficients.

Remark 3.4. Notice that in the particular cases of $|G| = 1$ and $E = 0$ the above definition restricts to that of the singular elliptic genus (up to a normalization factor) and orbifold elliptic genus; see [7]. However, the notion of orbifold elliptic class appears to be new.

Remark 3.5. The Kawamata log-terminality assures that we never divide by zero in the above formulas.

Our first goal is to show that the orbifold elliptic class is compatible with blowups.

THEOREM 3.6. *Let (X, E) be a Kawamata log-terminal G-normal pair and let Z be a smooth G-equivariant locus in X which is normal crossing to* $\mathrm{Supp}(E)$. *Let $f : \hat{X} \to X$ denote the blowup of X along Z. We define $\hat{E}$ by $\hat{E} = -\sum_k \delta_k \hat{E}_k - \delta \mathrm{Exc}(f)$ where $\hat{E}_k$ is the proper transform of E_k and δ is determined from $K_{\hat{X}} + \hat{E} = f^*(K_X + E)$. Then $(\hat{X}, \hat{E})$ is a Kawamata log-terminal G-normal pair and*

$$f_* \mathcal{ELL}_{\mathrm{orb}}(\hat{X}, \hat{E}, G; z, \tau) = \mathcal{ELL}_{\mathrm{orb}}(X, E, G; z, \tau).$$

Proof. It is clear that $(\hat{X}, \hat{E})$ is Kawamata log-terminal. Because of the normal crossing conditions on Z and $\mathrm{Supp}(E)$, the divisor $\mathrm{Supp}(\hat{E})$ has simple normal crossings. The G-normality is clearly preserved since the exceptional divisors do not intersect and any intersection of $\hat{E}_k$ on $\hat{X}$ induces an intersection of E_k on X.

We will prove the theorem by showing that for every pair (g, h) and every connected component $X^{g,h}$ the contributions to $f_* \mathcal{ELL}_{\mathrm{orb}}(\hat{X}, \hat{E}, G; z, \tau)$ of connected components $\hat{X}^{g,h}$ such that $f(\hat{X}^{g,h}) \subseteq X^{g,h}$ equals the contribution of $X^{g,h}$ to $\mathcal{ELL}_{\mathrm{orb}}(X, E, G; z, \tau)$. So from now on g, h and $X^{g,h}$ are fixed.

The set of connected components of the fixed point set of $\langle g, h \rangle$ that maps inside $X^{g,h}$ is described as follows. Let $Z^{g,h}$ denote the intersection of $X^{g,h}$ and Z. Since Z is G-equivariant, the intersection is a union of some connected components of $\langle g, h \rangle$-invariant points of Z. Locally at every point of the intersection, Z and $X^{g,h}$ intersect normally, since the normal spaces to $Z^{g,h}$ inside Z and $X^{g,h}$ have different characters. For simplicity, we assume that $Z^{g,h}$ is connected, and we will remark later on the general case.

If $X^{g,h} \neq Z^{g,h}$ then one of the $\hat{X}^{g,h}$ will be obtained as the proper preimage of $X^{g,h}$ under f and will be isomorphic to the blowup of X along $Z^{g,h}$. Other components will lie in the preimage of $Z^{g,h}$ and are described as follows. The restriction of the normal bundle to Z in X to $Z^{g,h}$ splits into character subbundles. For each character Λ the projectivization of the corresponding bundle over $Z^{g,h}$ is naturally embedded into the preimage of $Z^{g,h}$ under f (which is the projectivization of the whole normal bundle to Z restricted to $Z^{g,h}$).

We first concentrate on the case $X^{g,h} \neq Z^{g,h}$. Let N_1 be the subbundle of the normal bundle to $Z^{g,h}$ in X that is the image of the normal bundle of $Z^{g,h}$ in Z. Let N_2 be the subbundle of the normal bundle to $Z^{g,h}$ that is the image of the normal bundle of $Z^{g,h}$ in $X^{g,h}$. Finally let N_3 be the quotient of $NZ^{g,h}$ by the sum of N_1 and N_2. The transversality implies that it is also a bundle, i.e. the rank of the fibers is constant.

Let us calculate the contribution to $f_*\mathcal{ELL}_{\mathrm{orb}}(\hat{X}, \hat{E}, G; z, \tau)$ that comes from $\hat{X}_0^{g,h}$, which is the proper preimage of $X^{g,h}$, provided $N_2 \neq 0$. As in [7], we make a technical assumption that all bundles we consider are restrictions of some bundles defined on X. We will later explain why this assumption can be dropped. The calculation follows closely those of [7]. We have

$$c(T\hat{X}) = c(f^*TX)(1+\hat{z})\prod_i \frac{(1+f^*m_i-\hat{z})}{(1+f^*m_i)}$$

where $\hat{z}$ is the first Chern class of the exceptional divisor of f and $\prod_i(1+m_i)$ is the Chern class of the bundle on X whose restriction to Z is the normal bundle of Z in X. Similarly,

$$c(T\hat{X}_0^{g,h}) = c(f^*TX^{g,h})(1+\hat{z})\prod_i \frac{(1+f^*s_i-\hat{z})}{(1+f^*s_i)}$$

where $\hat{z}$ and f are restrictions to $X_0^{g,h}$ (mild abuse of notation) and $\prod_i(1+s_i)$ restricts to $c(N_2)$ on $Z^{g,h}$.

Thus the Chern class of the normal bundle to $\hat{X}_0^{g,h}$ is

$$c(N\hat{X}_0^{g,h}) = c(f^*NX^{g,h})\prod_i \frac{(1+f^*t_i-\hat{z})}{(1+f^*t_i)}$$

where $\prod_i(1+t_i)$ restricts to $c(N_3)$ on $Z^{g,h}$.

We will also need to know how the E_i change. For E_i that do not contain Z we have $\hat{E}_i = f^*E_i$, and for E_i that contain Z we have $\hat{E}_i = f^*E_i - \hat{Z}$.

As a result, the contribution of $\hat{X}_0^{g,h}$ to $\mathcal{ELL}_{\rm orb}(\hat{X},\hat{E},G;z,\tau)$ is

$$
\begin{aligned}
&(i_{X_0^{g,h}})_*\prod_i \frac{x_i\theta(\frac{x_i}{2\pi \mathrm{i}}-z)}{\theta(\frac{x_i}{2\pi \mathrm{i}})}\prod_{N_2}\frac{(f^*n_i-\hat{z})\theta(\frac{f^*n_i-\hat{z}}{2\pi \mathrm{i}}-z)}{\theta(\frac{f^*n_i-\hat{z}}{2\pi \mathrm{i}})}\frac{\theta(\frac{f^*n_i}{2\pi \mathrm{i}})}{(f^*n_i)\theta(\frac{f^*n_i}{2\pi \mathrm{i}}-z)}\\
&\times\frac{(\frac{\hat{z}}{2\pi \mathrm{i}})\theta(\frac{\hat{z}}{2\pi \mathrm{i}}-z)\theta'(0)}{\theta(\frac{\hat{z}}{2\pi \mathrm{i}})\theta(-z)}\\
&\times\prod_{N_1}\frac{\theta(\frac{f^*n_i}{2\pi \mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau-z)}{\theta(\frac{f^*n_i}{2\pi \mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau)}\mathrm{e}^{2\pi \mathrm{i}\lambda_i(h)z}\\
&\times\prod_{N_3}\frac{\theta(\frac{f^*n_i-\hat{z}}{2\pi \mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau-z)}{\theta(\frac{f^*n_i-\hat{z}}{2\pi \mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau)}\mathrm{e}^{2\pi \mathrm{i}\lambda_i(h)z}\\
&\times\prod_{E_i\supset Z}\frac{\theta(\frac{f^*e_i-\hat{z}}{2\pi \mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-(\delta_i+1)z)}{\theta(\frac{f^*e_i-\hat{z}}{2\pi \mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-z)}\frac{\theta(-z)}{\theta(-(\delta_i+1)z)}\mathrm{e}^{2\pi \mathrm{i}\delta_i\varepsilon_i(h)z}\\
&\times\prod_{E_i\not\supset Z}\frac{\theta(\frac{f^*e_i}{2\pi \mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-(\delta_i+1)z)}{\theta(\frac{f^*e_i}{2\pi \mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-z)}\frac{\theta(-z)}{\theta(-(\delta_i+1)z)}\mathrm{e}^{2\pi \mathrm{i}\delta_i\varepsilon_i(h)z}\\
&\times\frac{\theta(\frac{\hat{z}}{2\pi \mathrm{i}}-(\delta+1)z)}{\theta(\frac{\hat{z}}{2\pi \mathrm{i}}-z)}\frac{\theta(-z)}{\theta(-(\delta+1)z)}.
\end{aligned}
$$

In the above formula the first two lines account for the tangent bundle to $\hat{X}_0^{g,h}$, the next two lines account for the normal bundle to it, and the remaining three lines account for the divisors. We use the notation $\prod_{N_i}$ to indicate the product over the Chern roots of the corresponding bundle. Notice the normalization factor in the second line. The symbol $i_{X_0^{g,h}}$ denotes the embedding of $X_0^{g,h}$ into $\hat{X}$.

As in [7], we rewrite the above expression as a power series $\sum_n R_n\hat{z}^n$ in $\hat{z}$. Clearly, f_*R_0 is precisely the contribution of the $X^{g,h}$ to $\mathcal{ELL}_{\rm orb}(X,E,G;z,\tau)$. If we denote $r=rkN_2$, we have $f_*\hat{z}^{r+n}=i_*(s_n(i^*N_2))(-1)^{n+r-1}$ where i_* is the pushforward from $Z^{g,h}$ to $X^{g,h}$. We can therefore rewrite the contribution of $f_*R_{>0}$ as

$$(i_{Z^{g,h}})_*\sum_{n\geq 0}s_n(i^*N_2)(-1)^{n+r-1}(\text{Coeff. at }\hat{z}^{r+n})(\text{above expression})$$

where $i_{Z^{g,h}}$ is the embedding on $Z^{g,h}$ into X. Taking into account

$$\sum_{n\geq 0}s_n(i^*N_2)(-1)^n t^{-n}=\frac{t^r}{\prod_{N_2}(t-n_i)},$$

we can rewrite this as

$$
\begin{aligned}
&(-1)\mathrm{Res}_{t=0}(i_{Z^{g,h}})_*\frac{\theta(\frac{t}{2\pi\mathrm{i}}-(\delta+1)z)\theta'(0)}{(2\pi\mathrm{i})\theta(\frac{t}{2\pi\mathrm{i}})\theta(-(\delta+1)z)}\prod_{TZ^{g,h}}\frac{y_i\theta(\frac{y_i}{2\pi\mathrm{i}}-z)}{\theta(\frac{y_i}{2\pi\mathrm{i}})}\\
&\quad\times\prod_{N_1}\frac{\theta(\frac{f^*n_i}{2\pi\mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau-z)}{\theta(\frac{f^*n_i}{2\pi\mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau)}\mathrm{e}^{2\pi\mathrm{i}\lambda_i(h)z}\prod_{N_2}\frac{\theta(\frac{f^*n_i-t}{2\pi\mathrm{i}}-z)}{\theta(\frac{f^*n_i-t}{2\pi\mathrm{i}})}\\
&\quad\times\prod_{N_3}\frac{\theta(\frac{f^*n_i-t}{2\pi\mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau-z)}{\theta(\frac{f^*n_i-t}{2\pi\mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau)}\mathrm{e}^{2\pi\mathrm{i}\lambda_i(h)z}\\
&\quad\times\prod_{E_i\supset Z}\frac{\theta(\frac{f^*e_i-t}{2\pi\mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-(\delta_i+1)z)}{\theta(\frac{f^*e_i-t}{2\pi\mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-z)}\frac{\theta(-z)}{\theta(-(\delta_i+1)z)}\mathrm{e}^{2\pi\mathrm{i}\delta_i\varepsilon_i(h)z}\\
&\quad\times\prod_{E_i\not\supset Z}\frac{\theta(\frac{f^*e_i}{2\pi\mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-(\delta_i+1)z)}{\theta(\frac{f^*e_i}{2\pi\mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-z)}\frac{\theta(-z)}{\theta(-(\delta_i+1)z)}\mathrm{e}^{2\pi\mathrm{i}\delta_i\varepsilon_i(h)z}.
\end{aligned}
$$

We will denote the expression above by $F(t)$, to be thought of as a meromorphic function on $\mathbb{C}$ with values in the Chow group $A_*(Z^{g,h})$.

Let us now calculate the contributions from other components $\hat{X}^{g,h}$ that map inside $X^{g,h}$. As we have discussed earlier, these components correspond to nontrivial characters Λ that are present in N_3. We want to find the normal and tangent bundles of $X^{g,h}_\Lambda \cong \mathbb{P}N_\Lambda$ inside $\hat{X}$. The Chern class of the tangent bundle can be described as the restriction from $\hat{X}$ of

$$\prod_{N_\Lambda}(1+f^*n_i-\hat{z})\prod_{TZ^{g,h}}(1+f^*y_i),$$

so the normal bundle has Chern class which is a restriction of

$$(1+\hat{z})\prod_{N_1}(1+f^*n_i)\prod_{N_2\oplus N_3/N_\Lambda}(1+f^*n_i-\hat{z}).$$

Therefore, the contribution of $\hat{X}^{g,h}_\Lambda$ to $\mathcal{ELL}_{\mathrm{orb}}(\hat{X},\hat{E},G)$ is

$$
\begin{aligned}
&(i_{X^{g,h}_\Lambda})_*\frac{\theta'(0)}{2\pi\mathrm{i}\theta(-z)}\prod_{N_\Lambda}\frac{(f^*n_i-\hat{z})\theta(\frac{f^*n_i-\hat{z}}{2\pi\mathrm{i}}-z)}{\theta(\frac{f^*n_i-\hat{z}}{2\pi\mathrm{i}})}\prod_{TZ^{g,h}}\frac{f^*y_i\theta(\frac{f^*y_i}{2\pi\mathrm{i}}-z)}{\theta(\frac{f^*y_i}{2\pi\mathrm{i}})}\\
&\quad\times\frac{\theta(\frac{\hat{z}}{2\pi\mathrm{i}}+\Lambda(g)-\Lambda(h)\tau-z)}{\theta(\frac{\hat{z}}{2\pi\mathrm{i}}+\Lambda(g)-\Lambda(h)\tau)}\mathrm{e}^{2\pi\mathrm{i}\Lambda(h)z}\\
&\quad\times\prod_{N_1}\frac{\theta(\frac{f^*n_i}{2\pi\mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau-z)}{\theta(\frac{f^*n_i}{2\pi\mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau)}\mathrm{e}^{2\pi\mathrm{i}\lambda_i(h)z}\\
&\quad\times\prod_{N_2\oplus N_3/N_\Lambda}\frac{\theta(\frac{f^*n_i-\hat{z}}{2\pi\mathrm{i}}+(\lambda_i-\Lambda)(g)-(\lambda_i-\Lambda)(h)\tau-z)}{\theta(\frac{f^*n_i-\hat{z}}{2\pi\mathrm{i}}+(\lambda_i-\Lambda)(g)-(\lambda_i-\Lambda)(h)\tau)}\mathrm{e}^{2\pi\mathrm{i}(\lambda_i-\Lambda)(h)z}\\
&\quad\times\prod_{E_i\supset Z}\frac{\theta(\frac{f^*e_i-\hat{z}}{2\pi\mathrm{i}}+(\varepsilon_i-\Lambda)(g)-(\varepsilon_i-\Lambda)(h)\tau-(\delta_i+1)z)\theta(-z)}{\theta(\frac{f^*e_i-\hat{z}}{2\pi\mathrm{i}}+(\varepsilon_i-\Lambda)(g)-(\varepsilon_i-\Lambda)(h)\tau-z)\theta(-(\delta_i+1)z)}\mathrm{e}^{2\pi\mathrm{i}\delta_i(\varepsilon_i-\Lambda)(h)z}
\end{aligned}
$$

$$\times\prod_{E_i\not\supset Z}\frac{\theta(\frac{f^*e_i}{2\pi\mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-(\delta_i+1)z)}{\theta(\frac{f^*e_i}{2\pi\mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-z)}\frac{\theta(-z)}{\theta(-(\delta_i+1)z)}\mathrm{e}^{2\pi\mathrm{i}\delta_i\varepsilon_i(h)z}.$$

$$\times\frac{\theta(\frac{\hat z}{2\pi\mathrm{i}}+\Lambda(g)-\Lambda(h)\tau-(\delta+1)z)}{\theta(\frac{\hat z}{2\pi\mathrm{i}}+\Lambda(g)-\Lambda(h)\tau-z)}\frac{\theta(-z)}{\theta(-(\delta+1)z)}\mathrm{e}^{2\pi\mathrm{i}\delta\Lambda(h)z}$$

where $i_{X_\Lambda^{g,h}}$ is the embedding of $X_\Lambda^{g,h}$ into $\hat X$. Here we used the fact that the line bundle $\mathcal{O}(\hat Z)$ has character Λ on $X_\Lambda^{g,h}$. We again expand the integrand in terms of powers of $\hat z$ and use $f_*\hat z^{l-1+n}=s_n(N_\Lambda)(-1)^{l-1+n}$ where $l=rk(N_\Lambda)$, to rewrite the pushforward to X of the above as

$$(-1)\mathrm{Res}_{t=0}(i_{Z^{g,h}})_*\frac{\theta(\frac{t}{2\pi\mathrm{i}}+\Lambda(g)-\Lambda(h)\tau-(\delta+1)z)\theta'(0)}{(2\pi\mathrm{i})\theta(\frac{t}{2\pi\mathrm{i}}+\Lambda(g)-\Lambda(h)\tau)\theta(-(\delta+1)z)}\mathrm{e}^{2\pi\mathrm{i}(\delta+1)\Lambda(h)z}$$

$$\times\prod_{TZ^{g,h}}\frac{y_i\theta(\frac{y_i}{2\pi\mathrm{i}}-z)}{\theta(\frac{y_i}{2\pi\mathrm{i}})}$$

$$\times\prod_{N_1}\frac{\theta(\frac{f^*n_i}{2\pi\mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau-z)}{\theta(\frac{f^*n_i}{2\pi\mathrm{i}}+\lambda_i(g)-\lambda_i(h)\tau)}\mathrm{e}^{2\pi\mathrm{i}\lambda_i(h)z}$$

$$\times\prod_{N_2}\frac{\theta(\frac{f^*n_i-t}{2\pi\mathrm{i}}+\Lambda(g)-\Lambda(h)\tau-z)}{\theta(\frac{f^*n_i-t}{2\pi\mathrm{i}}+\Lambda(g)-\Lambda(h)\tau)}\mathrm{e}^{-2\pi\mathrm{i}\Lambda(h)z}$$

$$\times\prod_{N_3}\frac{\theta(\frac{f^*n_i-t}{2\pi\mathrm{i}}+(\lambda_i-\Lambda)(g)-(\lambda_i-\Lambda)(h)\tau-z)}{\theta(\frac{f^*n_i-t}{2\pi\mathrm{i}}+(\lambda_i-\Lambda)(g)-(\lambda_i-\Lambda)(h)\tau)}\mathrm{e}^{2\pi\mathrm{i}(\lambda_i-\Lambda)(h)z}$$

$$\times\prod_{E_i\supset Z}\frac{\theta(\frac{f^*e_i-t}{2\pi\mathrm{i}}+(\varepsilon_i-\Lambda)(g)-(\varepsilon_i-\Lambda)(h)\tau-(\delta_i+1)z)\theta(-z)}{\theta(\frac{f^*e_i-t}{2\pi\mathrm{i}}+(\varepsilon_i-\Lambda)(g)-(\varepsilon_i-\Lambda)(h)\tau-z)\theta(-(\delta_i+1)z)}\mathrm{e}^{2\pi\mathrm{i}\delta_i(\varepsilon_i-\Lambda)(h)z}$$

$$\times\prod_{E_i\not\supset Z}\frac{\theta(\frac{f^*e_i}{2\pi\mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-(\delta_i+1)z)\theta(-z)}{\theta(\frac{f^*e_i}{2\pi\mathrm{i}}+\varepsilon_i(g)-\varepsilon_i(h)\tau-z)\theta(-(\delta_i+1)z)}\mathrm{e}^{2\pi\mathrm{i}\delta_i\varepsilon_i(h)z}$$

which can be rewritten as

$$(-1)\mathrm{Res}_{t=\Lambda(g)-\Lambda(h)\tau}F(t)$$

because the additional exponential factors cancel due to $\delta=\sum_{E_i\supset Z}\delta_i+rk(N_2)+rk(N_3)-1$.

So in the case $X^{g,h}\neq Z^{g,h}$ all we need is to show that

$$\mathrm{Res}_{t=0}F(t)+\sum_{\Lambda}\mathrm{Res}_{t=\Lambda(g)-\Lambda(h)\tau}F(t)=0.$$

This follows from the observation that F is periodic with respect to $t\to t+2\pi\mathrm{i}$ and $t\to t+2\pi\mathrm{i}\tau$ and has poles at 0 and $\Lambda(g)-\Lambda(h)$ only. Indeed, the periodicity is a corollary of the transformation properties of θ and the definition of δ. The statement on poles follows from the fact that for every $E_i\supset Z$ the theta function in the denominator is precisely offset by of the theta functions in the numerator. Indeed, in view of the normal crossing condition on $\mathrm{Supp}(E)$ and Z, each E_k gives a quotient bundle of the normal bundle to Z and the sum over all E_k is (locally) a quotient of $N_2 \oplus N_3$. As a result, e_k is a Chern root of N_3 or N_2 depending on whether or not E_k contains $X^{g,h}$.

As in [7], we remark that we can ignore the assumption that the N_i come from bundles on X, because the expression for $F(t)$ makes sense without it and deformation to the normal cone can be used in general. We also observe that in the case when $Z^{g,h}$ has several connected components, the above calculation shows that the contributions of the components, other than $X_0^{g,h}$, to $f_*\mathcal{ELL}_{\mathrm{orb}}(\hat{X}, \hat{E}, G; z, \tau)$ cancel the $f_*R_{>0}$ contributions of the connected component $X_0^{g,h}$. The f_*R_0 contribution of $X_0^{g,h}$ is again the contribution of $X^{g,h}$ to $\mathcal{ELL}_{\mathrm{orb}}(X, E, G; z, \tau)$.

The case $X^{g,h} = Z^{g,h}$ is handled similarly. This time, the contributions to $\mathrm{Ell}(\hat{X}, \hat{E}, G; z, \tau)$ equal

$$-\int_{Z^{g,h}} \sum_{\Lambda} \mathrm{Res}_{t=\Lambda(g)-\Lambda(h)\tau} F(t) = \int_{X^{g,h}} \mathrm{Res}_{t=0} F(t)$$

which is precisely the contribution of $X^{g,h}$ to $\mathrm{Ell}(X, E, G; z, \tau)$. Indeed, since $N_2 = 0$, and no divisor E_i that contains Z can have $\varepsilon = 0$, $F(t)$ has a simple pole at $t = 0$ and the residue is easy to calculate. Similar calculation works at the elliptic class level. □

We will now use the invariance under blowups to define the orbifold elliptic genus and orbifold elliptic class for an arbitrary G-equivariant Kawamata log-terminal pair.

Definition 3.7. Let (Z, D) be an arbitrary G-equivariant Kawamata log-terminal pair with no additional conditions on its singularities. Let $\pi : X \to Z$ be a G-equivariant resolution of singularities of (Z, D), such that the corresponding pair (X, E) is G-normal. Then the orbifold elliptic class of (Z, D) in $A_*(Z)$ is defined as the pushforward π_* of the orbifold elliptic class of (X, E) and the orbifold elliptic genus of (Z, D) is defined as the orbifold elliptic genus of (X, E) or alternatively as the degree of the orbifold elliptic class.

Clearly, this definition does not make sense unless we can prove that it does not depend on the resolution π.

THEOREM 3.8. *Definition* 3.7 *makes sense*; *that is*, *the pushforwards of the orbifold elliptic classes do not depend on the resolution of singularities.*

Proof. In view of Theorem 3.6, it is enough to show that any two G-normal resolutions of singularities (X_-, E_-) and (X_+, E_+) of (Z, D) can be connected by a sequence of equivariant blowups and blowdowns among G-normal resolutions of singularities of (Z, D). This is a G-normality strengthening of the equivariant version of the Weak Factorization Theorem of [1]. The equivariant version itself assures that such a sequence of blowups and blowdowns exists in the category of simple normal crossing G-equivariant divisors E.

In order to get G-normality, observe that for every simple normal crossing G-equivariant divisor E on smooth X there is a canonical sequence of blowups that makes the preimage G-normal. Namely, this is the toroidal morphism that corresponds to the barycentric subdivision of the corresponding polyhedral complex (see Section 5.6 of [1]). In the notation of Section 4.3 of [1], we apply this procedure in the definition of $W_{i\pm}^{\mathrm{res}}$. Then the additional sequences of blowups $r_{i\pm}$ preserve G-normality and the statement is reduced to the case of the toroidal birational map φ_i^{can}. The group G acts by interchanging the vertices of the polyhedral complexes $\Delta_\pm$ of $W_{i\pm}^{\mathrm{can}}$. We apply the barycentric subdivision blowup to both of them, and then observe that all intermediate varieties in the toroidal version of weak factorization have G-normal divisors. Indeed, each of them comes from a subdivision Δ of $B\Delta_+$ or $B\Delta_-$, where B stands for barycentric subdivision, and we assume the former with no loss of generality. If a cone C in Δ maps to itself by some group element $g \in G$, then the same is true for the smallest cone C_+ in $B\Delta_+$ that contains its image. However as observed in Section 5.6 of [1], this implies that g acts trivially on the span of C^*, hence on C. This implies G-normality, since every fixed point of g comes from a stratum that corresponds to some cone of Δ. $\square$

Remark 3.9. The Weak Factorization Theorem also works in the category of G-strict divisors, defined by the condition that the translates of every irreducible component of E are either equal or disjoint. Indeed, the above argument works, since G-strictness is preserved under normal crossing G-equivariant blowups with smooth centers and the barycentric subdivision assures G-strictness, not just G-normality.

Remark 3.10. It is clear from the definition that the orbifold elliptic genus of a log-terminal G-variety is unchanged under equivariant crepant morphisms.

Remark 3.11. The arguments of this section clearly show that the contribution of each pair (g, h) of commuting elements of G to the orbifold elliptic class and genus is well-defined. Indeed, in the proof of Theorem 3.6 each pair was considered separately.

Remark 3.12. The orbifold elliptic genus for the product of triples (X_1, E_1, G_1) and (X_2, E_2, G_2) equals the product of elliptic genera. The product of the triples is defined as the product of the varieties, the sum of the pullbacks of the divisors and the direct product of the group actions.

We observe that our definition of orbifold elliptic genus is compatible with the definition of the orbifold string E-function of $E_{\mathrm{orb}}(X, E, G)$ of [5] in the sense that the limit of the orbifold elliptic genus as $\tau \to \mathrm{i}\infty$ recovers the orbifold string function analog of the χ_y-genus. For this, we will need the following easy lemma.

LEMMA 3.13. *Let X be a complete stratified G-variety with at most quotient singularities such that the action of G is effective and free and preserves the stratification. Let X_1 be any stratum of X and let G_1 be the subgroup of G that maps X_1 to itself. Then*

$$\chi_y(X_1/G_1) = \frac{1}{|G_1|}\chi_y(X_1).$$

Proof. We will argue by induction on the dimension of the stratum. In dimension zero the freeness of the action implies $|G_1| = 1$ and $\chi_y(X_1/G_1) = \chi_y(X_1) = 1$. For the induction step, it is enough to assume that $\overline{X_1} = X$ and X is connected. It is easy to see that the induction assumption allows us to consider X_1 to be a part of the nonsingular locus of X. After an equivariant desingularization, we may assume that X is smooth and X_1 is the open stratum. Notice that desingularization preserves the freeness of the action, which implies

$$\chi_y(\overline{X/G}) = \frac{1}{|G|}\chi_y(\overline{X}).$$

By additivity of χ_y, we can split the above identity according to the contributions of the strata. Each stratum Y_1 in $\overline{X/G}$ is a quotient of a stratum Y in X. If H is the subgroup of G that fixes Y, then there are $|G:H|$ disjoint strata of X that map to Y_1. By the induction assumption, $\chi_y(Y_1) = \frac{1}{|H|}\chi_y(Y) = \frac{1}{|G|}\sum_{\{gY\}}\chi_y(gY)$ where the sum is taken over the cosets of H. Consequently, the terms corresponding to smaller dimensional strata cancel, which finishes the proof of the lemma. We remark that the statement generally fails for free actions on noncomplete varieties. It is crucial that the action stays free on the completion of the stratum. $\square$

PROPOSITION 3.14. *Let $E_{\mathrm{orb}}(X, E, G; u, v)$ be defined as in* [5]. *Then*

$$\lim_{\tau\to \mathrm{i}\infty} \mathrm{Ell}_{\mathrm{orb}}(X, E, G; z, \tau) = y^{-\frac{\dim X}{2}} E_{\mathrm{orb}}(X, E, G; y, 1)$$

where $y = \mathrm{e}^{2\pi \mathrm{i} z}$.

Proof. From the product formula for θ, i.e.:

$$\theta(z, \tau) = q^{\frac{1}{8}}(2\sin\pi z)\prod_{l=1}^{l=\infty}(1 - q^l)\prod_{l=1}^{l=\infty}(1 - q^l\mathrm{e}^{2\pi \mathrm{i} z})(1 - q^l\mathrm{e}^{-2\pi \mathrm{i} z})$$

(see [9]), we have

$$\lim_{\tau\to \mathrm{i}\infty}\frac{\theta(u-\beta,\tau)}{\theta(u,\tau)} = \frac{(1-\mathrm{e}^{-2\pi \mathrm{i}(u-\beta)})}{(1-\mathrm{e}^{-2\pi \mathrm{i} u})}\mathrm{e}^{-\pi \mathrm{i}\beta}$$

and

$$\lim_{\tau\to i\infty}\frac{\theta(u-\alpha\tau-\beta,\tau)}{\theta(u-\alpha\tau,\tau)}=\mathrm{e}^{-\pi i\beta}$$

for $0<\alpha<1$. Hence, by taking the limit in Definition 3.2,

$$\begin{aligned}
&\lim_{\tau\to i\infty}\mathrm{Ell}_{\mathrm{orb}}(X,E,G;z,\tau)\\
&=\frac{1}{|G|}\sum_{g,h,gh=hg}\sum_{X^{g,h}}\int_{X^{g,h}}\prod_{\lambda(g)=\lambda(h)=0}x_\lambda\frac{(1-\mathrm{e}^{-x_\lambda+2\pi iz})}{(1-\mathrm{e}^{-x_\lambda})}\\
&\quad\times\mathrm{e}^{-\pi i(\dim X)z}\mathrm{e}^{2\pi i(\sum_\lambda\lambda(h))z}\prod_{\lambda(h)=0,\lambda(g)\neq0}\frac{(1-\mathrm{e}^{-x_\lambda-2\pi i\lambda(g)+2\pi iz})}{(1-\mathrm{e}^{-x_\lambda-2\pi i\lambda(g)})}\\
&\quad\times\mathrm{e}^{2\pi i\sum_k\delta_k\varepsilon_k(h)z}\prod_{k,\varepsilon_k(h)=0}\frac{(1-\mathrm{e}^{-e_k-2\pi i\varepsilon_k(g)+2\pi i(\delta_k+1)z})}{(1-\mathrm{e}^{-e_k-2\pi i\varepsilon_k(g)+2\pi iz})}\frac{(1-\mathrm{e}^{2\pi iz})}{(1-\mathrm{e}^{2\pi i(\delta_k+1)z})}\\
&\quad\times\prod_{k,\varepsilon_k(h)\neq0}\frac{(1-\mathrm{e}^{2\pi iz})}{(1-\mathrm{e}^{2\pi i(\delta_k+1)z})}\\
&=\frac{y^{-\frac{\dim X}{2}}}{|G|}\sum_{g,h,gh=hg}\sum_{X^{g,h}}\int_{X^{g,h}}\mathrm{td}(X^{g,h})y^{wt(h,X^h,E)}\\
&\quad\times\frac{\mathrm{ch}(\Lambda_{-y}\Omega^1_{X^h}|_{X^{g,h}}(g))}{\mathrm{ch}(\Lambda_{-1}N^*_{X^{g,h}\subseteq X^h}(g))}\prod_{E_k\not\supseteq X^h}\frac{(1-\mathrm{e}^{-e_k-2\pi i\varepsilon_k(g)}y^{\delta_k+1})}{(1-\mathrm{e}^{-e_k-2\pi i\varepsilon_k(g)}y)}\prod_{E_k}\frac{(1-y)}{(1-y^{\delta_k+1})}.
\end{aligned}$$

Here $wt(h,X^h,E)$ is the same weight as defined in [5] (cf. 6.1), for the irreducible component X^h of the fixed point set of h that contains $X^{g,h}$. We have also used

$$\mathrm{ch}(\Lambda_{-y}\Omega^1_{X^h}|_{X^{g,h}}(g))=\prod_{\lambda(h)=0}(1-\mathrm{e}^{-x_\lambda-2\pi i\lambda(g)+2\pi iz})$$

and

$$\mathrm{ch}(\Lambda_{-1}N^*_{X^{g,h}\subseteq X^h}(g))=\prod_{\lambda(h)=0,\lambda(g)\neq0}(1-\mathrm{e}^{-x_\lambda-2\pi i\lambda(g)}).$$

We use a trick to rearrange the product over $E_k\not\supseteq X^h$ as follows.

$$\begin{aligned}
&\lim_{\tau\to i\infty}\mathrm{Ell}_{\mathrm{orb}}(X,E,G;z,\tau)\\
&\quad=\frac{y^{-\frac{\dim X}{2}}}{|G|}\sum_{g,h,gh=hg}\sum_{X^{g,h}}\int_{X^{g,h}}\mathrm{td}(X^{g,h})y^{wt(h,X^h,E)}\frac{\mathrm{ch}(\Lambda_{-y}\Omega^1_{X^h}|_{X^{g,h}}(g))}{\mathrm{ch}(\Lambda_{-1}N^*_{X^{g,h}\subseteq X^h})}\\
&\qquad\times\prod_{E_k\not\supseteq X^h}\Big(1+\frac{(y-y^{\delta_k+1})(1-\mathrm{e}^{-e_k-2\pi i\varepsilon_k(g)})}{(y^{\delta_k+1}-1)(1-y\mathrm{e}^{-e_k-2\pi i\varepsilon_k(g)})}\Big)\prod_{E_k\supseteq X^h}\frac{(y-1)}{(y^{\delta_k+1}-1)}
\end{aligned}$$

$$= \frac{y^{-\frac{\dim X}{2}}}{|G|} \sum_{g,h,gh=hg} \sum_{X^{g,h}} \int_{X^{g,h}} \mathrm{td}(X^{g,h}) y^{wt(h,X^h,E)} \frac{\mathrm{ch}(\Lambda_{-y}\Omega^1_{X^h}|_{X^{g,h}}(g))}{\mathrm{ch}(\Lambda_{-1}N^*_{X^{g,h}\subseteq X^h})}$$

$$\times \sum_{J\subseteq I(X^h)} \prod_{k\in J} \frac{(y-y^{\delta_k+1})(1-\mathrm{e}^{-e_k-2\pi\mathrm{i}\varepsilon_k(g)})}{(y^{\delta_k+1}-1)(1-y\mathrm{e}^{-e_k-2\pi\mathrm{i}\varepsilon_k(g)})} \prod_{E_k\supseteq X^h} \frac{(y-1)}{(y^{\delta_k+1}-1)}$$

$$= \frac{y^{-\frac{\dim X}{2}}}{|G|} \sum_{g,h,gh=hg} \sum_{X^{g,h}} \sum_{J\subseteq I(X^h)} \int_{X^{g,h}\cap E_J} \mathrm{td}(X^{g,h}\cap E_J) y^{wt(h,X^h,E)}$$

$$\times \frac{\mathrm{ch}(\Lambda_{-y}\Omega^1_{X^h\cap E_J})|_{X^{g,h}\cap E_J}(g)}{\mathrm{ch}(\Lambda_{-1}N^*_{X^{g,h}\cap E_J\subseteq X^h\cap E_J})(g)} \prod_{k\in J} \frac{(y-y^{\delta_k+1})}{(y^{\delta_k+1}-1)} \prod_{E_k\supseteq X^h} \frac{(y-1)}{(y^{\delta_k+1}-1)}.$$

Here the set $I(X^h)$ is defined as the set of all k such that $E_k \not\supseteq X^h$ and $E_k\cap X^h \neq \emptyset$, which in particular implies that E_k is mapped to itself by h due to G-normality. We have also used the identities

$$\mathrm{td}(X^{g,h}\cap E_J) = \mathrm{td}(X^{g,h}) \prod_{k\in J, X^{g,h}\not\subseteq E_k} \frac{(1-\mathrm{e}^{-e_k})}{e_k}$$

$$\mathrm{ch}(\Lambda_{-y}\Omega^1_{X^h\cap E_J})|_{X^{g,h}\cap E_J}(g) = \mathrm{ch}(\Lambda_{-y}\Omega^1_{X^h})|_{X^{g,h}}(g) \prod_{k\in J} (1-y\mathrm{e}^{-e_k-2\pi\mathrm{i}\varepsilon_k(g)})^{-1}$$

$$\mathrm{ch}(\Lambda_{-1}N^*_{X^{g,h}\cap E_J\subseteq X^h\cap E_J})(g) = \mathrm{ch}(\Lambda_{-1}N^*_{X^{g,h}\subseteq X^h})(g) \prod_{k\in J, X^{g,h}\subseteq E_k} (1-\mathrm{e}^{-e_k-2\pi\mathrm{i}\varepsilon_k(g)}).$$

Changing the order of summation, one obtains

$$\lim_{\tau\to\mathrm{i}\infty} \mathrm{Ell}_{\mathrm{orb}}(X,E,G;z,\tau)$$

$$= \frac{y^{-\frac{\dim X}{2}}}{|G|} \sum_{h\in G} \sum_{X^h} y^{wt(h,X^h,E)} \sum_{J\subseteq I(X^h)} \sum_{g\in C(h,X^h,J)} \int_{X^{g,h}\cap E_J} \mathrm{td}(X^{g,h}\cap E_J)$$

$$\times \frac{\mathrm{ch}(\Lambda_{-y}\Omega^1_{X^h\cap E_J})|_{X^{g,h}\cap E_J}(g)}{\mathrm{ch}(\Lambda_{-1}N^*_{X^{g,h}\cap E_J\subseteq X^h\cap E_J})(g)} \prod_{k\in J} (\frac{y-1}{y^{\delta_k+1}-1}-1) \prod_{E_k\supseteq X^h} \frac{(y-1)}{(y^{\delta_k+1}-1)}$$

where the group $C(h,X^h,J)$ is defined as the subgroup of the centralizer of h that consists of group elements that map X^h to itself and preserve all elements of J. By the equivariant Riemann-Roch theorem the above expression equals

$$\frac{y^{-\frac{\dim X}{2}}}{|G|} \sum_{h,X^h,J\subseteq I(X^h)} y^{wt(h,X^h,E)} |C(h,X^h,J)| \chi_y(X^h\cap E_J/C(h,X^h,J))$$

$$\times \prod_{k\in J} (\frac{y-1}{y^{\delta_k+1}-1}-1) \prod_{E_k\supseteq X^h} \frac{(y-1)}{(y^{\delta_k+1}-1)}$$

$$= \frac{y^{-\frac{\dim X}{2}}}{|G|} \sum_{h,X^h,J\subseteq I(X^h)} y^{wt(h,X^h,E)} |C(h,X^h,J)| \chi_y(X^h \cap E_J / C(h,X^h,J))$$
$$\times \sum_{J_1\subseteq J} (-1)^{|J|-|J_1|} \prod_{k\in J_1} \frac{(y-1)}{(y^{\delta_k+1}-1)} \prod_{E_k\supseteq X^h} \frac{(y-1)}{(y^{\delta_k+1}-1)}.$$

We observe that we can replace the group $C(h, X^h, J)$ by a possibly bigger group $\hat{C}(h, X^h, J)$ characterized by the condition of fixing h and X^h and fixing J *as a set*. Indeed, the G-normality of E implies that the action of $\hat{C}(h, X^h, J)/C(h, X^h, J)$ on $X^h \cap E_J/C(h, X^h, J)$ is free and we can rewrite the above as

$$\frac{y^{-\frac{\dim X}{2}}}{|G|} \sum_{h,X^h,J\subseteq I(X^h)} y^{wt(h,X^h,E)} |\hat{C}(h,X^h,J)| \chi_y(X^h \cap E_J / \hat{C}(h,X^h,J))$$
$$\times \sum_{J_1\subseteq J} (-1)^{|J|-|J_1|} \prod_{k\in J_1} \frac{(y-1)}{(y^{\delta_k+1}-1)} \prod_{E_k\supseteq X^h} \frac{(y-1)}{(y^{\delta_k+1}-1)}.$$

The variety X^h is stratified by intersections with various E_J° which induces a stratification on $X^h \cap E_J/\hat{C}(h, X^h, J)$. Every $J_2 \supseteq J$ gives a stratum $X^h \cap E_{J_2}^\circ$ on $X^h \cap E_J$, but different such strata may map to the same stratum in $X^h \cap E_J/\hat{C}(h, X^h, J)$. In fact, the strata for all possible sets of J_2 from the same orbit of $\hat{C}(h, X^h, J)$-action on the set of J_2 that contain J will map to the same stratum S on $X^h \cap E_J/\hat{C}(h, X^h, J)$. This stratum S will be isomorphic to $X^h \cap E_{J_2}^\circ/\hat{C}(h, X^h, J \subseteq J_2)$ where $\hat{C}(h, X^h, J \subseteq J_2)$ is the subgroup of G that fixes h and X^h and fixes J and J_2 as sets. By Lemma 3.13, we get

$$\chi_y(S) = \frac{|\hat{C}(h, X^h, J_2)|}{|\hat{C}(h, X^h, J \subseteq J_2)|} \chi_y(X^h \cap E_{J_2}^\circ / \hat{C}(h, X^h, J_2)).$$

Indeed, both groups $\hat{C}(h, X^h, J \subseteq J_2)$ and $\hat{C}(h, X^h, J_2)$ act freely on the variety $X^h \cap E_{J_2}/C(h, X^h, J_2)$ and preserve the stratification which allows one to compare the χ_y-genera of the quotients. Using the additivity property of the χ_y-genus we now get

$$\chi_y(X^h \cap E_J/\hat{C}(h, X^h, J)) = \sum_{J_2\supseteq J} \frac{|\hat{C}(h, X^h, J_2)|}{|\hat{C}(h, X^h, J)|} \chi_y(X^h \cap E_{J_2}^\circ / \hat{C}(h, X^h, J_2))$$

with the rational coefficients included to account for the fact that the same stratum on the quotient may come from different strata on $X^h \cap E_J$. We notice that $\sum_{J, J_2 \supseteq J \supseteq J_1} (-1)^{|J|-|J_1|}$ equals 1 for $J_1 = J_2$ and equals zero otherwise, to get

$$\lim_{\tau\to \mathrm{i}\infty} \mathrm{Ell}_{\mathrm{orb}}(X,E,G;z,\tau)$$

$$= \frac{y^{-\frac{\dim X}{2}}}{|G|} \sum_{h,X^h,J\subseteq I(X^h)} y^{wt(h,X^h,E)}$$

$$\times |\hat{C}(h,X^h,J)|\chi_y(X^h\cap E_J^\circ/\hat{C}(h,X^h,J)) \prod_{k\in J}\frac{(y-1)}{(y^{\delta_k+1}-1)} \prod_{E_k\supseteq X^h}\frac{(y-1)}{(y^{\delta_k+1}-1)}$$

$$= y^{-\frac{\dim X}{2}} \sum_{\{h\},\{X^h\},\{J\}} y^{wt(h,X^h,E)}\chi_y(X^h\cap E_J^\circ/\hat{C}(h,X^h,J))$$

$$\times \prod_{k\in J}\frac{(y-1)}{(y^{\delta_k+1}-1)} \prod_{E_k\supseteq X^h}\frac{(y-1)}{(y^{\delta_k+1}-1)}.$$

Here we are summing over representatives h of conjugacy classes of G, then over representatives X^h of the orbits of the action of $C(h)$ on the components of the fixed point set of h and finally over the orbits of the action of $C(h,X^h,\emptyset)$ on the subsets of $I(X^h)$. This can be compared with Definitions 6.1 and 6.3 of [5]. Our sum over the subsets of the set of components fixed by h that contain the set of components E_k that contain X^h coincides with the set from the definition of [5] up to trivial contributions. Indeed, in Definition 6.1 of [5] W_J is empty unless J consists of the elements that correspond to divisors that intersect W and moreover contains all elements that correspond to the divisors that contain W.

However, it appears that we are summing over the orbits $\{J\}$ whereas Definition 6.3 of [5] contains the sum over all J. The extra factor is equal to the length of the orbit of J under the action of $C(h,X^h,\emptyset)$. This appears to be a typo in [5], which can be easily seen for a fixed point free action of G. □

Remark 3.15. Clearly, the comparison between the orbifold elliptic genus and the orbifold E-function follows from Theorem 5.3 and the main result of [5]. However, it would be strange to rely on such a roundabout way of proving it.

PROPOSITION 3.16. *Let X be a smooth G-variety and let E be a G-normal divisor on it such that (X,E) is Kawamata log-terminal. Let $m(K_X+E)$ be a trivial Cartier divisor for some integer m. Denote by n the order of the image of the homomorphism $G\to \mathrm{Aut}H^0(X,m(K_X+E))$, where the homomorphism can be defined due to G-invariance of E. Then $\mathrm{Ell}_{\mathrm{orb}}(X,E,G)$ is a weak Jacobi form of weight 0 and index $\dim X/2$ with respect to the subgroup of the Jacobi group Γ^J generated by the transformations*

$$(z,\tau)\to(z+mn,\tau),\ (z,\tau)\to(z+mn\tau,\tau),\ (z,\tau)\to(z,\tau+1),\ (z,\tau)\to(\frac{z}{\tau},-\frac{1}{\tau}).$$

Proof. As in the proof of Theorem 4.3 in [7], we introduce

$$\Phi(g,h,\kappa,z,\tau,x) := \frac{\theta(\frac{x}{2\pi\mathrm{i}}+\kappa(g)-\tau\kappa(h)-z)}{\theta(\frac{x}{2\pi\mathrm{i}}+\kappa(g)-\tau\kappa(h))}\mathrm{e}^{2\pi\mathrm{i}z\kappa(h)}$$

where κ is a character of the subgroup of G generated by g and h considered acting on a line bundle with the first Chern class x. Then the contribution of a connected component $X^{g,h}$ in Definition 3.2 is

$$\Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda\Big)\times\prod_\lambda \Phi(g,h,\lambda,z,\tau,x_\lambda)$$
$$\times\prod_k \frac{\Phi(g,h,\varepsilon_k,(1+\delta_k)z,\tau,e_k)}{\Phi(g,h,\varepsilon_k,z,\tau,e_k)}\frac{\theta(-z)}{\theta(-(\delta_k+1)z)}[X^{g,h}].$$

The proposition follows from the transformation properties of $\Phi(g,h,\kappa,z,\tau,x)$ proven in Theorem 4.3 of [7]. Note that these properties yield that the transformation $(z,\tau)\to(z+mn\tau)$ transforms $\mathrm{Ell}_{\mathrm{orb}}(X,E,G)$ as a Jacobi form provided: $\sum_\lambda x_\lambda+\sum\delta_k E_k=0$ and for any $g\in G$ one has

$$mn\left(\sum_\lambda \lambda(g)+\sum_k \delta_k\varepsilon_k(g)\right)\in\mathbf{Z}.$$

Those are the assumptions of the proposition. The Jacobi property for the transformation $(z,\tau)\to(z+1,\tau)$ also uses the above condition. The other two generators of Γ^J mentioned above transform the contribution of the pair (g,h) into the contribution for the pairs (gh^{-1},h) and (h,g^{-1}) respectively, multiplied by the corresponding Jacobi factor.

We remark that the result of this proposition also follows from the main Theorem 5.3 of this paper and [7, Prop. 3.8]. □

4. Toroidal morphisms of nonsingular pairs

The goal of this section is to derive pullback and pushforward formulas for functions of divisor classes for certain maps of varieties with simple normal crossing divisors on them.

Let Z be a smooth algebraic variety, together with an open set U_Z whose complement is a simple normal crossing divisor $D=\sum_{i\in I_Z} D_i$, where D_i are the irreducible components of D. To every subset $I\subseteq I_Z$ and every connected component $Z_{I;j}$ of $Z_I=\cap_{i\in I}D_i$ we associate a cone $C_{I;j}$ in the lattice $N_{I;j}\cong\mathbb{Z}^{|I|}$. We denote the standard basis of $N_{I;j}$ by $\{e_{k;j}\}, k\in I$. The cone $C_{I;j}$ is defined as $\oplus_{k\in I}\mathbb{R}_{\geq 0}e_{k;j}$. For any cone C its relative interior will be denoted by C°.

If $I_1\subseteq I_2$ and a connected component $Z_{I_1;j_1}$ contains a connected component $Z_{I_2;j_2}$ then we define a *face inclusion map* from $N_{I_1;j_1}$ to $N_{I_2;j_2}$ by mapping $e_{k;j_1}$ to $e_{k;j_2}$ for every $k\in I_1$. The image of the cone $C_{I_1;j_1}$ under this map is a face of $C_{I_2;j_2}$, which explains the terminology. In agreement with the terminology of [21] we define the *conical polyhedral complex* Σ_Z of (Z,D) as the union of all cones $C_{I;j}$ glued according to the face inclusion maps. We will often refer to it as the conical complex. This is the same as the conical polyhedral complex with an integral structure for the smooth toroidal embedding without self-intersection, in the terminology of [21]. We also observe that closed subvarieties $Z_{I;j}$ induce a stratification on Z. The corresponding locally closed strata will be denoted by $Z^\circ_{I;j}$.

We define piecewise linear (resp. polynomial) functions on Σ_Z as collections of linear (resp. polynomial) functions on each $C_{I;j}\in\Sigma_Z$ which are compatible with all face inclusions. We will analogously talk about formal power series on the conical complex by considering the completion of the space of polynomial functions by the degree filtration, i.e. the space of collections of formal power series on the vector space $N_{I;j}\otimes_{\mathbb{Z}}\mathbb{C}$ for each $Z_{I;j}$ that are compatible with the face inclusions. There is a natural ring structure on the space of formal power series, which we will denote by $\mathbb{C}[[\Sigma_Z]]$.

Another natural ring to consider is the partial semigroup ring defined by the conical complex Σ_Z. It is a vector space whose basis elements $[v]$ are in one-to-one correspondence with lattice points v of Σ_Z. For every pair of points $v_1,v_2\in\Sigma_Z$, the product $[v_1][v_2]$ is defined as follows:

$$[v_1][v_2]=\tilde{\sum_{\substack{C\in\Sigma_Z\\ v_1,v_2\in C}}}[v_1+v_2],$$

where $\tilde{\sum}$ means that the same point of Σ_Z that appears from different cones is counted only once. Alternatively, it is enough to consider the cones $C\ni v_1,v_2$ that do not contain any smaller such cone. In particular, the product is zero if there are no cones C that contain both v_1 and v_2. This ring will be denoted by $\mathbb{C}[\Sigma_Z]$. It can also be thought of as a subring of the direct sum of the semigroup rings $\oplus_{I;j}\mathbb{C}[C_{I;j}]$ that consists of collections that are compatible with the face inclusions. The identification is via mapping $[v]$ to the collection of $[v]$ for $C\ni v$ and 0 otherwise.

It will be crucial to our calculations to construct a natural isomorphism between the ring $\mathbb{C}[\Sigma_Z]$ and the subring of $\mathbb{C}[[\Sigma_Z]]$ that consists of piecewise polynomial functions. Namely, for every cone $C_{I;j}$ we denote by $x_{k;j}$ the linear functions on $N_{I;j}$ such that $x_{k;j}(e_{l;j})=\delta^l_k$, where δ is the Kronecker symbol. The element $[v]=[\sum_{k\in I}a_ke_{k;j}]$ of $\mathbb{C}[C_{I;j}]$ is mapped to the polynomial $\prod_{k\in I}(x_{k;j})^{a_k}$. If a collection of elements of $\mathbb{C}[C_{I;j}]$ is compatible with face restrictions, then so is the collection of the corresponding polynomial functions. Indeed for any face inclusion between $C_{I_1;j_1}$ and $C_{I_2;j_2}$ the linear functions $x_{k;j_2}$ restrict to $x_{k;j_1}$ if $k\in I_1$ and to 0 otherwise. It is straightforward to see that this identification is compatible with the product structure. The inverse map from piecewise polynomial functions on Σ_Z to $\mathbb{C}[\Sigma_Z]$ is easy to construct as well. In what follows we will frequently pass from one description of $\mathbb{C}[\Sigma]$ to the other.

Definition 4.1. We define a map $\rho\colon \mathbb{C}[[\Sigma_Z]] \to A^*(Z)$ as follows. For every lattice point $v \in C^\circ_{I;j}$ given by

$$v = \sum_{i\in I} k_i e_{i;j}, k_i \geq 1$$

we define by f the corresponding piecewise polynomial function on Σ_Z and set

$$\rho(f) = Z_{I;j} \cap (\cap_{i\in I}(D_i)^{k_i-1}).$$

We extend the definition of ρ to arbitrary piecewise polynomial functions by linearity. We extend it to arbitrary piecewise formal power series by noticing that that only v with $\sum_i k_i \leq \dim Z$ contribute nontrivially.

Proposition 4.2. *The map ρ defined above is a ring homomorphism.*

Proof. It is enough to calculate the image of the product of two monomial functions f_1 and f_2 that correspond to points v_1 and v_2 in the conical complex. If there is no cone $C_{I;j} \in \Sigma_Z$ that contains both v_1 and v_2, then $f_1 f_2 = 0$. On the other hand, in this case the components $Z_{I_1;j_1}$ and $Z_{I_2;j_2}$ do not intersect, so $\rho(f_1)\rho(f_2) = 0$.

In general, the product $f_1 f_2$ will correspond to

$$\sum_{C_{I;j}\supseteq C_{I_1;j_1}, C_{I;j}\supseteq C_{I_2;j_2}} [v_{C_{I;j}}]$$

where $I = I_1 \cup I_2$ and

$$v_{C_{I;j}} = \sum_{i\in I_1} k_{i,1} e_{k;j} + \sum_{i\in I_2} k_{i,2} e_{k;j}.$$

The cones $C_{I;j}$ are in one-to-one correspondence with the connected components of the intersection $Z_{I_1;j_1} \cap Z_{I_2;j_2}$. The image of each $f_{C_{I;j}}$ under ρ is

$$\rho(f_{C_{I;j}}) = Z_{I;j} \cap (\cap_{i\in I_1\cup I_2} D_i^{k_{i,1}+k_{i,2}-1})$$

where $k_{i,1}$ is defined to be zero for $i \notin I_1$ and similarly for $k_{i,2}$. On the other hand, the excess intersection formula [14] gives

$$Z_{I_1;j_1} \cap Z_{I_2;j_2} = \sum_j Z_{I;j} \cap (\cap_{i\in I_1\cap I_2} D_i)$$

in $A^*(Z)$. Then it is easy to see that $\rho(f_1 f_2) = \rho(f_1)\rho(f_2)$. $\square$

Remark 4.3. If all $Z_I = \cap_{i\in I} D_i$ are either empty or connected, then the above discussion simplifies greatly. Then the image of ρ is precisely the subring of $A^*(Z)$ generated by the classes of D_i. The difficulty was to somehow localize a polynomial in D_i to the correct connected component.

Remark 4.4. The relation between lattice points of Σ_Z and piecewise polynomial functions on Σ_Z becomes important for what follows. While the former are easier to describe, the latter behave better under pullbacks.

We now define a certain class of morphisms between two varieties $\hat{Z}$ and Z with the normal crossing divisors $\hat{D}$ and D respectively. This is a particular case of the general definition of [2].

Definition 4.5. We call a proper generically finite morphism $\mu\colon \hat{Z} \to Z$ *toroidal* if the following conditions hold.

- $\hat{D} = \mu^{-1}D$ and the morphism μ is finite and nonramified outside of $\hat{D}$.
- The image of the closure of any stratum of $\hat{Z}$ is the closure of a stratum in Z.
- For every pair of points $\hat{z} \in \hat{Z}$ and $z \in Z$ such that $\mu(\hat{z}) = z$ and every system of local analytic coordinates at z such that the components of D that pass through z are coordinate hyperplanes, there exists a system of local analytic coordinates at $\hat{z}$ such that the map μ is given by monomials.

We claim that locally in Z a toroidal morphism is given by a finite toric morphism. A local description of a finite toric morphism can be seen in Figure 1 where the positive orthant in one lattice is subdivided into cones of determinant 1 in a smaller lattice. We refer the reader to [15] for the background on toric geometry.

Remark 4.6. Let C be a positive orthant in a lattice N and let $\hat{N}$ be a finite index sublattice of N. Then to each subdivision Σ of C into cones of determinant one in $\hat{N}$ (see Figure 1) one can associate a proper generically finite toric morphism between the smooth toric variety that corresponds to $(\hat{N}, \Sigma)$ and the smooth toric variety $\mathbb{C}^{\mathrm{rk} N}$ that corresponds to (N, C).

As in [2], to every toroidal morphism we associate a map $\nu\colon \Sigma_{\hat{Z}} \to \Sigma_Z$ as follows. For a cone $C_{\hat{I},\hat{j}}$, pick a generic point $\hat{z}$ on the corresponding connected component $\hat{Z}_{\hat{I};\hat{j}}$ and consider the above monomial map between the neighborhoods of $\hat{z}$ and $z = \mu(\hat{z})$. The point z lies in the stratum $Z_{I;j}$ for some I and j. For every $\hat{i} \in \hat{I}$ consider the point in $C_{I;j}$ whose coordinates are the degrees of the local variables that correspond to $\hat{i}$ in the expressions of the variables that correspond to $D_i, i \in I$. This defines the image of the point $e_{\hat{i};\hat{j}}$ in $C_{I;j}$ and the

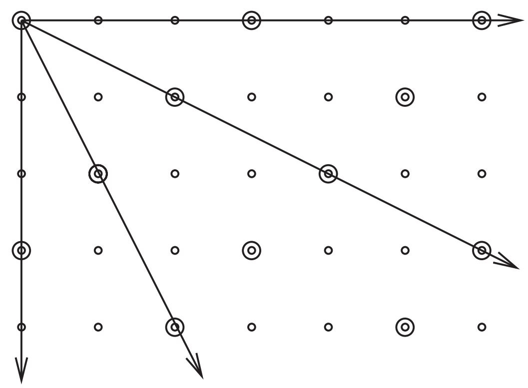

Figure 1

map is extended to the whole $\Sigma_{\hat{Z}}$ by linearity. This map is compatible with the face inclusions and is thus well-defined. Indeed, it encodes the coefficients of $\hat{D}_{\hat{i}}$ in the divisors $\mu^* D_i$.

Moreover, we can describe the preimage of any cone $C_{I;j}$ as follows. Let z be a generic point of the corresponding stratum $Z^\circ_{I;j}$ and let U be a small analytic neighborhood of z. Let $\hat{U}$ be one of the connected components of the preimage of U. We denote by U° and $\hat{U}^\circ$ the intersections of U and $\hat{U}$ with the complements of D and $\hat{D}$ respectively. Then μ induces a finite nonramified covering map from $\hat{U}^\circ$ to $\hat{U}$. Since the fundamental group of U° is naturally isomorphic to $N_{I;j}$, this covering gives a map from $N_{I;j}$ to a finite group. The kernel of this map is the fundamental group of $\hat{U}^\circ$. It is a finite index subgroup of $N = N_{I;j}$ which we denote by $\hat{N}$. The map from $\hat{U}$ to U can be factored through the singular variety U_1 which is the preimage of U under the natural map from $\mathrm{Spec}[C^*_{I;j} \cap \hat{N}] \to \mathrm{Spec}[C^*_{I;j} \cap N]$. Then the arguments of [21, Ch. 2, §2] show that the map from $\hat{U}$ to U_1 comes from a subdivision of the cone $C_{I;j}$ in $\hat{N}$ into cones of volume 1 as in Figure 1.

In general, it is possible that different connected components of the preimage of U are part of the same connected component of the preimage of $Z^\circ_{I;j}$. However, we have just shown that the preimage $\nu^{-1} C^\circ_{I;j}$ of the interior of $C_{I;j}$ is a union of connected components, each of which corresponds to a finite toric morphism. Namely, for each component, there is a sublattice $\hat{N}$ of finite index in the lattice $N = N_{I;j}$. The relative interior of cone $C_{I;j}$ is subdivided into several simplicial cones of volume 1 in the lattice $\hat{N}$ as in Figure 1. We will denote this subdivision by $\Sigma_{C_{I;j}}$, if it is clear from the discussion which connected component of $\nu^{-1} C^\circ_{I;j}$ we are referring to. Then every cone of $\Sigma_{C_{I;j}}$ is a cone of $\Sigma_{\hat{Z}}$ with the lattice $\hat{N}$ as the corresponding lattice. For the stratum $Z^\circ_{I;j}$ the restriction of μ to the preimage of an analytic neighborhood U of $Z^\circ_{I;j}$ is described by ν in the following sense. The connected components of this preimage are in one-to-one correspondence with the connected components of $\nu^{-1}C^\circ_{I;j}$. Let us fix one such connected component, which we will denote by $\hat U$. It is easily seen to be a neighborhood of the union of the strata $\hat Z^\circ_{\hat I;\hat j}$ for all $C_{\hat I;\hat j}\in\Sigma_{C_{I;j}}$. Moreover, it is a locally trivial fibration with fibers isomorphic to the preimage in $\mathbb{P}_{\hat N,\Sigma_{C_{I;j}}}$ of a disc around the origin under the map of Remark 4.6. The base is isomorphic to some smooth variety W which is a nonramified cover of degree $d_{i;j}$ of $Z^\circ_{I;j}$. The map μ on U is locally on Z isomorphic to a product of the map from Remark 4.6 and an identity map along. Here W can be taken to be any stratum on $\hat Z$ that corresponds to a maximum-dimensional cone in $\Sigma_{C_{I;j}}$. The numbers $d_{I;j}$ depend on the connected component $\hat U$ and they will be important in our description of the pushforward.

Our goal is to investigate the restriction of μ^* and μ_* to the subrings of $A^*(\hat Z)$ and $A^*(Z)$ which are images of $\hat\rho=\rho_{\hat Z}$ and $\rho=\rho_Z$.

Proposition 4.7. *The data of finite toroidal morphism define a map*

$$\nu^*\colon \mathbb{C}[[\Sigma_Z]]\to\mathbb{C}[[\Sigma_{\hat Z}]]$$

simply by pulling back the corresponding functions via ν. This map is a lifting of μ^* in the sense that

$$\hat\rho\circ\nu^*=\mu^*\circ\rho.$$

Proof. Let us first check this for a linear function on Σ_Z. Every such function corresponds to a divisor $\sum_i\alpha_iD_i$. Then locally this is a statement of toric geometry, namely that the preimage of a divisor under a map between two toric varieties is given by the same piecewise-linear function, which is apparent from the definition of ν.

It is then enough to check this statement for a function f that corresponds to a point $v=\sum_{i\in I}e_{i;j}$ in the relative interior of a cone $C_{I;j}\in\Sigma_Z$, since both ν^* and μ^* are ring homomorphisms. We can restrict our attention to a Zariski neighborhood U of $Z_{I;j}$ such that $U\cap Z_I=Z_{I;j}$. Then all we need is the above statement in toric geometry, restricted to U, together with the fact that μ^* and ν^* are ring homomorphisms. □

It is a bit more difficult to describe the pushforward.

Theorem 4.8. *Let ν_* be defined as follows. For every $f\in\mathbb{C}[[\Sigma_{\hat Z}]]$ and every cone $C_{I;j}\in\Sigma_Z$ consider the subdivision $\Sigma_{C_{I;j}}$ of each connected component of $\nu^{-1}C^\circ_{I;j}$. For each $C_{\hat I;\hat j}$ of $\Sigma_{C_{I;j}}$ with $|\hat I|=|I|$, the power series in the variables $x_{\hat i},\hat i\in\hat I$ that corresponds to the restriction of f to $C_{\hat I,\hat j}$ gives a power series in the variables $x_i,i\in I$ via the linear change of variables that corresponds to the inclusion of $C_{\hat I;\hat j}$ into $C_{I;j}$. This power series will be denoted* *by* $f_{\hat I;\hat j}$. *Then we define the element of* $\mathbb{C}((x_{i;j})), i\in I$

$$(\nu_* f)_{I;j} = \sum_{\Sigma_{C_{I;j}}} d_{I;j} \sum_{C_{\hat I;\hat j}\in\Sigma_{C_{I;j}}, |\hat I|=|I|} f_{\hat I;\hat j} \frac{\prod_{i\in I} x_{i;j}}{\prod_{\hat i\in\hat I} x_{\hat i;\hat j}} \tag{10}$$

where the outer sum is taken over all connected components of $\nu^{-1}C^\circ_{I;j}$. *These functions* $(\nu_* f)_{I;j}$ *are compatible with face restrictions and actually lie in* $\mathbb{C}[[x_{i;j}, i\in I]]$. *Thus they define a map* $\nu_*\colon \mathbb{C}[[\Sigma_{\hat Z}]]\to\mathbb{C}[[\Sigma_Z]]$. *Moreover,* ν_* *is a lift of* μ_* *in the sense that*

$$\mu_*\circ\hat\rho = \rho\circ\nu_*.$$

Proof. Let us check that the image of the function f is actually in $\mathbb{C}[[C_{I;j}]]$ for every given cone $C_{I;j}$. It is enough to consider one cone subdivision $\Sigma_{C_{I;j}}$. The function $\nu_* f$ may have simple poles over hyperplanes that correspond to cones of $\Sigma_{C_{I;j}}$ of dimension $|I|-1$. Each such cone has two adjacent cones of maximum dimension, and it is straightforward to see that the terms of (10) for two such cones will contribute opposite residues for this hyperplane, because of the compatibility condition on f.

Let us now show that $\nu_* f$ is well-defined as a map from $\mathbb{C}[[\Sigma_{\hat Z}]]$ to $\mathbb{C}[[\Sigma_Z]]$, that is, the definition of $(\nu_* f)_{I;j}$ is compatible with face inclusions. Let $C_{I_1;j_1}$ be a codimension one face of $C_{I_2;j_2}$. Hence $I_2 = I_1\cup\{i_0\}$. The cones $C_{\hat I_2;\hat j_2}\in\Sigma_{\hat Z}$ that map to the cone $C_{I_2;j_2}$ and have the same dimension may or may not contain a face $C_{\hat I_1;\hat j_1}$ that maps to $C_{I_1;j_1}$. In the latter case, the contribution of such $C_{\hat I_2,\hat j_2}$ to $(\nu_* f)_{I_2;j_2}$ is going to restrict to zero on the face $C_{I_1;j_1}$. Indeed, in equation (10), the restriction of the numerator to $C_{I_1;j_1}$ vanishes, because it contains the linear factor $x_{i_0;j_2}$, whereas the denominator does not vanish. In the former case, the factor $x_{i_0;j_2}$ will appear in the numerator while $x_{\hat i_0;\hat j_2}$ will appear in the denominator,where $\hat i_0$ is the linear function that vanishes on $C_{\hat I_1;\hat j_1}$. It is easy to see that

$$\frac{x_{i_0;j_2}}{x_{\hat i_0;\hat j_2}} = \frac{|N_{I_2;j_2} : N_{\hat I_2;\hat j_2}|}{|N_{I_1;j_1} : N_{\hat I_1;\hat j_1}|}.$$

The other factors in the fraction would restrict to those for the contribution of $C_{\hat I_1;\hat j_1}$ to $(\nu_* f)_{I_1;j_1}$. For each cone $C_{\hat I_1;\hat j_1}$ there may be several different cones $C_{\hat I_2;\hat j_2}$ as above, but we observe that for each $C_{\hat I_1;\hat j_1}$

$$\sum_{C_{\hat I_2;\hat j_2}} |N_{I_2;j_2} : N_{\hat I_2;\hat j_2}| d_{I_2;j_2} = |N_{I_1;j_1} : N_{\hat I_1;\hat j_1}| d_{I_1;j_1}. \tag{11}$$

Indeed, both sides depend only on the connected component of $\nu^{-1}C_{I_1;j_1}$ rather than the specific cone $C_{\hat I_1;\hat j_1}$. The right-hand side then describes the number of points in the preimage of a point in the neighborhood of the stratum $Z^\circ_{I_1;j_1}$ that lie near a certain connected component Y of $\mu^{-1}Z_{I_1;j_1}$. Indeed, for any

$z\in Z^\circ_{I_1;j_1}$ there is a neighborhood $U\ni z$ such that the preimage is isomorphic to a union of $d_{I_1;j_1}$ copies of a toric variety $\mathbb{P}_{N_{I_1;j_1},\Sigma_{C_{I_1;j_1}}}$. For each copy the map on the big open set is finite unramified of degree $|N_{I_1;j_1}:N_{\hat I_1;\hat j_1}|$. The left-hand side describes the sum of numbers of points in the preimage of a point in the neighborhood of $Z^\circ_{I_2;j_2}$ sorted by the connected component of its preimage. However, since we are summing over the connected components of the preimage of $Z_{I_2;j_2}$ that are part of the component of the preimage of $Z_{I_1;j_1}$ both sides are the same.

Next, we observe that ν_* is a module homomorphism with respect to the $\mathbb{C}[[\Sigma_Z]]$-algebra structure on $\mathbb{C}[[\Sigma_{\hat Z}]]$ induced by ν^*. Indeed, multiplication by a pullback of a function g on Σ_Z results in multiplication of all $f_{\hat I,\hat j}$ in equation (10) by g.

Because μ_* is also a module homomorphism, it is now enough to check that $\rho(\nu_*(f))=\mu_*(\hat\rho(f))$ for functions f from some generating set of $\mathbb{C}[\Sigma_{\hat Z}]$ as a module over $\mathbb{C}[\Sigma_Z]$. We claim that such a generating set can be taken to be the set of f that correspond to *minimal* lattice points in the interiors of cones. Indeed, for every cone $C_{\hat I;\hat j}\in\Sigma_{\hat Z}$, consider a cone $C_{I;j}\in\Sigma_Z$ that it maps into the interior of under ν. It is easy to see that products of the function f that correspond to the minimum interior point of $v=\sum_{\hat i\in I}e_{\hat i;\hat j}$ by pullbacks of polynomial functions on $C_{I;j}$ span precisely the space of functions that correspond to monomials from the interior of $C_{\hat I;\hat j}$. So it is now enough to consider the function f that comes from the minimum interior point, where we keep the notation as above. We recall that for such f we have $\hat\rho(f)=\hat Z_{\hat I;\hat j}$.

In the case of $|\hat I|<|I|$ the stratum $\hat Z_{\hat I,\hat j}$ has image of smaller dimension, so $\mu_*\hat\rho(f)=0$. Consider all cones $C_{\hat I_1;\hat j_1}$ in $\Sigma_{C_{I;j}}$ that contain $C_{\hat I;\hat j}$ and have dimension $|I_1|=|I|$. These cones form the star of the neighborhood of $C_{\hat I;\hat j}$ in the fan $\Sigma_{C_{I;j}}$. The contributions of the fractions of equation (10) times the corresponding f will be proportional to

$$\sum_{C_{\hat I_1;\hat j_1}}\prod_{\hat i\in\hat I_1,i\notin\hat I}\frac{1}{x_{\hat i;\hat j_1}}.$$

Since all these linear functions $x_{\hat i;\hat j_1}$ vanish on the span of $C_{\hat I;\hat j}$, one can work on the quotient, where Lemma 8.5 implies that $(\nu_*f)_{I;j}=0$.

We also need to check that $(\nu_*f)_{I_1;j_1}=0$ for all cones $C_{I_1;j_1}\in\Sigma_Z$ that contain $C_{I;j}$. These calculations are analogous and are left to the reader.

Let us now consider the case $|\hat I|=|I|$. In this case there will be only one contribution to $(\nu_*f)_{I;j}$ and we get

$$(\nu_*f)_{I;j}=d_{I;j}\prod_{i\in I}x_{i;j}.$$

We also claim that for every other cone $C_{I_1;j_1}$ that contains $C_{I;j}$ we will have

$$(\nu_* f)_{I_1;j_1} = d_{I;j} \prod_{i\in I} x_{i;j_1}.$$

Indeed, for every connected component of $\nu^{-1}(C_{I;j})$ the terms of equation (10) will be zero except for the cones $C_{\hat{I}_1;\hat{j}_1}$ with $|\hat{I}_1| = |I_1|$ that contain $C_{\hat{I};\hat{j}}$. If we again work in the quotient by the span of $C_{\hat{I},\hat{j}}$, we see that Lemma 8.3 implies that the total contribution of such $C_{\hat{I}_1;\hat{j}_1}$ is

$$d_{I_1;j_1} \frac{|N_{I_1;j_1} : N_{\hat{I}_1;\hat{j}_1}|}{|N_{I;j} : N_{\hat{I};\hat{j}}|} \prod_{i\in I} x_{i;j_1}.$$

Then an analog of equation (11) finishes the calculation of $(\nu_* f)_{I_1;j_1}$.

Then we conclude that $\nu_* f$ corresponds to $d_{I;j}$ times the minimum lattice point of $C_{I;j}$. On the other hand, the corresponding cycle $\hat{Z}_{\hat{I},\hat{j}}$ maps onto $Z_{I;j}$ and the morphism is is generically finite of degree $d_{I;j}$. Therefore, we get $\mu_* \hat{Z}_{\hat{I},\hat{j}} = d_{I;j} Z_{I;j}$, which finishes the proof. □

5. Main theorem

Our goal is to prove that any G-equivariant morphism $\mu : \hat{Z} \to Z$ of smooth varieties which is birational to a quotient by G has the property that the pushforward of the orbifold elliptic class in $A^*(\hat{Z})$ is the elliptic class in $A^*(Z)$. The strategy of the proof is to reduce the situation to a toroidal morphism.

Let $\mu : \hat{Z} \to Z$ be such a G-equivariant toroidal morphism. Let h be any linear function on the fan Σ_Z. The function h corresponds to the divisor

$$D_h = \sum_{i\in I_Z} \alpha_i D_i.$$

We will also consider the divisor $\hat{D}_h$ on $\hat{Z}$ such that

$$\mu^*(K_Z + D_h) = K_{\hat{Z}} + \hat{D}_h \tag{12}$$

and

$$\hat{D}_h = \sum_{\hat{i}\in I_{\hat{Z}}} \hat{\alpha}_{\hat{i}} \hat{D}_{\hat{i}}.$$

The set $I_{\hat{Z}}$ splits into two sets $I_{\hat{Z}}^{\rm exc}$ and $I_{\hat{Z}}^{\rm ram}$ according to whether or not D_i is contracted by μ to a smaller-dimensional variety. We will abuse notation and call the divisors $\hat{D}_{\hat{i}}$ for $i \in I_{\hat{Z}}^{\rm ram}$ ramification divisors, and assign to each of them the ramification index $r_{\hat{i}}$ (which may be equal to 1). We observe that if $\mu(\hat{D}_{\hat{i}}) = D_i$, then

$$\hat{\alpha}_{\hat{i}} + 1 = r_{\hat{i}}(\alpha_i + 1).$$

Since μ is birationally equivalent to a quotient morphism and is locally given by the map between two toric varieties, it is easy to see that locally the group G acts as a subgroup of the torus. The isotropy group of every point of the stratum that corresponds to the cone $C_{\hat{I};\hat{j}}$ is equal to the index of $\sum_{\hat{i}\in\hat{I}}\mathbb{Z}e_{\hat{i};\hat{j}}$ in the restriction of $N_{C_{I;j}}$ to the $\mathbb{Q}$-span of $e_{\hat{i};\hat{j}}$. We will denote this group by $G_{\hat{I};\hat{j}}$.

Consider the following function E of z,τ with values in $A^*(\hat{Z})$.

$$E(z,\tau)=\frac{1}{|G|}\sum_{g,h\in G;gh=hg}\sum_{Z^{g,h}=Z_{\hat{I};\hat{j}}} Z_{\hat{I};\hat{j}}\times\prod_{\hat{i}\in I_{\hat{Z}}}\frac{\hat{D}_{\hat{i}}\theta(\frac{\hat{D}_{\hat{i}}}{2\pi\mathrm{i}}-z)\theta'(0)}{2\pi\mathrm{i}\theta(\frac{\hat{D}_{\hat{i}}}{2\pi\mathrm{i}})\theta(-z)}$$
$$\times\prod_{\hat{i}\in\hat{I}}\frac{\theta(\frac{\hat{D}_{\hat{i}}}{2\pi\mathrm{i}}+g_{\hat{i}}-h_{\hat{i}}\tau-(\hat{\alpha}_{\hat{i}}+1)z)\theta(-z)\theta(\frac{\hat{D}_{\hat{i}}}{2\pi\mathrm{i}})}{\hat{D}_{\hat{i}}\theta(\frac{\hat{D}_{\hat{i}}}{2\pi\mathrm{i}}+g_{\hat{i}}-h_{\hat{i}}\tau)\theta(-(\hat{\alpha}_{\hat{i}}+1)z)\theta(\frac{\hat{D}_{\hat{i}}}{2\pi\mathrm{i}}-z)}\mathrm{e}^{2\pi\mathrm{i}(\hat{\alpha}_{\hat{i}}+1)h_{\hat{i}}z}$$
$$\times\prod_{\hat{i}\notin\hat{I}}\frac{\theta(\frac{\hat{D}_{\hat{i}}}{2\pi\mathrm{i}}-(\hat{\alpha}_{\hat{i}}+1)z)\theta(-z)}{\theta(\frac{\hat{D}_{\hat{i}}}{2\pi\mathrm{i}}-z)\theta(-(\hat{\alpha}_{\hat{i}}+1)z)}.$$

Here $g_{\hat{i}}$ and $h_{\hat{i}}$ are the rational numbers in the range $[0,1)$ which describe the characters of the action of g and h on the divisor $\hat{D}_{\hat{i}}$ at each point of $\hat{Z}_{\hat{I};\hat{j}}$.

The following theorem describes the pushforward of E to Z.

Theorem 5.1.

$$\mu_*E(z,\tau)=\prod_{i\in I_Z}\frac{D_i\theta'(0)\theta(\frac{D_i}{2\pi\mathrm{i}}-(\alpha_i+1)z)}{2\pi\mathrm{i}\,\theta(\frac{D_i}{2\pi\mathrm{i}})\theta(-(\alpha_i+1)z)}.$$

Proof. We will use Theorem 4.8 to reduce the statement to a combinatorial result. First, we observe that $E(z,\tau)$ can be obtained as $\hat{\rho}(F(z,\tau))$ where F is defined as follows.

Consider the cone $C_{\hat{I};\hat{j}}$ that is a part of the subdivision $\Sigma_{C_{I;j}}$. Denote by $G_{\hat{I};\hat{j}}$ the quotient of the intersection of the lattice $\oplus_{i\in I}\mathbb{Z}e_{i;j}$ with the rational span of $\hat{e}_{\hat{i};\hat{j}},\hat{i}\in\hat{I}$ by $\oplus_{\hat{i}\in\hat{I}}\mathbb{Z}\hat{e}_{\hat{i};\hat{j}}$. The value of $F(z,\tau)$ on this cone is

$$F_{\hat{I};\hat{j}}(z,\tau)=\frac{1}{|G|}\sum_{g,h\in G_{\hat{I};\hat{j}}}\prod_{\hat{i}\in\hat{I}}\frac{\hat{x}_{\hat{i};\hat{j}}\theta'(0)\theta(\frac{\hat{x}_{\hat{i};\hat{j}}}{2\pi\mathrm{i}}+g_{\hat{i}}-h_{\hat{i}}\tau-(\hat{\alpha}_{\hat{i}}+1)z)}{2\pi\mathrm{i}\,\theta(\frac{\hat{x}_{\hat{i};\hat{j}}}{2\pi\mathrm{i}}+g_{\hat{i}}-h_{\hat{i}}\tau)\theta(-(\hat{\alpha}_{\hat{i}}+1)z)}\mathrm{e}^{2\pi\mathrm{i}(\hat{\alpha}_{\hat{i}}+1)h_{\hat{i}}z}.$$

Indeed, for every $g,h\in G_{\hat{I};\hat{j}}$ there is a unique connected component of the fixed point set of g and h that contains $Z_{\hat{I};\hat{j}}$. Then we use the definition and multiplicative properties of ρ, together with the fact that $\hat{D}_{\hat{i}}$ corresponds to the function on $\Sigma_{\hat{Z}}$ that equals $x_{\hat{i},\hat{j}}$ for every $C_{\hat{I};\hat{j}}$ such that $\hat{I}\ni\hat{i}$ and equals zero otherwise.

Now we need to calculate $\nu_* F(z,\tau)$. Let us calculate the component of $\nu_* F(z,\tau)$ on a cone $C_{I;j}$. By the definition of ν_* we get

$$\nu_* F(z,\tau)_{I;j} = \left(\prod_{i\in I} x_{i;j}\right) \sum_{\Sigma_{C_{I;j}}} \frac{d_{I;j}}{|G|}$$

$$\sum_{C_{\hat{I};\hat{j}}\in\Sigma_{C_{I;j}},|\hat{I}|=|I|} \sum_{g,h\in G_{\hat{I};\hat{j}}} \prod_{\hat{i}\in\hat{I}} \frac{\theta'(0)\theta(\frac{\hat{x}_{\hat{i};\hat{j}}}{2\pi\mathrm{i}} + g_{\hat{i}} - h_{\hat{i}}\tau - (\hat{\alpha}_{\hat{i}}+1)z)}{2\pi\mathrm{i}\,\theta(\frac{\hat{x}_{\hat{i};\hat{j}}}{2\pi\mathrm{i}} + g_{\hat{i}} - h_{\hat{i}}\tau)\theta(-(\hat{\alpha}_{\hat{i}}+1)z)} \mathrm{e}^{2\pi\mathrm{i}(\hat{\alpha}_{\hat{i}}+1)h_{\hat{i}}z}.$$

We now apply Lemma 8.1. Indeed, it is easy to check that equation (12) implies that the values $(\alpha_{\hat{i}}+1)z$ are values of a linear function on $C_{I;j}$. As a result, we get

$$\begin{aligned}\nu_* F(z,\tau)_{I;j} &= \left(\prod_{i\in I} x_{i;j}\right) \sum_{\Sigma_{C_{I;j}}} \frac{d_{I;j}}{|G|} |N_{I;j} : N_{\hat{I};\hat{j}}| \prod_{i\in I} \frac{\theta'(0)\theta(\frac{x_{i;j}}{2\pi\mathrm{i}} - (\alpha_i+1)z)}{2\pi\mathrm{i}\,\theta(\frac{x_{i;j}}{2\pi\mathrm{i}})\theta(-(\alpha_i+1)z)} \\ &= \prod_{i\in I} \frac{x_{i;j}\theta'(0)\theta(\frac{x_{i;j}}{2\pi\mathrm{i}} - (\alpha_i+1)z)}{2\pi\mathrm{i}\,\theta(\frac{x_{i;j}}{2\pi\mathrm{i}})\theta(-(\alpha_i+1)z)}.\end{aligned}$$

Here we use $\sum_{\Sigma_{C_{I;j}}} d_{I;j}|N_{I;j} : N_{\hat{I},\hat{j}}| = |G|$, which follows from the count of the number of preimage points of a point close to the stratum $Z_{I;j}$.

We now use Theorem 4.8 to get

$$\mu_* E(z,\tau) = \mu_* \hat{\rho} F(z,\tau) = \rho\nu_* F(z,\tau) = \prod_{i\in I_Z} \frac{D_i\theta'(0)\theta(\frac{D_i}{2\pi\mathrm{i}} - (\alpha_i+1)z)}{2\pi\mathrm{i}\,\theta(\frac{D_i}{2\pi\mathrm{i}})\theta(-(\alpha_i+1)z)}.$$

Indeed, the calculation of $\rho\nu_* F(z,\tau)$ is accomplished by the multiplicativity of ρ and the fact that the power series in D_i has constant term 1. □

We will also need the following lemma that connects the Chern classes of $T\hat{Z}$ and $\mu^* TZ$.

LEMMA 5.2.

$$c(T\hat{Z}) = \prod_{\hat{i}\in I_{\hat{Z}}} (1 + D_{\hat{i}}) \prod_{i\in I} (1 + \mu^* D_i)^{-1} \mu^* c(TZ).$$

Proof. First of all, it is easy to see that the pullback of the bundle of logarithmic differentials on Z is the bundle of logarithmic differentials on $\hat{Z}$. Then it is straightforward to calculate the ratio of Chern classes for the bundles of logarithmic differentials and usual differentials for a variety with normal crossing divisor. The details are left to the reader. □

We are now ready to formulate and prove our main theorem.

THEOREM 5.3. *Let $(X;D_X)$ be a Kawamata log-terminal pair which is invariant under an effective action of G on X. Let $\psi\colon X \to X/G$ be the quotient morphism. Let $(X/G;D_{X/G})$ be the quotient pair; see Definition* 2.7. *Then*

$$\psi_*\mathcal{ELL}_{\mathrm{orb}}(X,D_X,G;z,\tau)=\mathcal{ELL}(X/G,D_{X/G};z,\tau).$$

Proof. The following lemma allows us to reduce the problem to the situation of a G-equivariant toroidal morphism.

LEMMA 5.4. *There exists a commutative diagram*

$$\begin{array}{cccc} \mu\colon & \hat{Z} & \to & Z \\ & \downarrow & & \downarrow \\ \psi\colon & X & \to & X/G \end{array}$$

where the vertical arrows are resolutions of singularities and μ is a G-equivariant toroidal morphism.

Proof. We define Z as a desingularization of $(X/G,D_{X/G})$. Consider the normalization of Z in the function field of X and the corresponding normalization morphism. By Abhyankar's lemma it is a (typically singular) toroidal embedding with the toroidal morphism to Z. Then toroidal desingularization finishes the job. See [3] for details. □

Proof of Theorem 5.3 *continues.* By Lemma 5.4, Definition 3.7 and composition properties of pushforwards, it is sufficient to prove the pushforward result for a G-equivariant toroidal morphism $\mu\colon \hat{Z}\to Z$ which is birational to ψ. By Lemma 5.2 and Definition 3.2 of the orbifold elliptic class $\mathcal{ELL}(\hat{Z},D_{\hat{Z}},G;z,\tau)$, we see that

$$\mathcal{ELL}(\hat{Z},D_{\hat{Z}},G;z,\tau)=E(z,\tau)\mu^*\left(\prod_k \frac{z_k\theta(\frac{z_k}{2\pi\mathrm{i}}-z)}{\theta(\frac{z_k}{2\pi\mathrm{i}})}\prod_{i\in I_Z}\frac{2\pi\mathrm{i}\theta(\frac{D_i}{2\pi\mathrm{i}})\theta(-z)}{D_i\theta'(0)\theta(\frac{D_i}{2\pi\mathrm{i}}-z)}\right).$$

Then Theorem 5.1 and the definition of the elliptic class $\mathcal{ELL}(Z,D;z,\tau)$ finishes the proof. □

Remark 5.5. Theorem 5.3 gives an affirmative answer to the conjecture of [7]. We call it the McKay correspondence for elliptic genera, analogously to the homological McKay correspondence for stringy E-functions.

6. DMVV formula for pairs

One of the motivations of the definition of orbifold elliptic genus in [7] was the formula for the generating functions of elliptic genera of symmetric products.

$$\sum_{n\geq 0} p^n \mathrm{Ell}_{\mathrm{orb}}(X^n, S_n; z, \tau) = \prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1-p^i y^l q^m)^{c(mi,l)}}. \tag{13}$$

Here X is a Kähler manifold, S_n is the symmetric group acting on the n-fold product and $c(m,l)$ are the coefficients of the elliptic genus $\sum_{m,l} c(m,l) y^l q^m$ of X.

This formula was originally derived in [11] by means of some string-theoretic arguments. In particular, the orbifold elliptic genus of a quotient of a variety X^n by the symmetric group S_n was defined as the trace of a certain operator over the Hilbert space of the conformal field theory quotient of $\mathcal{C}^n$, where $\mathcal{C}$ is the superconformal field theory conjecturally associated to X. In [7], DMVV formula was shown for the mathematically defined orbifold elliptic genus. Our goal now is to extend this result to singular varieties and more generally to arbitrary Kawamata log-terminal pairs.

THEOREM 6.1. *Let (X,D) be a Kawamata log-terminal pair. For every $n \geq 0$ consider the quotient of $(X,D)^n$ by the symmetric group S_n, which we will denote by $(X^n/S_n, D^{(n)}/S_n)$. Here we denote by $D^{(n)}$ the sum of pullbacks of D under n canonical projections to X. Then we have*

$$\sum_{n\geq 0} p^n \mathrm{Ell}(X^n/S_n, D^{(n)}/S_n; z, \tau) = \prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1-p^i y^l q^m)^{c(mi,l)}},$$

where the elliptic genus of (X,D) is

$$\sum_{m\geq 0} \sum_{l} c(m,l) y^l q^m$$

and $y = \mathrm{e}^{2\pi \mathrm{i} z}$, $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$.

Remark 6.2. In the case of smooth X with $D=0$, the Fourier coefficient of $\mathrm{Ell}(X,z,\tau)$ at q^m is a polynomial in $y^{\pm\frac{1}{2}}$. In general other rational l are possible, but more importantly, the coefficient at q^m is no longer a polynomial in $y^{\pm\frac{1}{d}}$, rather it is a rational function. However, we will always assume that this function is Laurent expanded around $y=0$, so we will be working in the field of formal power series in $y^{\pm\frac{1}{d}}$, where d is divisible by 2 and all denominators of the discrepancy coefficients for some resolution of (X,D). The issue of non-polynomiality was first raised in [4] at the q^0 level.

Proof. First of all, observe that the quotient of the tensor power of a Kawamata log-terminal pair is again a Kawamata log-terminal pair. Moreover, by Theorem 5.3, we can calculate the elliptic genus of $(X^n/S_n, D^{(n)}/S_n)$ as an orbifold elliptic genus of $(X^n, D^{(n)}, S_n)$. A resolution of singularities $(\hat{X}, \hat{D})$ of the pair (X, D) induces a birational morphism $\hat{X}^n \to X^n$ so we may assume that (X, D) is nonsingular, i.e. X is smooth and D is a normal crossing divisor $\sum_i \alpha_i D_i$ with $\alpha_i > -1$. While the divisor $D^{(n)}$ on X^n has simple normal crossings, it is not S_n-normal. Indeed the pullbacks of the same component D_i via different projections are group translates of each other and certainly intersect and are nontrivially permuted by the isotropy group of any such intersection point that lies on the main diagonal $X \subseteq X^n$. To rectify this situation we need to consider an appropriate blowup of X^n. By Remark 3.11, each pair of commuting elements (g, h) can be handled separately.

Let us describe the pairs of commuting elements $g, h \in S_n$ and the connected components of their fixed point set. If the cycle decomposition of h has a_j cycles of degree j, then the fixed point set of h on X^n is the product of $\sum_j a_j$ copies of X, embedded into X^n by the product of diagonal embeddings of X into X^j for each cycle of length j. Elements g of S_n that commute with h form a semidirect product of the group $C_h = \prod_j (\mathbb{Z}/j\mathbb{Z})^{a_j}$ which consists of the products of powers of cycle components of h and the group $B_h = \prod_j S_{a_j}$ which consists of the group that permutes cycles of the same length without disturbing the order in the cycle. A fixed point set of each such pair (g, h) consists of points on $X^{\sum_j a_j}$ that are preserved by the image of g in B_h. It is easy to see that the contribution of each such $(X^n)^{g,h}$ is the product of the contributions of each factor. As a result, it is enough to consider the contribution of the diagonal embedding of X into $X^{ij} = (X^j)^i$ where h acts by permuting the copies of X inside each X^j and g acts by a product of a cyclic permutation of i copies of X^j and some cyclic permutations within each X^j, that does not change the cyclic orders of the components of X^j. Then $g^i = h^s$ for some $0 \leq s \leq j-1$, and s determines the action uniquely. Namely, if $x_{k,l}, k \in \mathbb{Z}/i\mathbb{Z}, l \in \mathbb{Z}/j\mathbb{Z}$ denote the components of X^{ij}, then we may assume that h acts by $x_{k,l} \to x_{k,l+1}$ and g acts by $x_{k,l} \to x_{k+1,l}$ for $k = 0, \ldots, i-2$ and $x_{i-1,l} \to x_{0,l+s}$. We will denote by G the group generated by g and h. It is an abelian group of order ij given by the generators g, h and relations $gh = hg, g^i = h^s, h^j = 1$. We denote the corresponding product of ij copies of X by X^G, which indicates the action of G on it.

We now need to make $(X^G, D^{(G)})$ into a G-normal pair. Let $D_c, 1 \leq c \leq k$ be the irreducible components of D on X. We will denote by $D_{r,c}, r \in G$ the pullback of D_c under the r^{th} projection map $X^G \to X$. We will perform the following sequence of blowups to X^G. First, we blow up $\cap_{r\in G} D_{r,1}$, then we blow up the proper preimage of $\cap_{r\in G} D_{r,2}$, and so on. We can describe this blowup in terms of the subdivision of the conical complex that corresponds to the simple normal crossing divisor $D^{(G)}$ on X^G. For the sake of simplicity we assume that the intersection of every number of components D_c on X is connected. The general case is completely analogous, it can also be reduced to the connected case by further blowups of X. Every cone C of the conical complex Σ_{X^G} is generated by elements $e_{r,c}$ for some subset of $I_C \subseteq G \times \{1,\ldots,k\}$. We denote by J_C the subset of $\{1,\ldots,k\}$ that consists of all c for which $I_C \supseteq G \times \{c\}$. The subdivision of C that corresponds to this sequence of blowups is then the product of $\mathbb{Z}_{\geq 0} e_{r,c}$ for $(r,c) \in I_C, c \notin J_C$ and the product over all $c \in J_C$ of the subdivisions of $\sum_{r\in G} \mathbb{Z}_{\geq 0} e_{r,c}$ where the extra vertex $\sum_{r\in G} e_{r,C}$ is added and the cone is subdivided accordingly. It is clear that this is a well-defined subdivision of Σ_{X^G} and we denote the corresponding variety by $\hat{X}^G$ and the corresponding divisor by $\hat{D}^{(G)}$. We observe that there are k exceptional components of $\hat{D}^{(G)}$, which we will call E_c, and the rest are the proper preimages of the components of $D^{(G)}$.

We need to describe connected components of the fixed point set of G on $\hat{X}^G$. Every such fixed point maps to the diagonal $X \subseteq X^G$, and should lie on the stratum of the stratification by the intersections of components of $\hat{D}^{(G)}$ that is stable under the group action. Since the construction is local in X, we need to see what happens when X is a $\mathbb{C}^n$ with D given as a union of some coordinate hyperplanes $z_1 = 0, z_2 = 0, \ldots, z_l = 0$. The extra coordinates $z_{l+1}, \ldots, z_n$ will have an effect of tensoring the construction by an affine space, so it is enough to look at the $l = n$ case. Then we need to investigate the fixed point sets of the toric variety that corresponds to a certain blowup of the positive orthant in $\mathbb{Z}^{ijk}$ where the generators are denoted by $e_{r,c}, r \in G, 1 \leq c \leq l$. The group G acts by multiplication on the first component of the index of the coordinate. The rays of the fan of the blowup that are fixed under G correspond to $e_{*,c} = \sum_{r\in G} e_{r,c}$. Moreover, it is easy to see that the only strata that are preserved by G are the intersections of the corresponding divisors. In other words, we need to consider the faces of the l-dimensional cone C which is a part of the subdivision of the positive orthant and is the span of all $e_{*,c}$. This cone corresponds to the affine set which is isomorphic to

$$\mathbb{C}^l \times (\mathbb{C}^*)^{ijl-l}. \tag{14}$$

The coordinates on $(\mathbb{C}^*)^{ijl-l}$ are given by $x_{r,c} x_{r_1,c}^{-1}$ and the coordinates on $\mathbb{C}^l$ are given by $x_{0,c}$. Let P be a fixed point of G. For each c, $x_{r,c}/x_{r_1,c} = \exp(2\pi \mathrm{i} \lambda(r - r_1))$ for some character $\lambda\colon G \to \mathbb{Q}/\mathbb{Z}$. If λ is nontrivial then $x_{0,c}$ is zero, and otherwise arbitrary values of $x_{0,c}$ are allowed. Moreover, for each component of the fixed point set the map to $\mathbb{C}^n \subseteq (\mathbb{C}^n)^G$ is an embedding. Indeed, it is clear for each factor $\mathbb{C}^G$ that corresponds to the D_c. Basically, for each factor, the blowup locus intersects the main diagonal of $\mathbb{C}^G$ in codimension one, namely at the origin.

Returning to the global situation, the above description tells us that connected components of the fixed point set Y of $X^{(G)}$ correspond to the collections of characters $\lambda_c\colon G \to \mathbb{Q}/\mathbb{Z}$. The fixed point set for each such character is isomorphic to $D_I = \cap_{i\in I} D_i$ where I is the set of those components c for which λ_c is nontrivial. Indeed, this follows from the fact that locally the map from the component of the fixed point set to X^G is an embedding. We observe that for some combinations of characters we may have $D_I = \emptyset$.

We now need to calculate the tangent bundle to such a component, which we will denote by $Y_{\lambda_1,\ldots,\lambda_k}$. Notice that the divisors $\hat{D}_{r,c}$ do not intersect with Y. Indeed, every G-invariant point of $\hat{D}_{r,c}$ would belong to $\hat{D}_{r_1,C}$ for all r_1, but the intersection of all these divisors is empty since π factors through the blowup of the intersection of $D_{r,C}, r \in G$. As far as intersection with E_c is concerned, $Y_{\lambda_1,\ldots,\lambda_k}$ is contained in E_c for $\lambda_c \neq 0$ and intersects transversally the other E_c. For $\lambda_c = 0$ the intersection of E_c and $Y_{\lambda_1,\ldots,\lambda_k}$ can be identified with the intersection by D_c under the isomorphism $Y_{\lambda_1,\ldots,\lambda_k} \cong D_I$. The character of G that corresponds to $E_c \supseteq Y_{\lambda_1,\ldots,\lambda_k}$ is equal to λ_c.

The Chern classes of the tangent bundles of $\hat{X}^G$ and X^G are related by Lemma 5.2, namely

$$c(T\hat{X}^G) = \pi^* c(TX^G) \prod_{c=1}^{k} (1+E_c) \prod_{r\in G, 1\le c\le k} \frac{(1+\hat{D}_{r,c})}{(1+\pi^* D_{r,c})}$$

where $\hat{D}_{r,c}$ is the proper preimage of $D_{r,c}$. Notice that as classes in $A^*(\hat{X}^G)$, $\hat{D}_{r,c} = \pi^* D_{r,c} - E_c$. Moreover, we can write $c(TX^G)$ as $\oplus_{r\in G} TX_r$ where TX_r is the pullback of the tangent bundle of X under the r^{th} projection. Since $\hat{D}_{r,c}$ are disjoint from $Y_{\lambda_1\ldots,\lambda_k}$, we get

$$c(i^* T\hat{X}^G) = i^*\pi^* c(TX^G) \prod_{c=1}^{k} (1+i^* E_c)^{1-|G|}$$

where $i\colon Y_{\lambda_1,\ldots,\lambda_k} \to \hat{X}^G$ is the embedding. Notice that π restricts to an embedding on $Y_{\lambda_1,\ldots,\lambda_k}$ with image $D_I \subseteq X \subseteq X^G$ where I is the set of all c that for which λ_c is nontrivial. The following lemma describes $i^* T\hat{X}^G$ in more detail.

Lemma 6.3. *Let λ be a character of G. Then the λ-component V_λ of the restriction of $T\hat{X}^G$ to $Y_{\lambda_1,\ldots,\lambda_k}$, identified with $D_I \neq \emptyset$ can be described as follows. If $\lambda = 0$, then $V_\lambda = TD_I$. If $\lambda \neq 0$, then there is an exact sequence*

$$0 \to j^* TX_{\log} \to V_\lambda \to \bigoplus_{c,\lambda_c=\lambda} \mathcal{O}(D_c) \to 0$$

where j is the embedding $D_I \to X$ and $TX_{\log}$ is the dual to the bundle of log-differentials for (X, D).

Proof. We observe that $Y_{\lambda_1,\ldots,\lambda_k}$ is contained in the intersection of E_c for $\lambda_c \neq 0$, which induces a G-equivariant surjection from the restriction of $T\hat{X}^G$ to the restriction of $\oplus_{\lambda_c\neq 0}\mathcal{O}(E_c)$ with the kernel being the restriction of the tangent space to $\cap_{\lambda_c\neq 0}E_c$ to $Y_{\lambda_1,\ldots,\lambda_k}$. It is easy to see that under the identification of $Y_{\lambda_1,\ldots,\lambda_k}$ with D_I the restriction of $\mathcal{O}(E_c)$ is isomorphic to $\mathcal{O}(D_c)$ and has character λ_c.

So we now need to investigate the restriction of the tangent space of $\cap_{\lambda_c\neq 0}E_c$ and its eigenbundles. The $\lambda = 0$ case is clear, so it is enough to consider the normal bundle to $Y_{\lambda_1,\ldots,\lambda_k}$ in $\cap_{\lambda_c\neq 0}E_c$. Locally, in the notation of (14), this bundle is isomorphic to the restriction of the tangent bundle of $(\mathbb{C}^*)^{ijk-k}$. The cotangent bundle of $(\mathbb{C}^*)^{ijk-k}$ is generated by $\frac{dx_{r,c}}{x_{r,c}} - \frac{dx_{r_1,c}}{x_{r_1,c}}$, so its λ-eigenbundle is isomorphic to a bundle generated by $\frac{dx_c}{x_c}$, which is precisely the bundle of logarithmic differential forms. Even though (14) refers to the neighborhood of a point of the intersection of $\dim X$ divisors D_c, it is clear that the general case is obtained by a Cartesian product with a disc and the identification is still valid. It remains to notice that this identification behaves well under coordinate changes. $\square$

Proof of Theorem 6.1 *continues.* In view of Lemma 6.3, the contribution of (g,h) to the orbifold elliptic genus of $(X^G, D^{(G)})$ is

$$\sum_{\{\lambda_1,\ldots,\lambda_k\},\cap_{\lambda_c\neq 0}D_c\neq\emptyset} \int_X \Big(\prod_{c,\lambda_c\neq 0} D_c\Big) \prod_l \frac{x_l\theta(\frac{x_l}{2\pi\mathrm{i}} - z)}{\theta(\frac{x_l}{2\pi\mathrm{i}})} \prod_{c,\lambda_c\neq 0} \frac{\theta(\frac{D_c}{2\pi\mathrm{i}})}{D_c\theta(\frac{D_c}{2\pi\mathrm{i}} - z)}$$
$$\times \prod_{\lambda\neq 0}\Big(\prod_l \frac{\theta(\frac{x_l}{2\pi\mathrm{i}} + \lambda(g) - \lambda(h)\tau - z)}{\theta(\frac{x_l}{2\pi\mathrm{i}} + \lambda(g) - \lambda(h)\tau)} \mathrm{e}^{2\pi\mathrm{i}\lambda(h)z}$$
$$\times \prod_{c=1}^k \frac{\theta(\frac{D_c}{2\pi\mathrm{i}} + \lambda(g) - \lambda(h)\tau)\theta(\lambda(g) - \lambda(h)\tau - z)}{\theta(\frac{D_c}{2\pi\mathrm{i}} + \lambda(g) - \lambda(h)\tau - z)\theta(\lambda(g) - \lambda(h)\tau)}\Big)$$
$$\times \prod_{c,\lambda_c\neq 0} \frac{\theta(\frac{D_c}{2\pi\mathrm{i}} + \lambda_c(g) - \lambda_c(h)\tau - z)}{\theta(\frac{D_c}{2\pi\mathrm{i}} + \lambda_c(g) - \lambda_c(h)\tau)} \mathrm{e}^{2\pi\mathrm{i}\lambda_c(h)z}$$
$$\times \prod_{c,\lambda_c\neq 0} \frac{\theta(\frac{D_c}{2\pi\mathrm{i}} + \lambda_c(g) - \lambda_c(h)\tau - |G|(\alpha_c+1)z)\theta(-z)}{\theta(\frac{D_c}{2\pi\mathrm{i}} + \lambda_c(g) - \lambda_c(h)\tau - z)\theta(-|G|(\alpha_c+1)z)} \mathrm{e}^{2\pi\mathrm{i}(|G|\alpha_c + |G| - 1)\lambda_c(h)z}$$
$$\times \prod_{c,\lambda_c=0} \frac{\theta(\frac{D_c}{2\pi\mathrm{i}} - |G|(\alpha_c+1)z)\theta(-z)}{\theta(\frac{D_c}{2\pi\mathrm{i}} - z)\theta(-|G|(\alpha_c+1)z)}$$

where we have used the fact that the coefficients by E_c in the log-pair on $\hat{X}^G$ are $(|G|\alpha_c - |G| - 1)$ and other divisors do not intersect the fixed point set and are thus irrelevant. After observing that the formula gives 0 for the case

$D_I = \emptyset$, the above can be rewritten as

$$F_{i,j,s} = \sum_{\{\lambda_1,\dots,\lambda_k\}} \int_X \prod_l \Big(x_l \prod_\lambda \frac{\theta(\frac{x_l}{2\pi \mathrm{i}} + \lambda(g) - \lambda(h)\tau - z)}{\theta(\frac{x_l}{2\pi \mathrm{i}} + \lambda(g) - \lambda(h)\tau)} \mathrm{e}^{2\pi\mathrm{i}\lambda(h)z} \Big)$$

$$\times \prod_{\lambda\neq 0} \prod_{c=1}^{k} \frac{\theta(\frac{D_c}{2\pi\mathrm{i}} + \lambda(g) - \lambda(h)\tau)\theta(\lambda(g) - \lambda(h)\tau - z)}{\theta(\frac{D_c}{2\pi\mathrm{i}} + \lambda(g) - \lambda(h)\tau - z)\theta(\lambda(g) - \lambda(h)\tau)}$$

$$\times \prod_c \frac{\theta(\frac{D_c}{2\pi\mathrm{i}})\theta(\frac{D_c}{2\pi\mathrm{i}} + \lambda_c(g) - \lambda_c(h)\tau - |G|(\alpha_c+1)z)\theta(-z)}{\theta(\frac{D_c}{2\pi\mathrm{i}} - z)\theta(\frac{D_c}{2\pi\mathrm{i}} + \lambda_c(g) - \lambda_c(h)\tau)\theta(-|G|(\alpha_c+1)z)} \mathrm{e}^{2\pi\mathrm{i}|G|(\alpha_c+1)\lambda_c(h)z}.$$

We will use the following lemmas that take into account the specific form of G.

Lemma 6.4.

$$\prod_\lambda \frac{\theta(\frac{x_l}{2\pi\mathrm{i}} + \lambda(g) - \lambda(h)\tau - z)}{\theta(\frac{x_l}{2\pi\mathrm{i}} + \lambda(g) - \lambda(h)\tau)} \mathrm{e}^{2\pi\mathrm{i}\lambda(h)z} = \frac{\theta(\frac{ix_l}{2\pi\mathrm{i}} - iz, \frac{i\tau - s}{j})}{\theta(\frac{ix_l}{2\pi\mathrm{i}}, \frac{i\tau-s}{j})}.$$

Proof. First, we observe that the set of pairs $(\lambda(g), \lambda(h))$ can be taken to be the set of pairs $(\frac{m}{ij}, \frac{n}{j})$ such that $0 \le n \le j-1$, $0 \le m \le ij-1$ and $m = ns \mod j$.

Let us check the transformation properties of the left-hand side of the equation under $z \to z + 1/i$. The exponential factors contribute

$$\exp(2\pi\mathrm{i}\frac{1}{i}\sum_\lambda \lambda(h)) = \exp(2\pi\mathrm{i}\sum_{n=0}^{j-1}\frac{n}{j}) = (-1)^{j-1}.$$

For each $n = 0, \dots, j-1$, the set of $\lambda(g)$ is given by the fractional parts of $\frac{ns}{ij} + \frac{k}{i}, k = 0, \dots, i-1$. There will be exactly one such fractional part which is less than $\frac{1}{i}$. The transformation $z \to z + 1/i$ switches these fractions around except for the extra 1 for the fraction with $\lambda(g) < \frac{1}{i}$. As a result, we get the extra factor $(-1)^j$ from the numerator, so overall the left-hand side of the equation changes sign under $z \to z + \frac{1}{i}$, as does the right-hand side.

Now, let us check the transformation properties of the left-hand side under $z \to z + \frac{i\tau - s}{ij}$. This variable change amounts to $n \to n+1$, $m \to m+s$, which moves around the θs in the numerator, except for the cases when new values of m and n fall out of their prescribed ranges. In the case of m falling out of its range, the extra factor required to put it back in is (-1). It is easy to calculate the number of such occurrences, because the sum of all m is going to change by ijs which require s switches to put into the correct range. So the extra factor from the switches of m is $(-1)^s$. In the case of n, it falls out of the range when it goes from $(j-1)$ to j. In this case we get $m = 0 \mod j$, so $\lambda(g)=\frac{k}{i}, k=0,\ldots,i-1$. The extra factors come from the transformation properties of θ and equal

$$(-1)^i \mathrm{e}^{\sum_{k=0}^{i-1}(2\pi\mathrm{i}(\frac{x_l}{2\pi\mathrm{i}}+\frac{k}{i}-z)-\pi\mathrm{i}\tau)}=\mathrm{e}^{-\pi\mathrm{i}+ix_l-2\pi\mathrm{i}iz-\pi\mathrm{i}i\tau}.$$

The exponential factors contribute $\exp(\pi\mathrm{i}(j-1)\frac{(i\tau-s)}{j})$, so the overall factor is

$$\mathrm{e}^{-\pi\mathrm{i}+ix_l-2\pi\mathrm{i}iz-\pi\mathrm{i}i\tau+\pi\mathrm{i}(j-1)\frac{(i\tau-s)}{j}+\pi\mathrm{i}s}=-\mathrm{e}^{ix_l-2\pi\mathrm{i}iz-\pi\mathrm{i}\frac{(i\tau-s)}{j}}$$

which is exactly the effect of the transformation $z\to z+\frac{(is-\tau)}{ij}$ to the right-hand side of the equation.

It is straightforward to check that both sides have no poles and the same zeroes as functions of z, therefore their ratio is a holomorphic elliptic function, hence a constant. It remains to observe that both sides equal 1 for $z=0$. □

LEMMA 6.5.

$$\prod_{\lambda\neq 0}\frac{\theta(\frac{D_c}{2\pi\mathrm{i}}+\lambda(g)-\lambda(h)\tau)\theta(\lambda(g)-\lambda(h)\tau-z)}{\theta(\frac{D_c}{2\pi\mathrm{i}}+\lambda(g)-\lambda(h)\tau-z)\theta(\lambda(g)-\lambda(h)\tau)}$$
$$=\frac{\theta(\frac{iD_c}{2\pi\mathrm{i}},\frac{i\tau-s}{j})\theta(\frac{D_c}{2\pi\mathrm{i}}-z)\theta(-iz,\frac{i\tau-s}{j})\theta'(0)}{\theta(\frac{iD_c}{2\pi\mathrm{i}}-iz,\frac{i\tau-s}{j})\theta(\frac{D_c}{2\pi\mathrm{i}})i\theta'(0,\frac{i\tau-s}{j})\theta(-z)}.$$

Proof. We use the result of Lemma 6.4 with x_l replaced by D_c and the limit of the same calculation as $x_l\to 0$. □

Proof of Theorem 6.1 *continues.* By Lemmas 6.4 and 6.5, we can rewrite $F_{i,j,s}$ as

$$F_{i,j,s}=\sum_{\{\lambda_1,\ldots,\lambda_k\}}\int_X\prod_l\left(x_l\frac{\theta(\frac{ix_l}{2\pi\mathrm{i}}-iz,\frac{i\tau-s}{j})}{\theta(\frac{ix_l}{2\pi\mathrm{i}},\frac{i\tau-s}{j})}\right)\prod_{c=1}^k\mathrm{e}^{2\pi\mathrm{i}ij(\alpha_c+1)\lambda_c(h)z}$$
$$\times\prod_{c=1}^k\frac{\theta(\frac{iD_c}{2\pi\mathrm{i}},\frac{i\tau-s}{j})\theta(-iz,\frac{i\tau-s}{j})\theta'(0)\theta(\frac{D_c}{2\pi\mathrm{i}}+\lambda_c(g)-\lambda_c(h)\tau-ij(\alpha_c+1)z)}{\theta(\frac{iD_c}{2\pi\mathrm{i}}-iz,\frac{i\tau-s}{j})i\theta'(0,\frac{i\tau-s}{j})\theta(\frac{D_c}{2\pi\mathrm{i}}+\lambda_c(g)-\lambda_c(h)\tau)\theta(-ij(\alpha_c+1)z)}.$$

We will use the following lemma.

LEMMA 6.6.

$$\sum_\lambda\frac{\theta(u+\lambda(g)-\lambda(h)\tau-v)}{\theta(u+\lambda(g)-\lambda(h)\tau)}\mathrm{e}^{2\pi\mathrm{i}\lambda(h)v}=i\frac{\theta'(0,\frac{i\tau-s}{j})\theta(-v)\theta(iu-\frac{v}{j},\frac{i\tau-s}{j})}{\theta'(0)\theta(-\frac{v}{j},\frac{i\tau-s}{j})\theta(iu,\frac{i\tau-s}{j})}.$$

Proof. We use the following basic formula which is essentially contained in [6], where the right-hand side converges for $\Im(\tau)>\Im(u)>0$.

$$-\frac{\theta(u+z)\theta'(0)}{2\pi\mathrm{i}\theta(u)\theta(z)}=\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{2\pi\mathrm{i}ku}}{1-\mathrm{e}^{2\pi\mathrm{i}z}\mathrm{e}^{2\pi\mathrm{i}k\tau}}.$$

We also recall the description of $(\lambda(g),\lambda(h))$ from Lemma 6.4. Not that the quotient depends on the choice of $\lambda(g) \mod 1$ only, so we can assume that $\lambda(g)=\frac{ns}{ij}+\frac{m}{i}, m\in\mathbb{Z}/i\mathbb{Z}$. Then,

$$\begin{aligned}
\sum_{\lambda}&\frac{\theta(u+\lambda(g)-\lambda(h)\tau-v)}{\theta(u+\lambda(g)-\lambda(h)\tau)}\mathrm{e}^{2\pi\mathrm{i}\lambda(h)v}\\
&=\sum_{m=0}^{i-1}\sum_{n=0}^{j-1}\frac{\theta(u-v+\frac{m}{i}+\frac{ns}{ij}-\frac{n}{j}\tau)}{\theta(u+\frac{m}{i}+\frac{ns}{ij}-\frac{n}{j}\tau)}\mathrm{e}^{2\pi\mathrm{i}v\frac{n}{j}}\\
&=-\frac{2\pi\mathrm{i}\theta(-v)}{\theta'(0)}\sum_{m=0}^{i-1}\sum_{n=0}^{j-1}\sum_{k\in\mathbb{Z}}\mathrm{e}^{2\pi\mathrm{i}v\frac{n}{j}}\frac{\mathrm{e}^{2\pi\mathrm{i}k(u+\frac{m}{i}+\frac{ns}{ij}-\frac{n}{j}\tau)}}{1-\mathrm{e}^{-2\pi\mathrm{i}v}\mathrm{e}^{2\pi\mathrm{i}k\tau}}\\
&=-i\frac{2\pi\mathrm{i}\theta(-v)}{\theta'(0)}\sum_{n=0}^{j-1}\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{2\pi\mathrm{i}v\frac{n}{j}}\mathrm{e}^{2\pi\mathrm{i}kiu}\mathrm{e}^{-2\pi\mathrm{i}kn\frac{i\tau-s}{j}}}{1-\mathrm{e}^{-2\pi\mathrm{i}v}\mathrm{e}^{2\pi\mathrm{i}ki\tau}}\\
&=-i\frac{2\pi\mathrm{i}\theta(-v)}{\theta'(0)}\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{2\pi\mathrm{i}kiu}}{(1-\mathrm{e}^{-2\pi\mathrm{i}v}\mathrm{e}^{2\pi\mathrm{i}ki\tau})}\frac{(1-\mathrm{e}^{2\pi\mathrm{i}v}\mathrm{e}^{-2\pi\mathrm{i}k(i\tau-s)})}{(1-\mathrm{e}^{2\pi\mathrm{i}v\frac{1}{j}}\mathrm{e}^{-2\pi\mathrm{i}k\frac{i\tau-s}{j}})}\\
&=i\frac{2\pi\mathrm{i}\theta(-v)}{\theta'(0)}\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{2\pi\mathrm{i}kiu}\mathrm{e}^{2\pi\mathrm{i}v}\mathrm{e}^{-2\pi\mathrm{i}ki\tau}}{(1-\mathrm{e}^{2\pi\mathrm{i}v\frac{1}{j}}\mathrm{e}^{-2\pi\mathrm{i}k\frac{i\tau-s}{j}})}\\
&=-i\mathrm{e}^{2\pi\mathrm{i}v(1-\frac{1}{j})}\frac{2\pi\mathrm{i}\theta(-v)}{\theta'(0)}\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{2\pi\mathrm{i}k(iu-(j-1)\frac{i\tau-s}{j})}}{(1-\mathrm{e}^{-2\pi\mathrm{i}v\frac{1}{j}}\mathrm{e}^{2\pi\mathrm{i}k\frac{i\tau-s}{j}})}\\
&=i\mathrm{e}^{2\pi\mathrm{i}v(1-\frac{1}{j})}\frac{\theta(-v)\theta'(0,\frac{i\tau-s}{j})\theta(iu-(j-1)\frac{i\tau-s}{j}-\frac{v}{j},\frac{i\tau-s}{j})}{\theta'(0)\theta(iu-(j-1)\frac{i\tau-s}{j},\frac{i\tau-s}{j})\theta(-\frac{v}{j},\frac{i\tau-s}{j})}\\
&=i\frac{\theta'(0,\frac{i\tau-s}{j})\theta(-v)\theta(iu-\frac{v}{j},\frac{i\tau-s}{j})}{\theta'(0)\theta(-\frac{v}{j},\frac{i\tau-s}{j})\theta(iu,\frac{i\tau-s}{j})}.
\end{aligned}$$

In the above calculations the series are absolutely convergent, as long as $\Im(\tau)>0$ and $1>\frac{\Im(u)}{\Im(\tau)}>\frac{j-1}{j}$. Then analytic continuation finishes the proof. $\square$

Proof of Theorem 6.1 *continues.* By Lemma 6.6 we can rewrite

$$F_{i,j,s}=\int_X\prod_l\left(\frac{x_l\theta(\frac{ix_l}{2\pi\mathrm{i}}-iz,\frac{i\tau-s}{j})}{\theta(\frac{ix_l}{2\pi\mathrm{i}},\frac{i\tau-s}{j})}\right)\prod_{c=1}^{k}\frac{\theta(-iz,\frac{i\tau-s}{j})\theta(\frac{iD_c}{2\pi\mathrm{i}}-i(\alpha_c+1)z,\frac{i\tau-s}{j})}{\theta(\frac{iD_c}{2\pi\mathrm{i}}-iz,\frac{i\tau-s}{j})\theta(-i(\alpha_c+1)z,\frac{i\tau-s}{j})}.$$

We notice that when we calculate $\int_X$, we only pick up the polynomials of degree $\dim X$ in x_l and D_c, which allows us to conclude that

$$F_{i,j,s}(z,\tau)=\mathrm{Ell}(X,D;iz,\frac{i\tau-s}{j}).$$

We now recall that the contribution of the commuting pair of elements $g,h\in S_n$ to the orbifold elliptic genus of $(X^n,D^{(n)})$ is $\frac{1}{n!}$ times the product of several

$F_{i,j,s}$, each one corresponding to an orbit of the action of $\langle g,h\rangle$ on $\{1,\dots,n\}$. Every such orbit I_m will have i_m, j_m and $s_m\in\mathbb{Z}/j_m\mathbb{Z}$ uniquely specified. So we have

$$\begin{aligned}
&\sum_{n\geq 0} p^n \mathrm{Ell}(X^n/S_n, D^{(n)}/S_n; z,\tau)\\
&\qquad = \sum_{n\geq 0} p^n \sum_{gh=hg, g,h\in S_n} \frac{1}{n!}\prod_{I_m} F_{i_m,j_m,s_m}(z,\tau)\\
&\qquad = \sum_{r\colon Z_{>0}\times Z_{>0}\to Z_{\geq 0}} \frac{p^{\sum_{i,j} ijr(i,j)}}{\prod_{i,j} r(i,j)!(ij)^{r(i,j)}} \prod_{i,j}\left(\sum_{s=0}^{j-1} F_{i,j,s}(z,\tau)^{r(i,j)}\right).
\end{aligned}$$

In this calculation we have used the fact that for $n=\sum_{i,j} ijr(i,j)$ there are

$$\frac{1}{\prod_{i,j} r(i,j)!}\frac{n!}{\prod_{i,j}((ij)!)^{r(i,j)}}$$

ways to split $\{1,\dots,n\}$ into groups of subsets so that there $r(i,j)$ subsets of "type (i,j)". Then for each set of type (i,j) there are $\frac{(ij)!}{ij}$ different ways to define the action of the g and h conjugate to the standard action we have discussed earlier. We now conclude that

$$\begin{aligned}
\sum_{n\geq 0} p^n \mathrm{Ell}(X^n/S_n, D^{(n)}/S_n; z,\tau) &= \exp\left(\sum_{i,j>0}\sum_{s=0}^{j-1} \frac{p^{ij}}{ij} F_{i,j,s}(z,\tau)\right)\\
&= \exp\left(\sum_{i,j>0}\sum_{s=0}^{j-1} \frac{p^{ij}}{ij}\mathrm{Ell}(X,D; iz, \frac{i\tau-s}{j})\right)\\
&= \exp\left(\sum_{i,j>0}\sum_{m,l}\sum_{s=0}^{j-1} c(m,l)\frac{p^{ij}}{ij} y^{il} q^{\frac{im}{j}} \mathrm{e}^{2\pi\mathrm{i}\frac{ms}{j}}\right)\\
&= \exp\left(\sum_{i,j>0} c(mj,l)\frac{p^{ij}}{i} y^{il} q^{im}\right)\\
&= \prod_{j=1}^{\infty}\prod_{m,l} \exp\left(c(mj,l)\sum_{i>0}\frac{p^{ij}}{i} y^{il}q^{im}\right)\\
&= \prod_{j=1}^{\infty}\prod_{m,l} \exp(-c(mj,l)\ln(1-p^j y^l q^m))\\
&= \prod_{j=1}^{\infty}\prod_{m,l} (1-p^j y^l q^m)^{-c(mj,l)},
\end{aligned}$$

which finishes the proof. $\square$

Corollary 6.7. *Let X be a complex projective surface and $X^{(n)}$ be the Hilbert scheme of subschemes of X of length n. Let $\sum_{m,l} c(m,l)y^l q^n$ be the elliptic genus of X. Then*

$$\sum_{n\geq 0} p^n \mathrm{Ell}(X^{(n)};z,\tau) = \prod_{i=1}^{\infty}\prod_{l,m} \frac{1}{(1-p^i y^l q^m)^{c(mi,l)}}.$$

Proof. By Theorem 5.3, the orbifold elliptic genus of the symmetric power X^n/S_n equals the elliptic genus of its crepant resolution, which is provided by $X^{(n)}$ in the surface case. □

Remark 6.8. As a corollary of our work we easily deduce the analog of the DMVV conjecture for wreath products; see [32].

7. Open questions

In this section we mention possible directions in which the results of this paper could be extended.

The biggest drawback of our technique is that it does not establish the elliptic genus of a Kawamata log-terminal pair as a graded dimension of some natural vector space. In the smooth nonequivariant case such a description is provided by (3). Even more interesting is the description of the elliptic genus as the graded dimension of the vertex algebra which is the cohomology of the chiral de Rham complex of [27]; see [6]. This is still open even in the nonequivariant case. This would be very interesting even at the $q = 0$ level, since it may give a vector space that realizes the stringy Hodge numbers of a singular variety X.

It would also be interesting to try to somehow extend the results of this paper to more general orbifolds (smooth stacks). The definition of orbifold elliptic genus (no divisor) was extended to this generality in [13]. While our paper focuses on the global quotient case, it is possible that its techniques may still apply to the case of an algebraic variety with at most quotient singularities. Indeed, the toroidal techniques are in some sense local. In a related remark, we do believe that the analog of our main theorem holds for the orbifold elliptic classes of (X, E, G) and $(X/G_1, E/G_1, G/G_1)$ where G is an arbitrary normal subgroup of G.

The birational properties of elliptic genus mean that it is preserved under K-equivalence (cf. [20], [31]). It is conjectured in [20] that K-equivalent varieties have equivalent derived categories. This therefore points to a possible connection between elliptic classes considered above and derived categories. It is however more likely that both objects are a part of a bigger structure of a conformal field theory which somehow behaves well under K-equivalence.

This is largely speculative at this point, but it would be interesting to define mathematically an invariant of a variety which would encompass both its derived category and its elliptic genus. The situation is even more murky for Kawamata log-terminal pairs, since it is unclear what the correct definition of the derived category of the pair may be.

A mirror symmetric analog of a resolution of singularities is a deformation to a smooth variety. Unfortunately, this theory is not nearly as developed as the theory of birational morphisms. It would be interesting to define an analog of a crepant resolution in this setting and to try to check the invariance of the elliptic genus.

It is known that the elliptic genus for smooth manifolds has a rigidity property. Recently, this property has been extended to the orbifold case in [13]. It is reasonable to try to extend this property to the case of Kawamata log-terminal pairs. It is possible that the framework of pairs that consist of an orbifold and an equivariant bundle over it (see [13]) will be useful.

It would be also interesting to see how the orbifold elliptic class of a singular variety X compares to the Mather Chern class of X; see for example [14].

8. Appendix. Assorted toric lemmas

In this appendix we collect several combinatorial statements which are useful in our study of toroidal morphisms.

LEMMA 8.1. *Let Σ be a simplicial fan in the first orthant of a lattice $N = \bigoplus_i \mathbb{Z} e_i$. Moreover, let $\hat{N}$ be a sublattice of N of finite index. We denote the quotient group $N/\hat{N}$ by G. We further assume that each cone C of Σ is generated by a part of a basis of $\hat{N}$. We denote by x_i the linear functions on $N_{\mathbb{C}}$ that are dual to e_i. For each cone C of maximum dimension we denote by $\{x_{i;C}\}$ the linear combinations of x_i which are dual to the generators of C. Let a be a linear function on N which takes values a_i on e_i and values $a_{i;C}$ on the generators of C. Then*

$$\sum_{C\in\Sigma,\dim C=\operatorname{rk}N}\ \sum_{g,h\in G}\prod_i \frac{\theta'(0)\theta(\frac{x_{i;C}}{2\pi\mathrm{i}}+g_{i;C}-h_{i;C}\tau-a_{i;C})}{2\pi\mathrm{i}\,\theta(\frac{x_{i;C}}{2\pi\mathrm{i}}+g_{i;C}-h_{i;C}\tau)\theta(-a_{i;C})}\mathrm{e}^{2\pi\mathrm{i}a_{i;C}h_{i;C}}$$
$$= |N:\hat{N}|\prod_i \frac{\theta'(0)\theta(\frac{x_i}{2\pi\mathrm{i}}-a_i)}{2\pi\mathrm{i}\,\theta(\frac{x_i}{2\pi\mathrm{i}})\theta(-a_i)}$$

where $g_{i;C}$ and $h_{i;C}$ denote rational numbers in the range $[0,1)$ that are fractional parts of the coordinates of the lifts of g and h to N in the basis of C.

In the case when the lattice N is one-dimensional one obtains the following identity not involving toric data:

Corollary 8.2.

$$\frac{1}{d}\sum_{0\le i,j<d}\frac{\theta(\frac{x}{2\pi \mathrm{i} d}+\frac{i}{d}-\frac{j}{d}\tau-z)}{\theta(\frac{x}{2\pi \mathrm{i} d}+\frac{i}{d}-\frac{j}{d}\tau)\theta(-z)}e^{\frac{2\pi i z j}{d}}=\frac{\theta(\frac{x}{2\pi \mathrm{i}}-\frac{z}{d})}{\theta(\frac{x}{2\pi \mathrm{i}})\theta(-\frac{z}{d})}.$$

The scheme of the proof is the same as in the general case below: one checks that both sides have the pole of order 1 for $x=0$ the residues are the same and, moreover, the ratio of both sides is an elliptic function with respect to $x\to x+2\pi \mathrm{i}, x\to x+2\pi \mathrm{i}\tau$.

Proof. We will argue by induction on $\mathrm{rk}N$, with $\mathrm{rk}N=0$ being the trivial base of the induction (or checking first 8.2 as outlined above).

Let us study transformation properties of both sides of the equation under the translations $x_1\to x_1+2\pi\mathrm{i}$ and $x_1\to x_1+2\pi\mathrm{i}\tau$. Under the transformation $x_1\to x_1+2\pi\mathrm{i}$ the term of the sum that corresponds to C,g,h changes into the term that corresponds to $C,g+e_1,h$. Indeed, the coefficients of e_1 in the basis of the cone C are the same as the coefficients of x_1 in the linear functions $x_{i;C}$. As a result, both sides of the equation are unchanged under $x_1\to x_1+2\pi\mathrm{i}$. Under the transformation $x_1\to x_1+2\pi\mathrm{i}\tau$ the term that corresponds to C,g,h changes into the term that corresponds to $C,g,h-e_1$ times $\mathrm{e}^{2\pi\mathrm{i}a_1}$. Indeed, the extra factor comes from the exponential terms since a_1 is the difference between the value of a on h and $h-e_1$. We also observe that the terms of the product are such that any lift of h to N gives the same value, so the fact that some of the coefficients of $h-e_1$ in the basis of C are not in $[0,1)$ is not a problem. Clearly, the right-hand side of the equation has the same transformation properties.

We will now show that the left-hand side of the equation of the lemma has only simple poles at $x_1=2\pi\mathrm{i}(\mathbb{Z}+\mathbb{Z}\tau)$, considered as a function of x_1 with fixed generic values of other parameters. By the above transformation argument, it is enough to show there are no poles at the solutions to linear equations on x_1 given by $x_{i;C}=0$. We only need to worry about such $x_{i;C}$ that define a noncoordinate hyperplane which corresponds to some cone $\bar{C}$ of dimension $\mathrm{rk}N-1$ in the interior of the first orthant. This cone $\bar{C}$ is contained in two cones C and C' of maximum dimension and we argue that the contributions of these cones to the singular part of the Laurent expansion around $x_{i;C}=0$ cancel. Let $v_1,\ldots,v_{\mathrm{rk}N-1},v$ be the generators of C and $v_1,\ldots,v_{\mathrm{rk}N-1},v'$ be the generators of C'. It is easy to see that $v+v'=\sum_{i=1}^{\mathrm{rk}N-1}c_iv_i$ for some integer c_i and

$$x_{i;C}=x_{i;C'}+c_ix_{\mathrm{rk}N;C},\ 1\le i\le \mathrm{rk}N-1;\quad x_{\mathrm{rk}N;C}=-x_{\mathrm{rk}N;C'}.$$

There are similar transformation formulas for $g_{i;C}$ and $h_{i;C}$. The poles at $x_{\mathrm{rk}N;C}=0$ can be of order at most 1, and they can only occur in the case $g_{\mathrm{rk}N;C}=h_{\mathrm{rk}N;C}=g_{\mathrm{rk}N;C'}=h_{\mathrm{rk}N;C'}=0$. As a result, we only need to calculate the residue at this pole. The residue of the term that corresponds to C', g, h is equal to

$$-\frac{1}{c}\prod_{i=1}^{\mathrm{rk}N-1}\frac{\theta'(0)\theta(\frac{x_{i;C}}{2\pi\mathrm{i}}+g_{i;C}-h_{i;C}\tau-a_{i;C})}{2\pi\mathrm{i}\,\theta(\frac{x_{i;C}}{2\pi\mathrm{i}}+g_{i;C}-h_{i;C}\tau)\theta(-a_{i;C})}\mathrm{e}^{2\pi\mathrm{i}a_{i;C}h_{i;C}}$$

where c is the coefficient of x_1 in $x_{\mathrm{rk}N;C}$. This cancels the residue of the term that corresponds to C, g, h.

By a standard argument from the theory of elliptic functions we conclude that the left-hand side of the equation of the lemma has simple zeros at $x_1 = 2\pi\mathrm{i}(a_1+\mathbb{Z}+\mathbb{Z}\tau)$ and at no other points. Moreover, the ratio of the two sides of the equation is independent of x_1. It is therefore enough to verify that the residues at $x_1 = 0$ of both sides are same. Only the terms with cones C that have a face $\bar{C}$ of dimension $\mathrm{rk}N-1$ that lies in the side of the orthant spanned by $e_{>1}$ can contribute to the residue. We will denote the generator of C that does not lie in $\bar{C}$ by e_1. The residue occurs only for $g_{1;C} = h_{1;C} = 0$ and then it equals

$$\frac{1}{c}\prod_{i>1}\frac{\theta'(0)\theta(\frac{x_{i;\bar{C}}}{2\pi\mathrm{i}}+g_{i;\bar{C}}-h_{i;\bar{C}}\tau-a_{i;\bar{C}})}{2\pi\mathrm{i}\,\theta(\frac{x_{i;\bar{C}}}{2\pi\mathrm{i}}+g_{i;\bar{C}}-h_{i;\bar{C}}\tau)\theta(-a_{i;\bar{C}})}\mathrm{e}^{2\pi\mathrm{i}a_{i;\bar{C}}h_{i;\bar{C}}}$$

where c is the coefficient of x_1 in $x_{1;C}$. Here we have observed that $x_{i;C}$ restricts to $x_{i;\bar{C}}$ on $x_1 = 0$, and similarly for $g_{i;C}$ and $h_{i;C}$. If the intersection of $\hat{N}$ and N with the span of $e_{>1}$ are lattices $\hat{N}_1$ and N_1 respectively, then

$$c = \frac{|N_1 : \hat{N}_1|}{|N : \hat{N}|}.$$

It remains to apply the induction hypothesis to the fan Σ_1 induced by Σ on the span of $e_{>1}$. $\square$

Lemma 8.3. *Let Σ be a simplicial fan in the first orthant of a lattice $N = \oplus_i\mathbb{Z}e_i$. Moreover, let $\hat{N}$ be a sublattice of N of finite index. We further assume that each cone C of Σ is generated by a part of a basis of $\hat{N}$. We denote by x_i the linear functions on $N_{\mathbb{C}}$ that are dual to e_i. For each cone C of maximum dimension we denote by $\{x_{i;C}\}$ the linear combinations of x_i which are dual to the generators of C. Then*

$$\sum_{C\in\Sigma,\dim C=\mathrm{rk}N}\frac{1}{\prod_{i=1}^{\mathrm{rk}N}x_{i,C}} = |N:\hat{N}|\frac{1}{\prod_{i=1}^{\mathrm{rk}N}x_i}.$$

Proof. By Lemma 8.1,

$$\sum_{C\in\Sigma,\dim C=\mathrm{rk}N}\sum_{g,h\in G}\prod_i\frac{\theta'(0)\theta(\frac{\varepsilon x_{i;C}}{2\pi\mathrm{i}}+g_{i;C}-h_{i;C}\tau-a_{i;C})}{2\pi\mathrm{i}\,\theta(\frac{\varepsilon x_{i;C}}{2\pi\mathrm{i}}+g_{i;C}-h_{i;C}\tau)\theta(-a_{i;C})}\mathrm{e}^{2\pi\mathrm{i}a_{i;C}h_{i;C}}$$

$$= |N:\hat{N}|\prod_i\frac{\theta'(0)\theta(\frac{\varepsilon x_i}{2\pi\mathrm{i}}-a_i)}{2\pi\mathrm{i}\,\theta(\frac{\varepsilon x_i}{2\pi\mathrm{i}})\theta(-a_i)}.$$

It remains to look at the coefficient by $\varepsilon^{-\mathrm{rk}N}$ in the Laurent expansion of both sides around $\varepsilon = 0$. $\square$

Example 8.4. In the case of Figure 1 the identity of Lemma 8.3 is

$$\frac{1}{x_2(\frac{x_1-2x_2}{3})} + \frac{1}{(\frac{2x_2-x_1}{3})(\frac{2x_1-x_2}{3})} + \frac{1}{x_1(\frac{x_2-2x_1}{3})} = \frac{3}{x_1x_2}.$$

LEMMA 8.5. *Let Σ be a simplicial fan in a lattice N such that the union of all of its cones is a product of a subspace and a positive orthant. In addition, we assume that all maximum-dimensional cones of Σ are generated by a basis of N. Then*

$$\sum_{C\in\Sigma,\dim C=\mathrm{rk}N} \frac{1}{\prod_{i=1}^{\mathrm{rk}N} x_{i;C}} = 0$$

where $x_{i;C}$ denote the basis of linear forms dual to the lattice generators of C.

Proof. By Lemma 8.3, applied to the case $\hat{N} = N$, the function $\frac{1}{\prod_{i=1}^{\mathrm{rk}N} x_{i;C}}$ is additive on Σ, so we can replace Σ by any of its subdivisions with the same properties. After an appropriate subdivision, we can assume that each cone of Σ sits in one of the orthants and the support of Σ is $\oplus_{i\notin I}\mathbb{R}_{\geq 0}e_i + \oplus_{i\in I}\mathbb{R}e_i$ for some basis $\{e_i\}$ and some nonempty set I. Then we apply Lemma 8.3 again to show that

$$\sum_{C\in\Sigma,\dim C=\mathrm{rk}N} \frac{1}{\prod_{i=1}^{\mathrm{rk}N} x_{i;C}} = \sum_{\{\sigma_i\}\in\{1,-1\}^I} \prod_{i\in I}\frac{1}{\sigma_i x_i}\prod_{i\notin I}\frac{1}{x_i} = 0. \qquad \square$$

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