

# MULTIVARIABLE HODGE THEORETICAL INVARIANTS OF GERMS OF PLANE CURVES I

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ABSTRACT. We describe methods for calculation of polytopes of quasiadjunction for plane curve singularities which are invariants giving a Hodge theoretical refinements of the zero sets of multivariable Alexander polynomials. In particular we identify some hyperplanes on which all polynomials in multivariable Bernstein ideal vanish.

## 1. INTRODUCTION

The purpose of this paper is to study Hodge theoretical invariants of local systems on the complements to germs of plane curve singularities. These invariants, called the faces of quasiadjunction, yield a refinement for the multivariable Alexander polynomial of a link of isolated singularity or, more precisely, for the characteristic varieties associated with the homology of universal abelian covers of the complements to a germ of plane curve. We develop algorithmic methods for calculation of these Hodge theoretical invariants in terms of combinatorics of defining equations of germs and to use them to study distributions of polytopes for all germs of plane curves.

While a multivariable Alexander polynomial is a product of factors of the form  $t_1^{m_1} \dots t_r^{m_r} - \omega$  where  $\omega = e^{2\pi\sqrt{-1}\alpha}$  is a root of unity (i.e.  $\alpha, \alpha \in \mathbf{Q}$ ), the faces of quasiadjunction are subsets of the union of hyperplanes of the form  $\sum m_i x_i = \alpha + k$ ,  $k \in \mathbf{Z}$  and are related to the zero set of the Alexander polynomial via the exponential map. Faces of quasiadjunction for links of plane curve singularities were defined in [8], [9] using adjunction conditions for the abelian covers of germs. In these papers they were related to the cohomology of the finite order local systems of the complements. Here we show that faces of quasiadjunction can be defined entirely in terms of Mixed Hodge structure, described in [12], on the cohomology of local systems (non necessarily of finite order) on the complement to the germ: they parametrize the unitary local systems with the dimension of a Hodge component having a fixed value. In particular the families of (unitary) local systems on the complement to a link with fixed Hodge data have a linear structure

and form polytopes of various dimensions in the universal covering of the space of unitary local systems. This description of the Mixed Hodge structure on the local systems in terms of log-resolution of the germ provides first approach to calculations of faces of quasiadjunction.

The second approach discussed in this paper is based on the relation of faces of quasiadjunction with the spectrum of non reduced singularities (cf.3.10). Below we show that “most” local systems with non vanishing Hodge numbers correspond to eigenvalues of the (semi-simple part of) the monodromy acting on graded vector space associated with limit Hodge filtration on the cohomology of the Milnor fiber of the non-reduced singularity of the form  $f_1^{a_1} \cdot \dots \cdot f_r^{a_r}$ . Another technical result which makes calculations more effective shows that while the definition of faces of quasiadjunction describes them in terms of multiplicities of pull backs of defining equations and differentials along the exceptional curves, this data corresponding only to curves having at least three intersection points with other exceptional curves suffices. This can be viewed as an extension of what was known previously in uni-branched case and in the case of multivariable Alexander polynomials (cf. also related result which concerns the log-canonical thresholds [17]).

We start with review in section 2 of the definitions of the polytopes and ideals of quasiadjunction, multiplier ideals and their basic properties. In section 3 we show the connection with the Arnold-Steenbrink spectrum of non reduced singularities. As an application we show in section 4 that all multivariable Bernstein polynomials, i.e. the polynomials  $b(s_1, \dots, s_r)$  for which there exist a differential operator  $P$  such that  $b(s_1, \dots, s_r)f_1^{s_1} \cdot \dots \cdot f_r^{s_r} = Pf_1^{s_1+1} \cdot \dots \cdot f_r^{s_r+1}$  are vanishing on hyperplanes containing codimension one faces of quasiadjunction.

The part II of this paper ([2]) develops methods for calculations of the ideals and polytopes of quasiadjunction from defining equations of the germs. The main technical tool are the Newton trees introduced by the first named author. The Newton trees are very close to Eisenbud and Neumann diagrams. Moreover we introduce a partial resolution of singularity of the germ of a pair  $(\mathbb{C}^2, D)$  which we call Newton space and which is a toroidal morphism  $\tilde{C}^2 \rightarrow \mathbb{C}^2$  of the space  $\mathbb{C}^2$  having only quotient singularities. The attractive feature of the Newton space is that it has canonical construction in terms of Newton tree and does not involve choice (while classically used in this context resolutions do). A byproduct of this construction is the fact that only the divisors in the resolution which intersect at least three other divisors appear in the computation of the polytopes of quasiadjunction. A subsequent paper also describes application of these methods to a study of the polytopes

of quasiadjunction and spectrum at infinity which serve as refinements of the studied earlier the Alexander polynomials at infinity. Part II (cf. [2]) also includes several explicit examples of calculation of polytopes and ideals of quasiadjunction.

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## 2. IDEALS OF QUASIADJUNCTION AND MULTIPLIER IDEALS

The ideals and polytopes of quasiadjunction, which are the main objects of this paper, are closely related to other invariants, i.e. the multiplier ideals, which recently attracted much attention. We shall briefly review their main properties relevant to this work. To describe them, let  $X$  be a smooth complex affine variety of dimension  $n$  and  $\mathbf{a}$  an ideal in  $\mathbb{C}[X]$ . A log-resolution of  $\mathbf{a}$  is a proper birational map  $\mu : Y \rightarrow X$  whose exceptional locus is a divisor  $E$  such that

- (1)  $Y$  is non singular
- (2)  $\mathbf{a}\mathcal{O}_Y = \mathcal{O}_Y(-\mathbf{F})$  with  $F = \sum r_i E_i$  an effective divisor,
- (3)  $F + E$  has simple normal crossing support.

Consider the relative canonical divisor  $\kappa_{Y/X} = \kappa_Y - \mu^* \kappa_X$ , and write  $\kappa_{Y/X} = \sum b_i E_i$ .

For every rational number  $c \geq 0$  the multiplier ideal of the ideal  $\mathbf{a}$  with exponent  $c$  is

$$\mathfrak{S}(\mathbf{a}^c) = \{\mathbf{h} \in \mathbb{C}[\mathbf{X}] \mid \text{ord}_{E_i}(\mu^* \mathbf{h}) \geq [\mathbf{c}r_i] - \mathbf{b}_i, \forall i\}$$

where  $[ \ ]$  is the round down operation. The definition is independent of the chosen resolution.

More generally one can consider the jumping numbers of  $\mathbf{a} = (\mathbf{f})$  i.e. one can prove that there exists an increasing sequence of rational numbers

$$0 = \xi_0 < \xi_1 < \xi_2 < \dots$$

such that  $\mathfrak{S}(\mathbf{a}^c)$  is constant on  $\xi_i \leq c < \xi_{i+1}$ , and  $\mathfrak{S}(\mathbf{a}^{\xi_i}) \neq \mathfrak{S}(\mathbf{a}^{\xi_{i+1}})$ .

The numbers  $\xi_i$  are called the *jumping numbers* of  $(f)$ . If  $f$  has isolated singularity at the origin. It follows from [13] that if  $\alpha \in (0, 1]$ , then  $\alpha$  belongs to the Arnold-Steenbrink-Hodge spectrum of  $f$  if and only if  $\alpha$  is a jumping number.

Moreover, Ein, Lazarfeld, Smith and Varolin (cf.[5]) have shown that if  $\xi$  is a jumping number of  $f$  in  $(0, 1]$ , then  $b_f(-\xi) = 0$  where  $b_f(s)$  is the Bernstein polynomial of  $f$ .

In this paper we are interested in more general multiplier ideals, called the *mixed multiplier ideals*. Let  $\mathbf{a}_1, \dots, \mathbf{a}_t$  be ideals in  $\mathbb{C}[X]$  and  $c_1, \dots, c_t$  be positive numbers and consider a log resolution of  $\mathbf{a}_1 \cdots \mathbf{a}_t$ . We write  $\mathbf{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-\mathbf{F}_i)$  with  $F_i = \sum r_{i,j} E_j$ , then

$$\mathfrak{S}(\mathbf{a}_1^{c_1} \cdots \mathbf{a}_t^{c_t}) = \{\mathbf{h} \in \mathbb{C}[\mathbf{X}] \mid \text{ord}_{E_j}(\mu^* \mathbf{h}) \geq [\mathbf{c}_1 \mathbf{r}_{1,j} + \cdots + \mathbf{c}_t \mathbf{r}_{t,j}] - \mathbf{b}_j, \forall j\}$$

Again the definition does not depend on the chosen resolution. We will restrict ourselves to the case where the ideals  $\mathbf{a}_i = (\mathbf{f}_i)$  are principal.

It turns out that the mixed multiplier ideals coincide with the ideals of quasiadjunction (cf. [9]). For these ideals there is a notion of “jumping numbers” which in this case are the *faces of quasiadjunction* referred to earlier. There is also a notion of log canonical threshold. In this paper we focus on the case where  $n = 2$ .

Now let us review the definition of ideals and polytopes of quasiadjunction. Let  $B$  be a small ball about the origin in  $\mathbb{C}^2$  and let  $\mathcal{C}$  be a germ of a plane curve having at 0 a singularity with  $r$  branches. Let  $f_1(x, y) \cdots f_r(x, y) = 0$  be a local equation of this curve (each  $f_i$  is assumed to be irreducible). An abelian cover of type  $(m_1, \dots, m_r)$  of  $\partial B$  is the link of complete intersection surface singularity:

$$(1) \quad V_{m_1, \dots, m_r} : z_1^{m_1} = f_1(x, y), \dots, z_r^{m_r} = f_r(x, y)$$

The covering map is given by  $p : (z_1, \dots, z_r, x, y) \rightarrow (x, y)$ .

An *ideal of quasiadjunction* of type  $(j_1, \dots, j_r \mid m_1, \dots, m_r)$  is the ideal in the local ring of the singularity of  $\mathcal{C}$  consisting of germs  $\phi$  such that the 2-form:

$$(2) \quad \omega_\phi = \frac{\phi z_1^{j_1} \cdots z_r^{j_r} dx \wedge dy}{z_1^{m_1-1} \cdots z_r^{m_r-1}}$$

extends to a holomorphic form on a resolution of the singularity of  $V_{m_1, \dots, m_r}$ .

An *ideal of log-quasiadjunction* (resp. *ideal of weight one log-quasiadjunction*) of type  $(j_1, \dots, j_r \mid m_1, \dots, m_r)$  is the ideal in the same local ring consisting of germs  $\phi$  such that the 2-form  $\omega_\phi$  extends to a log-form (resp. weight one log-form) on a resolution of the same singularity. We shall denote the ideal of quasiadjunction, (resp. log-quasiadjunction, resp. weight one log-quasiadjunction <sup>1)</sup>) corresponding

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<sup>1</sup>recall that a form  $\omega$  on a complement to a divisor  $\cup E_i$  with normal crossings has weight  $w$  if in a local coordinates near a point where  $E_i$  is given by  $z_i = 0$  one has  $\omega = f \cdot \frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_w}}{z_{i_w}} \wedge dz_{i_{w+1}} \dots$

to  $(j_1, \dots, j_r | m_1, \dots, m_r)$  as  $\mathcal{A}(j_1, \dots, j_r | m_1, \dots, m_r)$   
(resp.  $\mathcal{A}''(j_1, \dots, j_r | m_1, \dots, m_r)$ , resp.  $\mathcal{A}'(j_1, \dots, j_r | m_1, \dots, m_r)$ ).

We have

$$\begin{aligned} & \mathcal{A}(j_1, \dots, j_r | m_1, \dots, m_r) \subseteq \\ & \mathcal{A}'(j_1, \dots, j_r | m_1, \dots, m_r) \subseteq \mathcal{A}''(j_1, \dots, j_r | m_1, \dots, m_r) \end{aligned}$$

**Proposition 2.1.** *For any germ with  $f_1 \cdot \dots \cdot f_r = 0$  there are finitely many ideals of quasiadjunction (and log-quasiadjunction)*

*Proof.* Let  $E_1, \dots, E_k, \dots (k \in K)$  be the collection of exceptional components of a resolution  $\pi$  of this germ. For each map  $\Theta : K \rightarrow \mathbf{N}_+$  one can define the ideal  $\mathcal{A}_\Theta = \{\phi \in \mathcal{O}_{0, \mathbb{C}^2} | \text{mult}_{E_k} \pi^*(\phi) \geq \Theta(k)\}$ . We shall call  $\Theta$  coherent (resp. log-coherent) if there exist  $(x_1, \dots, x_r)$   $0 \leq x_i < 1$  such that  $\Theta(k) = \sum a_{k,i} x_i - c_k$  (resp.  $\Theta(k) = \sum a_{k,i} x_i - c_k - 1$ ); other notations are as in [9] for all  $k \in K$ . For each  $k$  there is an integer  $N_k$  such that for all  $k$  and and coherent (or log-coherent)  $\Theta$  one has  $\Theta(k) \leq N_k$ . The ideals of quasiadjunction are the ideals  $\mathcal{A}_\Theta$  corresponding to coherent  $\Theta$ . There are at most  $\prod_k N_k$  coherent  $\Theta$  and hence the claim follows. This proposition reduces calculation of the ideals of quasiadjunction to calculation of the ideals corresponding to valuations and identifying those among them which correspond to coherent  $\Theta$ .  $\square$

It is proved in [9], Remark 2.6 that  $\mathcal{A}(j_1, \dots, j_r | m_1, \dots, m_r)$  is equal to  $\mathfrak{S}(\mathbf{a}_1^{c_1} \cdots \mathbf{a}_r^{c_r})$ , where  $\mathbf{a}_i = (\mathbf{f}_i)$  and  $c_i = 1 - \frac{j_i+1}{m_i}$  for all  $i$ .

Let

$$\mathcal{U} = \{(x_1, \dots, x_r) \in \mathbb{R}^r, 0 \leq x_i < 1\}$$

be the unit cube with coordinates corresponding to  $f_1, \dots, f_r$ .

**Proposition 2.2.** *Let  $\mathcal{A}$  be an ideal of quasiadjunction. There is a unique polytope  $\mathcal{P}(\mathcal{A})$  open subset in  $\mathcal{U}$  such that: For  $(m_1, \dots, m_r) \in \mathbb{Z}^r$  and  $(j_1, \dots, j_r) \in \mathbb{Z}^r$  with  $0 \leq j_i < m_i, 1 \leq i \leq r$*

$$\mathcal{A} \subseteq \mathcal{A}(j_1, \dots, j_r | m_1, \dots, m_r) \Leftrightarrow \left( \frac{j_1+1}{m_1}, \dots, \frac{j_r+1}{m_r} \right) \in \mathcal{P}(\mathcal{A})$$

A face of quasiadjunction is a face of the boundary of the polytope  $\mathcal{P}(\mathcal{A})$ . It follows that it can be characterized as follows. Let  $E_i$  be the exceptional curves of an embedded resolution  $\pi : \tilde{\mathbb{C}}^2 \rightarrow \mathbb{C}^2$  of  $f_1 \cdot \dots \cdot f_r = 0$ . Let  $a_{i,k} = \text{mult}_{E_k} \pi^*(f_i)$  be the multiplicity of pullback of  $f_i$  to  $\tilde{\mathbb{C}}^2$ ,  $c_k = \text{mult}_{E_k} \pi^*(dx \wedge dy)$  and for a germ  $\phi \in \mathcal{O}_O$ ,  $e_k(\phi) = \text{mult}_{E_k} \pi^*(\phi)$ . Then the face of quasiadjunction containing  $\wp = \left( \frac{j_1+1}{m_1}, \dots, \frac{j_r+1}{m_r} \right) \in \mathcal{U}$  is the face  $\alpha$  of the boundary of the set of points satisfying:

$$(3) \quad \sum_i a_{i,k} x_i > \sum_i a_{i,k} - e_k(\phi) - c_k - 1$$

for all  $\phi$  in the ideal of quasiadjunction  $\mathcal{A}(j_1, \dots, j_r | m_1, \dots, m_r)$  (and such that  $\wp \in \alpha$ ). In particular for  $(j'_1, \dots, j'_r | m'_1, \dots, m'_r)$  for which the corresponding point satisfies (3) the form  $\omega_\phi$  extends over all  $E_i$ . However for  $(j'_1, \dots, j'_r | m'_1, \dots, m'_r)$  on the face itself there exist  $\phi$  in the ideal of quasiadjunction for which  $\omega_\phi$  has pole on one of the exceptional curves.

An intrinsic interpretation of the cube  $\mathcal{U}$  is as follows (cf. [12]). Consider the map  $exp : \mathcal{U} \rightarrow \mathbb{C}^{*r}$  where the target is identified with the group of characters of the fundamental group  $\pi_1(B - \mathcal{C})$ . Its image consists of the unitary characters. The characters of the Galois group of the covering map  $p : V_{m_1, \dots, m_r} \rightarrow \mathbb{C}^2$  form a finite subgroup in  $exp(\mathcal{U}) \subset Char\pi_1(B - \mathcal{C})$ . A pull back of the multivalued form  $\omega_\phi$  given by (2) is single-valued on  $V_{m_1, \dots, m_r} - p^{-1}(\mathcal{C})$  and is an eigenform for the action of the Galois group of  $(V_{m_1, \dots, m_r} - p^{-1}(\mathcal{C})) / (B - \mathcal{C})$ . If loop  $\gamma_i$  is defined by the condition that it has linking number with  $f_i = 0$  (resp.  $f_j = 0$ ) in  $B$  equal to one (resp. zero) then this character is given by:

$$(4) \quad \chi(\gamma_i) = exp(2\pi\sqrt{-1}\frac{j_i + 1}{m_i})$$

The condition that  $\phi$  belongs to the ideal of quasiadjunction is equivalent to the condition that  $\omega_\phi$  represents the zero cohomology class in  $H^1(V_{m_1, \dots, m_r} - \mathcal{C})$ . The condition that  $\phi$  belongs to the ideal log-quasiadjunction (resp. weight 2 log-quasiadjunction) is equivalent to the condition that  $\omega_\phi$  has logarithmic singularities along the exceptional divisor of the resolution of singularities of  $V_{m_1, \dots, m_r}$  induced by the resolution of  $f_1 \cdot \dots \cdot f_r = 0$  (resp. is a weight 2 logarithmic form).

### 3. FACES OF QUASIADJUNCTION AND SPECTRUM

In this section we shall describe a relationship between the faces of quasiadjunction and the spectrum. For this we need to reinterpret the elements of faces of quasiadjunction in terms of the cohomology of local systems on  $B - \mathcal{C}$ . Recall first the definitions from [9] and [12]. One has  $H_1(S^3 - L, \mathbb{Z}) = \mathbb{Z}^r$  with generators given by the classes of the loops  $\gamma_i$  described in just above (4). In particular the local systems on  $S^3 - L$  are parametrized by a torus of characters  $\mathbb{C}^{*r}$ . For a fixed rank one local system  $\chi : H^1(S^3 - L, \mathbb{Z}) \rightarrow \mathbb{C}^*$ , the cohomology groups  $H^1(S^3 - L, \chi)$  support a mixed Hodge structure (cf. [12]) and we need a description of the polytopes of quasiadjunction in terms of this mixed Hodge structure.

### 3.1. Mixed Hodge structure on the cohomology of a local system on a complement to a link.

Let us describe the mixed Hodge structure on  $H^i(S^3 - L, \chi)$ , which is a slight modification of construction in [12] and method for calculating the Hodge numbers of this mixed Hodge structure in this particular case of local systems on the complement to a germ of plane curve. First we shall select a resolution of singularity of a germ  $D$  given by  $f_1 \cdot \dots \cdot f_r = 0$ . Denote the collection of exceptional curves  $E_i$  (as in the proof of Prop. 2.1). We shall identify the 3-sphere  $S^3$  about the origin with the boundary  $\partial T(\cup E_i)$  of a regular neighborhood  $T(\cup E_i)$  of the exceptional set  $\cup E_i$ . If  $D^*$  is the proper preimage of  $D$  for selected resolution then  $\partial T(\cup E_i) \cap D^* = S^3 \cap D = L$ . In particular one obtains the identification:

$$(5) \quad S^3 - L = \left( \bigcup_i \partial T(E_i) \right) - D^*$$

The set  $\cup E_i \cup D^*$  has the canonical stratification with one dimensional (over  $\mathbb{C}$ ) strata  $E_i^\circ$  and zero dimensional strata  $E_i \cap E_j$ , or  $E_i \cap D^*$ . This induces the decomposition of  $(\bigcup_i \partial T(E_i)) - D^*$  into a union of open 3-dimensional (over  $\mathbb{R}$ ) manifolds each being a circle bundle over  $E_i^\circ$  which we denote  $\partial T(E_i^\circ)$ . The intersection of any two such manifolds is a product of two punctured disks which we shall denote  $\partial_{i,j}$  (it is empty unless  $E_i \cap E_j \neq \emptyset$ ). One has the embeddings  $\phi_{i,j} : \partial_{i,j} \rightarrow \partial T(\cup_k E_k) - D^*$  each being either of the compositions  $\partial_{i,j} \rightarrow \partial T(E_i^\circ) \rightarrow T(\cup_k E_k) - D^*$  or  $\partial_{i,j} \rightarrow \partial T(E_j^\circ) \rightarrow T(\cup_k E_k) - D^*$ . Given a character  $\chi \in \text{Char}(\pi_1(S^3 - L))$  we denote by  $\phi_{i,j}^*(\chi)$  the pull back of  $\chi$  to  $\partial_{i,j}$ .

We shall start by describing the mixed Hodge structure on the cohomology of local systems on each  $\partial T(E_i^\circ)$  and then show how they yield the Hodge numbers of the MHS on the union. We can deal only with the case of non trivial local systems  $\chi \in \text{Char}\pi_1(\partial \cup_i T(E_i) - D^*) = \text{Char}\pi_1(S^3 - L)$  since the case  $\chi = 1$  dealt with in [4].

Let

$$(6) \quad \theta : \partial T(E_i^\circ) \rightarrow E_i^\circ$$

be the canonical circle fibration and let  $\gamma_{E_i}$  be the class of the fiber of the latter in  $H_1(\partial T(E_i^\circ))$ .

Notice that  $H^i(\partial T(E_i^\circ), \chi) = 0$  for any  $i$  unless  $\chi(\gamma_{E_i}) = 1$ . This follows from Leray spectral sequence:

$$E_2^{p,q} = H^p(E_i^\circ, R^q \theta_*(\chi)) \Rightarrow H^{p+q}(\partial T(E_i^\circ), \chi)$$

since for a non trivial local system  $\chi$  on a circle one has  $H^i(S^1, \chi) = 0$ .

From now on we assume that  $\chi(\gamma_i) = 1$  i.e. that the local system, on which cohomology we want to construct a mixed Hodge structure, is the pullback  $\theta^*(\chi)$  of a local system on  $E_i^\circ$ . Notice that  $R^q\theta_*(\mathbb{C})$  are trivial local systems since circle fibrations over  $E_i^\circ$  are restrictions of the circle fibrations over  $E_i$  and  $\pi_1(E_i) = 0$ . Leray spectral sequence, which is a sequence of MHS (cf. [1] for a similar statement) has the term  $E_2^{p,q}$  as follows:

$$(7) \quad \begin{array}{ccccc} H^0(E_i^\circ, R^2\theta_*(\mathbb{C}) \otimes \chi) & & 0 & & 0 \\ H^0(E_i^\circ, R^1\theta_*(\mathbb{C}) \otimes \chi) & H^1(E_i^\circ, R^1\theta_*(\mathbb{C}) \otimes \chi) & & & 0 \\ H^0(E_i^\circ, R^0\theta_*(\mathbb{C}) \otimes \chi) & H^1(E_i^\circ, R^0\theta_*(\mathbb{C}) \otimes \chi) & H^2(E_i^\circ, R^0\theta_*(\mathbb{C}) \otimes \chi) & & \end{array}$$

It yields hence:

$$(8) \quad H^2((\partial T(E_i^\circ), \theta^*(\chi))) = H^1(E_i^\circ, R^1\theta_*(\mathbb{C}) \otimes \chi) = H^1(E_i^\circ, \chi)(-1)$$

(the Tate twist is the contribution of  $R^1\theta_*(\mathbb{C})$  i.e. a shift of weight by 2) and

$$(9) \quad 0 \rightarrow H^0(E_i^\circ, R^1\theta_*(\mathbb{C}) \otimes \chi) \rightarrow H^1(\partial T(E_i^\circ), \theta^*(\chi)) \rightarrow H^1(E_i^\circ, \chi) \rightarrow 0$$

We can assume that  $\chi \in \text{Char}\pi_1(E_i^\circ)$  is non trivial since the case  $\chi = 1$  was treated in [3]. If  $\chi \neq 1$  then  $H^0(E_i^\circ, \chi) = 0$  and the above exact sequence shows that the MHS on the cohomology of  $\partial T(E_i^\circ)$  with coefficients in a local system coincides with the MHS on the cohomology of  $E_i^\circ$ :

$$(10) \quad H^1(\partial T(E_i^\circ), \theta^*(\chi)) = H^1(E_i^\circ, \chi)$$

The MHS on the cohomology of local systems on a quasiprojective manifolds and relevant MHC were constructed in [19].

To obtain MHS on  $(\bigcup_i \partial T(E_i)) - D^*$  we first observe that

$$(11) \quad H^i((\bigcup_i \partial T(E_i)) - D^*, \chi) = H^i((\bigcup_i \partial T(E'_i)) - D^* - \bigcup E''_i, \chi)$$

where  $E'_i$  are the components such that  $\chi(\gamma_{E'_i}) = 1$  and  $E''_i$  are the components such that  $\chi(\gamma_{E''_i}) \neq 1$ . Indeed, the Mayer Vietoris spectral sequences calculating the cohomology of the union in both sides of (11) coincide. These spectral sequences are respectively

$$(12) \quad E_2^{p,q} = H^p((\partial T(E_i^\circ) - D^*)^{[q]}, \chi) \Rightarrow H^{p+q}((\bigcup_i \partial T(E_i)) - D^*, \chi)$$

and

(13)

$$E_2^{p,q} = H^p((\partial T(E_i^\circ) - D^* - \bigcup E_i'')^{[q]}, \chi) \Rightarrow H^{p+q}((\bigcup_i \partial T(E_i')) - D^* - \bigcup E_i'', \chi)$$

Here the superscript  $\cdot^{[q]}$  denotes a  $q + 1$ -fold intersections. In our case these intersections are empty if  $q = 2$  and for  $q = 1$  they are equivalent to the products of punctured disks denoted earlier as  $\partial_{i,j}$ . Notice that we denoted by  $E'^\circ$  a non singular stratum of stratification of the union of components  $E' = \bigcup_i E'_i$  for which  $\chi(\gamma_{E'_i}) = 1$  and that one obvious identity  $\bigcup_i E_i^\circ = \bigcup E_i'^\circ - \bigcup E_i''$  yields the identification of the terms  $E_2$  (and differentials  $d_2$ ) of the above spectral sequences.

On the other hand the local system on  $(\bigcup_i \partial T(E_i')) - D^* - \bigcup E_i''$  (as a consequence of  $\chi(\gamma_{E'_i}) = 1$ ) is the pullback of a local system on  $(\bigcup_i T(E_i')) - D^* - \bigcup E_i''$ . Next take a projective surface  $Z$  containing  $\bigcup_i T(E_i')$  and extend the unitary connection corresponding to the local system on  $(\bigcup_i T(E_i')) - D^* - \bigcup E_i''$  from  $(\bigcup_i T(E_i')) - D^* - \bigcup E_i''$  to a meromorphic connection on  $Z$ . Then the complement  $Z^\circ$  to the polar set of this connection is a quasiprojective manifold, containing  $(\bigcup_i T(E_i')) - D^* - \bigcup E_i''$  and supporting unitary local system extending the local system  $\chi$  on  $(\bigcup_i \partial T(E_i')) - D^* - \bigcup E_i''$ . Now one applies the same construction of MHC calculating the cohomology of local system  $\chi$  as in [3] so that the following Mayer-Vietoris sequence is a sequence of MHSs (we denote by the same letter  $\chi$  the extensions of local system on  $(\bigcup_i \partial T(E_i')) - D^* - \bigcup E_i''$ ):

$$(14) \quad \rightarrow H^i((\bigcup_i \partial T(E_i')) - D^* - \bigcup E_i'', \chi) \rightarrow$$

$$H^i(Z^\circ - \bigcup E_i^\circ, \chi) \oplus H^i((\bigcup_i T(E_i')) - D^* - \bigcup E_i'', \chi) \rightarrow H^i(Z^\circ, \chi) \rightarrow$$

To obtain the Hodge numbers of the mixed Hodge structure on the cohomology of the union  $H^*(\partial T(\bigcup_i E_i^\circ))$  we use the Mayer-Vietoris spectral sequence which since all triple intersections are empty becomes the exact sequence:

(15)

$$\begin{aligned} \oplus_{i,j} H^0(\partial_{i,j}, \phi_{i,j}^*(\chi)) &\rightarrow H^1(\partial T(\bigcup_i E_i^\circ), \chi) \rightarrow \oplus_i H^1(\partial T(E_i^\circ), \phi_i^*(\chi)) \rightarrow \\ &\rightarrow \oplus_{i,j} H^1(\partial_{i,j}, \phi_{i,j}^*(\chi)) \end{aligned}$$

Note that  $H^1(\mathbb{C}^*) = H^{1,1}(\mathbb{C}^*) = \mathbb{C}$  and hence  $H^1(\partial_{i,j}, \phi_{i,j}^*(\chi))$  (which is non zero only if  $\phi_{i,j}^*(\chi)$  is trivial) has pure Hodge structure of weight

2. Moreover, the map

$$(16) \quad W_2/W_1(\oplus_i H^1(\partial T(E_i^\circ), \phi_i^*(\chi))) \rightarrow \oplus_{i,j} H^1(\partial_{i,j}, \phi_{i,j}^*(\chi))$$

is injective since the classes of weight 2 correspond to the cohomology of the fiber of the map  $\partial T(E_i^\circ) \rightarrow E_i^\circ$  i.e. the term  $H^0(E_i^\circ, R^1\theta_*(\mathbb{C}) \otimes \chi)$  in (9).

Hence we obtain the following exact sequence which determines the Hodge numbers of  $H^1(S^3 - L, \chi)$ :

$$(17) \quad \oplus_{i,j} H^0(\partial_{i,j}, \phi_{i,j}^*(\chi)) \rightarrow H^1(\partial T(\bigcup_i E_i^\circ), \chi) \rightarrow W_1(\oplus_i H^1(\partial T(E_i^\circ), \phi_i^*(\chi))) \rightarrow 0$$

We shall sum up the above discussion in following

**Proposition 3.1.** *The isomorphism (10) and exact sequence (17) of MHS determine for the MHS on  $H^1(S^3 - L, \chi)$  the only non zero Hodge numbers  $h^{p,q,1}(S^3 - L, \chi)$  ( $0 \leq p+q \leq 1$ ) uniquely. Moreover, the Hodge numbers  $h^{p,q,1}(S^3 - L, \chi)$  with  $p+q = 1$  depend only on the Hodge structure on  $\oplus_{i \in R} H^1(\partial T(E_i^\circ), \phi_i^*(\chi))$  where  $R$  is the set of components of  $E$  which are intersected by at least three other components of exceptional divisor while  $h^{0,0,1}(S^3 - L, \chi)$  depends on the cohomology (of dimension zero) of pair  $(\partial T(\bigcup_{i \in R} E_i), \partial T(\bigcup_{i \in R} E_i \cap \bigcup_{i \notin R} E_i))$*

The last part which concerns the components with at least three intersections with remaining ones follows from the exact sequence (17) since for the component with only two intersections the curve  $E_i^\circ$  is homotopy equivalent to a circle and hence

$$H^1(\partial T(E_i^\circ), \phi^*(\chi)) = H^1(E_i^\circ, \chi) = 0$$

for non trivial  $\chi$  while for trivial  $\chi$  one has weight two Hodge structure also does not contribute to the cohomology of link.

**REMARK 3.2.** *In [9] the MHS on the cohomology of local systems of finite order was obtained from the MHS on the finite abelian branched cover (cf. Prop. 3.3).*

**3.2. Depth of Characters.** In this subsection we relate the ideals of quasiajunction (cf. section 2), and specifically  $\dim \mathcal{A}''/\mathcal{A}'$  and  $\dim \mathcal{A}'/\mathcal{A}$  to the dimensions of  $Gr_F^p Gr_q^W H^1(L_\chi)$ . Here  $L_\chi$  is the local system corresponding to the character  $\chi$  of the local fundamental group of the complement to the germ  $\mathcal{C}$  taking the value  $\exp(2\pi\sqrt{-1}\frac{j_i+1}{m_i})$  on the generator  $\gamma_i \in \pi_1(S^3 - L)$  corresponding to factor  $f_i$  (cf. Prop. 3.3).

Recall (cf. [9]) that depth of  $\chi \in \text{Char}(\pi_1(S^3 - L))$  is the dimension  $\dim H^1(S^3 - L, \chi)$  of the cohomology of the complement with the coefficients in the rank one local system corresponding to  $\chi$ . The data of  $\dim Gr_F^p Gr_q^W H^1(L_\chi)$  determines the Hodge theoretical refinement of the depth (cf. 3.7) which depends only on the data of ideals of quasiadjunction. We show how this data (determined by the ideals of quasiadjunction) yields the depth of a character  $\chi$ .

**Proposition 3.3.** *Let  $\phi : \mathbb{Z}^r \rightarrow \bigoplus_{i=1}^{i=r} \mathbb{Z}/m_i$  be the surjection and let  $\partial V_{m_1, \dots, m_r}$  be the corresponding covering space of  $S^3$  branched over the link  $L$  which we view as the link of singularity (cf. (1)).*

*Let  $\chi$  be a character belonging to the image of the embedding:  $\text{Char} \bigoplus_{i=1}^{i=r} \mathbb{Z}/m_i \rightarrow \text{Char} \mathbb{Z}^r$  induced by  $\phi$ . Assume that  $\chi(\gamma_i) \neq 1$  for any  $i = 1, \dots, r$ . Then*

(a)  $H^1(S^3 - L, L_\chi) = \{v \in H^1(\partial V_{m_1, \dots, m_r}) \mid g \cdot v = \chi(g)v\}$  and

(b) *the mixed Hodge structure on  $H^1(S^3 - L, L_\chi)$  is induced by the mixed Hodge structure on  $H^1(\partial V_{m_1, \dots, m_r})$*

*Proof.* This proposition can be deduced from the following two lemmas.

**Lemma 3.4.** *Let  $G_i = \bigoplus_{j=1}^{j=i} \mathbb{Z}/m_j$  and let  $G_r$  acts freely on a CW-complex  $X$  and  $X_i = X/G_i$ . Let  $\chi \in \text{Hom}(G, \mathbb{C}^*)$  be such that  $\chi_i = \chi|_{G_i}$  is non-trivial. Let  $L_{\chi_i}$  be the local system on  $X_i$  induced by  $\chi_i$ . Then  $H^1(X_{i+1}, L_{\chi_{i+1}})$  is the eigenspace of the generator of  $G_{i+1}/G_i$  acting on  $H^1(X_i, L_{\chi_i})$ .*

**Lemma 3.5.** *Let  $U_{m_1, \dots, m_r} \subset \partial V_{m_1, \dots, m_r}$  be the complement to the branching locus of the map*

$$\pi(m_1, \dots, m_r) : \partial V_{m_1, \dots, m_r} \rightarrow S^3.$$

*Let  $\chi \in \text{Char} \pi_1(S^3 - L)$  such that  $\chi(\gamma_i) \neq 1$ . Then  $H^1(\partial V_{m_1, \dots, m_r}) \rightarrow H^1(U_{m_1, \dots, m_r})$  corresponding to the embedding induces an isomorphism of  $\chi$ -eigenspaces.*

Indeed, (a) in the proposition 3.3 follows by applying 3.4 to the subgroups  $G_i = \mathbb{Z}/m_1 \oplus \dots \oplus \mathbb{Z}/m_i$  of the Galois cover  $\partial V_{m_1, \dots, m_r} \rightarrow S^3$ . Part (b) follows since the embedding in lemma 3.5 is a morphism of mixed Hodge structures and the deck transformation preserves MHS on  $H^1(U_{m_1, \dots, m_r})$  since they are induced by holomorphic maps.

In order to show 3.4 consider the exact sequence of chain complexes of  $G/G_{i+1}$ -modules in which we view  $C_*(X_i) = C_*(X)^{G_i}$  as  $G/G_i$ -module and in which the left map is induced by the multiplication by  $t - \chi_{i+1}(\delta_{i+1})$  where  $\delta_{i+1}$  is a generator of the cyclic group  $G_{i+1}/G_i$  and  $t$  is the generator of  $\mathbb{C}[G_i/G_{i+1}]$ :

$$(18) \quad 0 \rightarrow C_*(X_i) \otimes_{\chi_i} \mathbb{C} \rightarrow C_*(X_i) \otimes_{\chi_i} \mathbb{C} \rightarrow C_*(X_{i+1}) \otimes_{\chi_{i+1}} \mathbb{C} \rightarrow 0$$

This yields the corresponding cohomology sequence:

$$(19) \quad \begin{aligned} H^0(X_i, L_{\chi_i}) &\xrightarrow{t^{-\chi_i(\delta_i)}} H^0(X_i, L_{\chi_i}) \rightarrow \\ H^1(X_{i+1}, L_{\chi_{i+1}}) &\rightarrow H^1(X_i, L_{\chi_i}) \xrightarrow{t^{-\chi_i(\delta_i)}} H^1(X_i, L_{\chi_i}) \end{aligned}$$

The map in the upper line is surjective since  $\chi_i(\delta_i) \neq 1$  and the lemma 3.4 follows.

To show lemma 3.5 let us consider the exact sequence:

$$(20) \quad \begin{aligned} H^1(\partial V_{m_1, \dots, m_r}, U_{m_1, \dots, m_r}) &\rightarrow H^1(\partial V_{m_1, \dots, m_r}) \rightarrow H^1(U_{m_1, \dots, m_r}) \\ &\rightarrow H^2(\partial V_{m_1, \dots, m_r}, U_{m_1, \dots, m_r}) \end{aligned}$$

of the pair  $(\partial V_{m_1, \dots, m_r}, U_{m_1, \dots, m_r})$ . Denoting by  $T(\pi(m_1, \dots, m_r)^{-1}(Br))$  the regular neighborhood of the ramification locus of  $\pi(m_1, \dots, m_r)$  and by  $\partial\pi(m_1, \dots, m_r)^{-1}(Br)$  its boundary, we have the Galois equivariant identification of the relative cohomology:

$$(21) \quad \begin{aligned} H^i(\partial V_{m_1, \dots, m_r}, U_{m_1, \dots, m_r}) &= \\ H^i(T(\pi(m_1, \dots, m_r)^{-1}(Br)), \partial\pi(m_1, \dots, m_r)^{-1}(Br)) &= \\ = H^{3-i}(\partial\pi(m_1, \dots, m_r)^{-1}(Br)) \end{aligned}$$

For an element in the last group, corresponding to the component of the branching locus given by  $f_i = 0$  the action of  $\gamma_i$  on it is trivial. Hence the map:

$$(22) \quad H^1(\partial V_{m_1, \dots, m_r})_\chi \rightarrow H^1(U_{m_1, \dots, m_r})_\chi$$

is an isomorphism for a character  $\chi$  which that  $\chi(\gamma_i) \neq 1$  for all  $i$ .  $\square$

**REMARK 3.6.** *Proposition 3.3 is closely related to Prop. 4.5 in [10]. A different type of relation (only for certain Hodge components) between cohomology of branched and unbranched covers of quasiprojective manifolds is given in [11]*

**Proposition-Definition 3.7.** *Let  $a$  belongs to the fundamental domain of  $H_1(S^3 - L, \mathbf{Z})$  acting on the universal cover of the torus of unitary characters  $\text{Char}^u(\pi_1(S^3 - L))$ . The holomorphic weight one depth of  $\chi = \exp(2\pi ia)$  (denoted  $d^h(\chi)$ ) is the dimension of vector space  $\text{Gr}_F^1 \text{Gr}_1^W H^1(L_\chi)$ . The weight zero depth of a character  $\chi$  is the dimension vector space  $\text{Gr}_0^W H^1(L_\chi)$ . We denote it as  $d^0(\chi)$ . An integer  $w = 0, 1$  is a weight of  $\chi$  if  $d_0(\chi) \neq 0$  (resp.  $d^h(\chi) \neq 0$ ). Total depth of  $\chi$  is  $d(\chi) = d^h(\chi) + d^0(\chi)$ .*

*The value of the depth (resp. holomorphic depth) of a character  $\exp(2\pi ia)$  where  $a$  belongs to the interior of the face is independent of  $a$  and is called the depth (resp. holomorphic depth) of the face of quasiadjunction.*

*Proof.* For the characters of finite order, it follows from the identification of the holomorphic depths with quotients of ideals of quasiadjunction since the ideals of quasiadjunction themselves depend only on the face of quasiadjunction. This and the definition of MHS for infinite order characters yields the claim in general.  $\square$

We have the following.

**Proposition 3.8.** *If the values of character  $\chi$  are  $\pm 1$  then the depth of a character  $\chi$  can be calculated as follows:*

$$d^h(\chi) + d^h(\bar{\chi}) + d_0(\chi)$$

where  $\bar{\chi}$  is the conjugate character. Otherwise the depth is equal to:

$$d^h(\chi) + d^h(\bar{\chi}) + d_0(\chi) + d_0(\bar{\chi})$$

*Proof.* Since the action of the Galois group acting on  $Gr_1^W = H^{1,0} \oplus H^{0,1}$  preserves this direct sum decomposition one has  $(Gr_1^W)_\chi = H_\chi^{1,0} + H_{\bar{\chi}}^{1,0}$ . The vector space  $Gr_0^W$  is defined over  $\mathbb{Z}$  and hence the eigenspaces corresponding to conjugate characters contribute for  $d_0(\chi) + d_0(\bar{\chi})$  unless the characters take values  $\pm 1$  in which case the dimension of eigenspace will be  $d_0(\chi)$   $\square$

**REMARK 3.9.** *This proposition is closely related to Prop. 3.2 in [9]. The contribution of characters for which  $d_0(\bar{\chi}) \neq 0$  is missing there however i.e. the sum in 3.2 should have the term  $\dim \mathcal{A}''_\Sigma / \mathcal{A}'_\Sigma$  for  $\chi$  taking values  $\neq \pm 1$ .*

**Example** (cf.[9]) Consider the singularity  $x^r - y^r = 0$ . Faces of quasiadjunction are hyperplanes  $H_l : x_1 + \dots + x_r = l$  ( $l = 1, \dots, r-2$ ) where the holomorphic depth of  $H_l$  is  $r-l-1$ . If  $\chi = \exp(2\pi i a)$  and  $a \in H_l$  then  $\bar{\chi} = \exp(2\pi i \bar{a})$  where  $\bar{a} \in H_{r-l}$ . Hence by above proposition the depth of any character in  $t_1 \cdots t_r = 1$  is equal  $r-l-1 + r - (r-l) - 1 = r-2$ .

**3.3. Spectrum and faces of quasiadjunction.** Next let us consider the relation between Arnold-Steenbrink spectrum and the cohomology of local systems. Let us fix the zero set of a germ of holomorphic function:

$$(23) \quad f_1^{a_1} \cdot \dots \cdot f_r^{a_r} = 0$$

in a ball  $B_\epsilon : z_1 \bar{z}_1 + z_2 \bar{z}_2 \leq \epsilon$  and denote it  $Z(f_1, \dots, f_r)$ . Let  $\gamma_i$  be the standard generators of  $H_1(B_\epsilon - Z(f_1, \dots, f_r), \mathbb{Z}) = \mathbb{Z}^r$  used earlier (i.e. given by oriented loops having linking number with the germ of  $f_i$  equal to +1).

Let  $\mathbb{U} \subset \mathbb{C}^*$  denotes the group of complex numbers having modulus one. For any  $\xi \in \mathbb{U}$ , let  $\chi_\xi$  be the local system given by  $\chi_\xi(\gamma_i) = \xi^{a_i}$ . As already mentioned, the cohomology  $H^1(B_\epsilon - Z(f_1, \dots, f_r), \chi_\xi)$  support a mixed Hodge structure (cf. [12]).

Consider the Milnor fiber  $M(a_1, \dots, a_r)$  of the singularity (23) and the mixed Hodge structure on  $H^1(M(a_1, \dots, a_r))$  defined by [18]. Recall that the multiplicity of a spectral pair  $(\alpha, w)$  where  $-\alpha = n - p - \beta$ ,  $0 \leq \beta < 1$ ,  $\alpha \notin \mathbf{Z}$  is the dimension of the eigenspace with the eigenvalue  $\exp(2\pi\sqrt{-1}\alpha)$  of the semi-simple part of the monodromy acting on  $H^n$  of the Milnor fiber of an  $n$ -dimensional singularity. Moreover, the eigenvalues corresponding to the Jordan blocks of size  $1 \times 1$  (resp.  $2 \times 2$ ) of the monodromy appear in the action of the semi-simple part of the monodromy  $T_s$ <sup>2</sup> on  $Gr_1^W$  (resp.  $Gr_0^W$  and  $Gr_2^W$ ) (cf. [18]).

The multiplicity of the eigenvalue of the monodromy is related to the cohomology of local systems as follows:

**Proposition 3.10.** *Let  $m_\xi$  denotes the multiplicity of  $\xi$  as the eigenvalue of the semi-simple part of the monodromy of singularity (23) acting on  $Gr_F^0 H^1(M(a_1, \dots, a_r))$ . Then*

$$m_\xi = d(\chi_\xi)$$

*Proof.* We shall use the following Steenbrink-Wang sequence of mixed Hodge structure on the semi-stable reduction i.e. finite degree covering of the base  $\Delta_s^* \rightarrow \Delta^*$  of the map  $F : B_\epsilon - Z(f_1, \dots, f_r) \rightarrow \Delta^*$  such that the pullback of (23)  $(B_\epsilon - Z(f_1, \dots, f_r))_s \rightarrow \Delta_s^*$  has multiplicity one along components of the exceptional set. We have the sequence of MHS (cf. [15], p.275):

$$(24) \quad H^1((B_\epsilon - Z(f_1, \dots, f_r))_s) \rightarrow H^1(M(a_1, \dots, a_r)) \xrightarrow{\log(T_u - I)} H^1(M(a_1, \dots, a_r))(-1)$$

The left term, which is the cohomology of punctured neighborhood, on semi-stable reduction of a resolution, has the MHS defined in terms of the log-complex associated with exceptional divisor (cf. [18], [15]). The two remaining terms have the limit Mixed Hodge structure with the weight filtration given in terms of the monodromy (cf. [18]). Both, the cohomology of local system as in proposition and the multiplicity of the eigenspace of  $T_s$  have the same description in terms of  $H^1((B_\epsilon - Z(f_1, \dots, f_r))_s)$  compatible with the MHS. Indeed, the mixed Hodge structure on  $B_\epsilon - Z(f_1, \dots, f_r)_s$  together with action of the deck transformation, which is holomorphic and hence preserves MHS, yields the MHS

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<sup>2</sup>i.e. one has  $T = T_s T_u$  where  $T_s$  is semi-simple and  $T_u$  is unipotent

on  $B_\epsilon - Z(f_1, \dots, f_r)$ . The same expression takes place for the relation between the MHS on Milnor fiber and on semi-stable reduction. More precisely, one has the commutative diagram comparing the two rows of (24):

$$(25) \quad \begin{array}{ccccc} 0 & \rightarrow & H^1((B_\epsilon - Z(f_1, \dots, f_r))_s) & \rightarrow & H^1(M(a_1, \dots, a_r)) \\ & & \begin{array}{c} T - \lambda I \\ \downarrow \end{array} & & \begin{array}{c} T_s - \lambda I \\ \downarrow \end{array} \\ 0 & \rightarrow & H^1((B_\epsilon - Z(f_1, \dots, f_r))_s) & \rightarrow & H^1(M(a_1, \dots, a_r)) \end{array}$$

where  $T_s$  is the semi-simple part of the monodromy acting on the Milnor fiber and  $T$  is the (holomorphic deck transformation). The cokernel of the left map yields the cohomology of the local system corresponding to  $\lambda$  and cohomology of the right is the  $\lambda$ -eigenspace of the semi-simple part of the monodromy acting on the cohomology of Milnor fiber. Since the horizontal maps are the maps of MHS we obtain the claim.  $\square$

Now we shall describe the faces of quasiadjunction of  $f_1 \cdots f_r$  in terms of faces of quasiadjunction of functions  $f_1^{a_1} \cdots f_r^{a_r}$ . We shall use only collections  $(a_1, \dots, a_r) \in \mathbf{Z}_+^r$  for which  $\gcd(a_1, \dots, a_r) = 1$  i.e. for which the Milnor fiber is connected. Let us consider the subgroup  $\oplus(a_i \mathbf{Z}) \subset \mathbf{Z}^r$  where  $a_i \mathbf{Z}$  is the subgroup of index  $a_i$ . If  $\mathcal{U}$  is the unit cube in  $\mathbf{R}^r$ , which we view as the fundamental domain of  $\mathbf{Z}^r$  acting on  $\mathbf{R}^r$  via translations, then the fundamental domain of  $\oplus(a_i \mathbf{Z})$  is the union of  $\prod a_i$  unit cubes i.e. the subset of  $\mathbf{R}^r$  given by:

$$\{(x_1, \dots, x_r) \in \mathbf{R}^r \mid 0 \leq x_i \leq a_i\}$$

We shall denote this subset of  $\mathbf{R}^r$  by  $\mathcal{U}(a_1, \dots, a_r)$  or  $\mathcal{U}(\mathbf{a})$ . Each face of quasiadjunction via translations yields  $\prod a_i$  corresponding polytopes in  $\mathcal{U}(\mathbf{a})$ . The collection of translates of a face  $F$  in  $\mathcal{U}(\mathbf{a})$  we shall denote  $F(\mathbf{a})$ .

**Proposition 3.11.** *The rational numbers  $s \in (0, 1)$  belongs to the spectrum of singularity  $\Pi f_i^{a_i}$  iff  $(a_1 s, \dots, a_r s)$  belongs to a translate  $F' \in F(\mathbf{a}) \subset \mathcal{U}(\mathbf{a})$  for some face of quasiadjunction  $F$ . Moreover the multiplicity of  $s$  in the spectrum is the depth of the face which translate the segment  $L_{\mathbf{a}} = (a_1 t, \dots, a_r t)$  ( $t \in (0, 1)$ ) intersects at the point corresponding to  $t = s$ .*

*Vice versa, a face of quasiadjunction of depth  $k$  is a connected component of closure of points  $(b_1, \dots, b_r) \in \mathcal{U} \cap \mathbf{Q}$  such that for some  $s \in (0, 1)$  and  $(a_1, \dots, a_r) \in \mathbf{Z}_+^r$  ( $\gcd(a_i) = 1$ ) one has  $b_i = a_i s$  and  $s$  is an element of the spectrum of singularity  $\Pi f_i^{a_i}$  having multiplicity  $k$ .*

**Proof.** This proposition follows from the relation between Betti numbers or ranks of Hodge groups of local systems and faces of quasiadjunction. Recall that the local system corresponding to the character  $\chi, \chi(\gamma_i) = \xi_i$  has non vanishing cohomology iff  $(\xi_1, \dots, \xi_r)$  belongs to the characteristic variety (cf. remark 3.13 below). Similarly,  $(s_1, \dots, s_r), 0 < s_i < 1$  has the dimension of Hodge group equal  $k$  if  $(s_1, \dots, s_r)$  is in the face of quasiadjunction of depth  $k$ . The Milnor fiber of the singularity  $\Pi f_i^{a_i}$  is homotopy equivalent to the infinite cyclic cover of  $S^3 - L$  with the Galois group having the characters which when viewed as the characters of  $\pi_1(S^3 - L)$  have the form  $(t^{a_1}, \dots, t^{a_r})$  for some  $t$ . By proposition (3.10) we can identify an element of spectrum (with multiplicity) in  $(0, 1)$  with the element of the universal cover of the torus of characters which exponent is the local system with non vanishing  $Gr_0^F H^1(S^3 - L, L_\xi)$  (of dimension equal to multiplicity). Hence spectrum of  $\Pi f_i^{a_i}$  consists of  $s$  such that  $(a_1 s \bmod 1, \dots, a_r s \bmod 1)$  belongs to a face of quasiadjunction. This is equivalent to the description of the relation between the spectrum and the faces of quasiadjunction given in the proposition.

**REMARK 3.12.** *Similarly, the elements of the spectrum in  $(-1, 0)$  can be obtained as intersections of lines  $(a_1 s, \dots, a_r s)$  with conjugates of translates of faces of quasiadjunction. Indeed, the elements of the spectrum  $(-1, 0)$  characterized by the property that  $\exp(2\pi\sqrt{-1}\alpha)$  are the eigenvalues on the semi-simple part of the monodromy acting on  $Gr_1^W(F^0 \cap \bar{F}^1)H^1(M(a_1, \dots, a_r))$ . On the other hand  $Gr_1^W(F^0 \cap \bar{F}^1)$  is the conjugate of  $Gr_1^W F^1 \cap \bar{F}^0 = Gr_1^W Gr_F^1$  (since  $H^1 = F^0 H^1, F^2 H^1 = 0$ ).*

**REMARK 3.13.** *If  $\Pi(t_1^{m_1} \dots t_r^{m_r} - \omega)$  is the multivariable Alexander polynomial of  $f_1 \dots f_r$  then the Alexander polynomial of  $\Pi f_i^{a_i}$  is given by  $(t - 1)\Pi(t^{a_1 m_1 + \dots + a_r m_r} - \omega)$ . This follows from an extension of the classical relation between multivariable and one variable Alexander polynomial (cf. [20]).*

In particular one obtains a relation between the faces of quasiadjunction of  $f_1 \dots f_r$  and the spectrum of corresponding singularity as follows.

**Corollary 3.14.** *A rational  $\xi \in (0, 1)$  belongs to the spectrum of  $f_1 \dots f_r$  if and only if*

$$(26) \quad (\xi, \dots, \xi)$$

*belongs to a face of quasiadjunction. Moreover the multiplicity of  $\xi$  in the spectrum is the holomorphic depth of the character  $\chi_{\gamma_i}$  given by  $\chi_{\gamma_i} = e^{2\pi\sqrt{-1}\xi}$ . In particular the number of faces of quasiadjunction*

intersecting the line in  $\mathcal{U}$  given by points (26) is equal to the number of elements in the spectrum of  $f_1 \cdots f_r$ .

The Milnor number of  $f_1 \cdots f_r$  is related to the depths of faces of quasiadjunction as follow. If  $\dim Gr_0^W H^1(L_\chi) = 0$  for any  $\chi$  which is the exponent of (26) then

$$(27) \quad \mu = (r - 1) + 2 \sum_F d^h(F)$$

where summation is over faces of quasiadjunction containing points of the form (26) and extra  $r - 1$  is due to the fact that this is the difference between the Betti number of finite branch and unbranched cover of  $S^3 - L$ ; the second part of (27) calculates the Betti number of branched covering of  $S^3$ .

**Example** The situation in which the number of faces of quasiadjunction coincides with the number of elements in the spectrum comes up in cases  $(x^2 + y^3)(y^2 + x^3)$ . For the singularity  $x^r - y^r = 0$  the sum of the depths of all faces is  $\frac{(r-1)(r-2)}{2}$  and (\*\*\*) translates to  $r - 1 + 2 \frac{(r-1)(r-2)}{2} = (r - 1)^2$

**REMARK 3.15.** *One may conjecture the following method to calculate faces of quasiadjunction. Let  $\exp : \mathcal{U} \rightarrow \text{Char}^u H_1(S^3 - Z(f_1, \dots, f_r))$  be the universal covering map of the group of unitary characters restricted to the unit cube  $\mathcal{U} \subset \mathbb{R}^r$ . The subset of  $\mathcal{U}$  which is the preimage of the zero set of the multivariable Alexander polynomial in  $\text{Char}^u H_1(S^3 - Z(f_1, \dots, f_r))$  is a union of hyperplanes. Consider the stratification of this union in which each stratum consists of points belonging to the same collection of hyperplanes. The conjecture is that holomorphic depth is constant on each stratum and hence the faces of quasiadjunction are the strata of this stratification with positive holomorphic depth.*

*In particular, in the case of two branches  $V_1$  is a union of segments and  $V_k$  with  $k \geq 2$  are union of isolated points. The holomorphic depth of each point  $(\alpha_1, \alpha_2)$  ( $\alpha_i \in \mathbb{Q}$ ) (e.g. a component of the zero dimensional stratum of the preimage of the zero set of Alexander polynomial) is the multiplicity of  $\exp(2\pi it)$  of the singularity  $f_1^{\alpha_1} f_2^{\alpha_2} = 0$  where  $\alpha_1/\alpha_2 = a_1/a_2$  (here  $a_1, a_2$  are relatively prime integers and  $t \in \mathbb{Q}$  is determined from  $\alpha_i = a_i t$ ).*

#### 4. FACES OF QUASIADJUNCTION AND BERNSTEIN POLYNOMIALS

Recall a definition of the multivariable Bernstein ideal following [16] and [6]. Consider the collection of polynomials  $b(s_1, \dots, s_r) \in$

$\mathbb{C}[s_1, \dots, s_r]$  and differential operators  $P(s_1, \dots, s_r) \in \mathcal{D}[s_1, \dots, s_r]$  such that:

$$(28) \quad P \cdot f_1^{s_1+1} \cdot \dots \cdot f_r^{s_r+1} = b(s_1, \dots, s_r) f_1^{s_1} \cdot \dots \cdot f_r^{s_r}$$

where  $f_1, \dots, f_r$  are the germs of plane curves as in section 2. Such polynomials  $b$  form an ideal  $\mathcal{B}_{f_1, \dots, f_r}$  in  $\mathbb{C}[s_1, \dots, s_r]$ . This ideal is a non-zero ideal (cf. [16]). In the case  $r = 1$  the monic generator of it (uniquely defined) is called the Bernstein polynomial of germ  $f$ . The goal of this section is to show the following:

**Theorem 4.1.** *Let  $\mathcal{P}$  be the product of linear forms  $L_i(s_1+1, \dots, s_r+1)$  where  $L_i$  runs through linear forms vanishing on  $(r-1)$ -dimensional faces of polytopes of quasiadjunction corresponding to a germ with  $r$  irreducible components  $f_1, \dots, f_r$ . Then any  $b \in \mathcal{B}_{f_1, \dots, f_r}$  is divisible by  $\mathcal{P}$ .*

*Proof.* Let  $L_\phi$  be the equation of a face of quasiadjunction corresponding to a germ  $\phi \in \mathcal{A}$  and let  $\omega_\phi$  be the form (2). Using relation (1) we have:

$$(29) \quad \omega_\phi = \phi \cdot f_1^{\frac{j_1+1}{m_1}-1} \cdot \dots \cdot f_r^{\frac{j_r+1}{m_r}-1} dx \wedge dy$$

This form extends over the exceptional locus of the resolution of singularity (1) if and only if  $\omega_\phi$  is  $L^2$ -form (cf. [14]). It follows from the definition of an ideal of quasiadjunction that the integral of form (29) (over a small compact ball  $B$  about the origin and a test function  $\psi$  on  $B$ ):

$$(30) \quad \int_B |\phi|^2 |f_1|^{2(\frac{j_1+1}{m_1}-1)} \cdot \dots \cdot |f_r|^{2(\frac{j_r+1}{m_r}-1)} \psi$$

converges for all test functions  $\psi$  and all  $\phi$  in the ideal of quasiadjunction corresponding to a face given by  $L(x_1, \dots, x_r) = 0$  provided that  $L(\frac{j_1+1}{m_1}, \dots, \frac{j_r+1}{m_r}) > 0$  and if  $L(\frac{j_1+1}{m_1}, \dots, \frac{j_r+1}{m_r}) = 0$  then there is  $\phi$  in this ideal of quasiadjunction for which the integral diverges.

On the other hand we have:

$$(31) \quad |b(s_1, \dots, s_r)|^2 |\phi|^2 |f_1|^{2s_1} \cdot \dots \cdot |f_r|^{2s_r} \psi = P\bar{P} |f_1|^{2s_1+1} \cdot \dots \cdot |f_r|^{2s_r+1} |\phi|^2 \psi$$

since holomorphic and anti-holomorphic differential operators commute (as above,  $\psi$  is a test function). Hence integrating over  $B$  both sides of (31) yields

$$(32) \quad |b(s_1, \dots, s_r)|^2 \int_B |\phi|^2 |f_1|^{2s_1} \cdot \dots \cdot |f_r|^{2s_r} \psi = \int P\bar{P} |f_1|^{2(s_1+1)} \cdot \dots \cdot |f_r|^{2(s_r+1)} |\phi|^2 \psi$$

Applying this to  $\phi$  for which integral (30) diverges for  $s_i = \frac{j_i+1}{m_i} - 1$  where  $\frac{j_i+1}{m_i}$  is on the face of quasiadjunction we obtain that the meromorphic function of  $(s_1, \dots, s_r)$  given by either side of (32) when we approach to any point on the face of quasiadjunction right hand side is holomorphic in  $s_1, \dots, s_r$  while the integral in the left hand side has pole. Hence  $b(s_1, \dots, s_r)$  must vanish on  $L(s_1 + 1, \dots, s_r + 1) = 0$ .  $\square$

**EXAMPLE 4.2.** *The hyperplanes containing the faces of quasiadjunction of singularity  $(x^2 + y^3)(x^3 + y^2) = 0$  are  $6x_1 + 4x_2 = 1, 3, 5$  and  $4x_1 + 6x_2 = 1, 3, 5$  (cf. [2] and [9]). Hence polynomials in Bernstein ideal are vanishing on  $6s_1 + 4s_2 + k = 0, 4s_1 + 6s_2 + k = 0, k = 5, 7, 9$ . In fact according calculations of A.Leykin using SINGULAR (private communication) the Bernstein ideal for this singularity is principal with generator:*

$$(s_1 + 1)(s_1 + 1)\prod_{k=5,7,9,11,13}(4s_1 + 6s_2 + k)(6s_1 + 4s_2 + k)$$

**REMARK 4.3.** *Similar to theorem 4.1 result, i.e. an identification of the faces of quasi-adjunction with the zeros of polynomials in the multi-variable Bernstein ideal can be shown for isolated non normal crossibgs singularities discussed in [10]. The details will appear in a forthcoming publication.*

## REFERENCES

- [1] D.Arapura, The Leray spectral sequence is motivic. Invent. Math. 160 (2005), no. 3, 567–589.
- [2] P.Cassou Nogues, A.Libgober, Multivariable Hodge theoretical invariants of germs of plane curves II, in preparation.
- [3] A.Durfee, Mixed Hodge structures on punctured neighborhoods. Duke Math. J. 50 (1983), no. 4, 1017–1040.
- [4] A.Durfee, R.Hain, Mixed Hodge structures on the homotopy of links. Math. Ann. 280 (1988), no. 1, 69–83.
- [5] L.Ein, R.Lazarsfeld, K.Smith, D.Varolin, Jumping coefficients of multiplier ideals. Duke Math. J. 123 (2004), no. 3, 469–506.
- [6] A.Gyoja, Bernstein-Sato’s polynomial for several analytic functions. J. Math. Kyoto Univ. 33 (1993), no. 2, 399–411.
- [7] J.Kollar, Singularities of pairs. Algebraic geometry—Santa Cruz 1995, 221–287, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997.

- [8] A.Libgober, Characteristic varieties of algebraic curves. Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), 215–254, NATO Sci. Ser. II Math. Phys. Chem., 36, Kluwer Acad. Publ., Dordrecht, 2001.
- [9] A.Libgober, Hodge decomposition of Alexander invariants. *Manuscripta Math.* 107 (2002), no. 2, 251–269.
- [10] A.Libgober, Isolated non-normal crossings. *Real and complex singularities*, 145–160, *Contemp. Math.*, 354, Amer. Math. Soc., Providence, RI, 2004.
- [11] A.Libgober Homotopy groups of complements to ample divisors. *Singularity theory and its applications*, 179–204, *Adv. Stud. Pure Math.*, 43, Math. Soc. Japan, Tokyo, 2006.
- [12] A.Libgober, Non vanishing loci of Hodge numbers of local systems, *Manuscripta mathematica*, 2009.
- [13] F.Loeser, Vaquié M. The Alexander polynomial of a projective plane curve. *Topology* 29 (1990), no. 2, 163–173.
- [14] M.Merle, B. Tossier, Conditions d’adjonction, d’après Du Val; pp. 229–247 in *Séminaire sur les Singularités des Surfaces*. Centre de Mathématiques de l’Ecole Polytechnique, Palaiseau, 1976–1977. Edited by Michel Demazure, Henry Charles Pinkham and Bernard Teissier. *Lecture Notes in Mathematics*, 777. Springer, Berlin, 1980.
- [15] C.Peters, J.Steenbrink, Mixed Hodge structures. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 52*. Springer-Verlag, Berlin, 2008.
- [16] C.Sabbah, Proximité évanescence. I. La structure polaire d’un  $D$ -module. *Compositio Math.* 62 (1987), no. 3, 283–328, Proximité évanescence. II. Equations fonctionnelles pour plusieurs fonctions analytiques. *Compositio Math.* 64 (1987), no. 2, 213–241.
- [17] K.Smith, H.Thompson, Irrelevant exceptional divisors for curves on a smooth surface. *Algebra, geometry and their interactions*, 245–254, *Contemp. Math.*, 448, Amer. Math. Soc., Providence, RI, 2007.
- [18] Steenbrink, J. H. M. Mixed Hodge structure on the vanishing cohomology. *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pp. 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
- [19] K.Timmerscheidt, Mixed Hodge theory for unitary local systems. *J. Reine Angew. Math.* 379 (1987), 152–171.
- [20] Turaev, V. G. Reidemeister torsion in knot theory. *Uspekhi Mat. Nauk* 41 (1986), no. 1(247), 97–147, 240.

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