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Mordell–Weil groups of elliptic threefolds and the Alexander module of plane curves

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Abstract. We establish a correspondence between the rank of Mordell–Weil group of the complex elliptic threefold associated with a plane curve $\mathcal{C} \subset \mathbb{P}^2(\mathbb{C})$ with equation $F = 0$, certain roots of the Alexander polynomial associated with the fundamental group $\pi_1(\mathbb{P}^2(\mathbb{C}) \setminus \mathcal{C})$ and the polynomial solutions for the functional equation of type

$$h_1^p F_1 + h_2^q F_2 + h_3^r F_3 = 0$$

where $F = F_1 F_2 F_3$. This correspondence is obtained for curves in a certain class which includes the curves having introduced here δ-essential singularities and in particular for all curves with ADE singularities.

As a consequence we find a linear bound for the degree of the Alexander polynomial in terms of the degree of $\mathcal{C}$ for curves with δ-essential singularities and in particular arbitrary ADE singularities.

1. Introduction

The Alexander polynomial of singular curves in $\mathbb{P}^2_{\mathbb{C}}$ provides an effective way to relate the fundamental group of the complement of such curves to the topology and geometry of their singularities. In this paper we show that the Mordell–Weil groups of certain elliptic threefolds with constant j-invariant are closely related to the Alexander modules of the zero sets of their discriminants. As a byproduct of this relation we obtain a bound on the degree of the Alexander polynomials of plane curves. The bound is linear in the degree of the curve and gives a new restriction on the groups which can be fundamental groups of the complements to plane curves.

Let $\mathcal{C}$ be a (possibly reducible with components $\mathcal{C}_i$) curve in $\mathbb{P}^2$ of degree d. Its Alexander polynomial is defined in terms of purely topological data, $G = \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C})$, where $\mathcal{C}_0$ is a line at infinity, and a surjection $\varepsilon : G \to \mathbb{Z}$ (cf. Section 2.1 for a definition). On the other hand, dependence of the Alexander polynomial of $\mathcal{C}$, on its degree, the local type of its singularities, and their *position*, relating the latter to the topology, has been known for some time

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(cf. [32, 34]). For example, for irreducible curves having nodes and cusps as the only singularities, the degree of the Alexander polynomial $\Delta_{\mathcal{C},\varepsilon}(t)$ equals $\operatorname{rk} G'/G''$ where G' and G'' are respectively the first and second commutators of G (in this case there is only one choice, up to sign, of ε). Moreover, $\Delta_{\mathcal{C},\varepsilon}(t)$ is not trivial only if $6 \mid d$, in which case $\Delta_{\mathcal{C},\varepsilon}(t) = (t^2 - t + 1)^s$, where s is the superabundance of the curves of degree $d - 3 - \frac{d}{6}$ passing through the cusps of $\mathcal{C}$. For a given curve, this provides a purely geometric method for calculation of the Alexander polynomial. However, how big can this superabundance be for a special cuspidal curve is still not known (cf. [36]). The largest known value of s for irreducible cuspidal curves, to our knowledge, is 3. This occurs for the dual curve to a non-singular cubic, that is a sextic with nine cusps. In this paper we give an example of an irreducible curve with nodes and cusps as only singularities, for which the superabundance of the set of cusps is equal to 4.

One of our main results is the inequality (cf. Corollary 3.13)

$$\deg \Delta_{\mathcal{C},\varepsilon} \leq \frac{5}{3}d - 2. \tag{1.1}$$

For reducible curves and general ε, the explicit relation with the fundamental group comes from the equality (cf. [32])

$$\dim(G'/G'' \otimes \mathbb{Q}) \otimes_\Lambda \Lambda/(t_1 - t^{\varepsilon(\gamma_1)}, \dots, t_r - t^{\varepsilon(\gamma_r)}) = \deg \Delta_{\mathcal{C},\varepsilon}, \tag{1.2}$$

where $\Lambda := \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$, $r := \operatorname{rk}(G/G')$ (if $\mathcal{C}$ is a curve in $\mathbb{C}^2$, then r is the number of irreducible components of $\mathcal{C}$), $\varepsilon : \mathbb{Z}^r \to \mathbb{Z}$ is an epimorphism, and γ_i represents the class of a meridian around the i-th irreducible component of $\mathcal{C}$, i.e. its abelian image in H_1 is t_i.

The main idea in this note is to relate the degree of the Alexander polynomial of a curve $\mathcal{C}$ to the rank of Mordell–Weil group of the elliptic curve over field $\mathbb{C}(x, y)$ with constant j-invariant and having the curve $\mathcal{C}$ as the zero set of its discriminant. An important consequence is the relation between the Alexander module of $\mathcal{C}$ and rational pencils of elliptic type corresponding to $\mathcal{C}$ (cf. Definition 2.3).

The relationship between Alexander invariants and pencils, in the case of line arrangements and reducible curves is discussed in [35, 39]: the positive dimensional components of characteristic varieties induce maps between their complements and the complements to $p \geq 3$ points in $\mathbb{P}^1$. Here we show that non-vanishing of Alexander polynomial of a curve yields existence of special pencils with *non-reduced* fibers. These pencils are such that they induce rational maps from $\mathbb{P}^2$ onto $\mathbb{P}^1$ with an orbifold structure and take the curve $\mathcal{C}$ onto a finite set of points. The pencils are orbifold elliptic pencils in the following sense (cf. Definitions 2.3 and 2.4): each rational map $\mathbb{P}^2 \to \mathbb{P}^1$ has three non-reduced fibers of the form $m_i D_i + F_i$, where F_i divides F and $F = 0$ is an equation of $\mathcal{C}$. The multiplicities m_i are such that $\sum \frac{1}{m_i} = 1$ and thus the orbifold structure is given by assigning multiplicities m_i to those three points in $\mathbb{P}^1$.

The possible orbifold structures (m_1, m_2, m_3) depend on the local type of singularities of $\mathcal{C}$. For example for irreducible curves with only nodes and cusps as singularities, we relate the global Alexander polynomial of $\mathcal{C}$ to the following functional relation:

$$P(x, y, z)^2 + Q(x, y, z)^3 + F(x, y, z) = 0, \tag{1.3}$$

where $F = 0$ is an equation of $\mathcal{C}$ and $P(x, y, z)$, $Q(x, y, z)$ are homogeneous polynomials. Such a relation is equivalent to the existence of a rational map from $\mathbb{P}^2$ onto $\mathbb{P}^1$ with the orbifold structure $(2, 3, 6)$. This orbifold structure can be obtained considering the global orbifold in the usual sense (cf. [2]) corresponding to the action of a cyclic group of order 6 on an elliptic curve with non-trivial stabilizers at three points; their orders being equal to $2, 3$ and 6 respectively.

The correspondence between the Alexander modules and orbifold elliptic pencils is established in two rather different steps. On one hand, the Alexander module of $\mathcal{C}$ can be related to the Mordell–Weil group of the elliptic threefold

$$u^2 + v^3 = F(x, y, 1) \tag{1.4}$$

over the field $\mathbb{C}(x, y)$ of rational functions in two variables having j-invariant equal to zero. We have the following (cf. Theorem 3.1):

Theorem 1.1. *Let $\mathcal{C}$ be an irreducible curve in $\mathbb{P}^2$ having ordinary nodes and cusps as the only singularities. Let $F(x, y, 1) = 0$ be a (reduced) equation of the affine part of $\mathcal{C}$. Then the $\mathbb{Z}$-rank of the Mordell–Weil group of the elliptic threefold* (1.4) *is equal to the degree of the Alexander polynomial of the curve $\mathcal{C}$.*

The Mordell–Weil group here is the group of rational sections of the elliptic threefolds (see for instance [30, 31] and Section 3 for further discussion). The rank of the Mordell–Weil group of the threefold (1.4) was recently studied in [27] for the case $\deg \mathcal{C} = 6$ using different methods (cf. also [11,24]). These results follow immediately from the correspondence between Alexander polynomials and Mordell–Weil groups in this paper since the Alexander polynomials of sextic curves considered in [27] are readily available.

On the other hand, each element of the Mordell–Weil group of the aforementioned elliptic threefold defines a functional relation of the type (1.3). This can be summarized as follows (cf. Theorem 4.7):

Theorem 1.2. *For any irreducible plane curve $\mathcal{C} = \{F = 0\}$ whose only singularities are nodes and cusps the following statements are equivalent:*

(1) *$\mathcal{C}$ admits a quasi-toric relation of elliptic type* $(2, 3, 6)$*,*

(2) *$\mathcal{C}$ admits an infinite number of quasi-toric relations of elliptic type* $(2, 3, 6)$*,*

(3) *$\Delta_{\mathcal{C},\varepsilon}(t)$ is not trivial, i.e. $\Delta_{\mathcal{C},\varepsilon}(t) \neq 1$.*[1)]

Moreover, the set of quasi-toric relations of $\mathcal{C}$,

$$\{(f, g, h) \in \mathbb{C}[x, y, z]^3 \mid f^2 + g^3 + h^6 F = 0\},$$

has a group structure and it is isomorphic to $\mathbb{Z}^{2q}$, where $\Delta_{\mathcal{C}}(t) = (t^2 - t + 1)^q$. Also, $\mathcal{C}$ admits an infinite number of primitive quasi-toric relations unless $q = 1$, in which case $\mathcal{C}$ only has one primitive quasi-toric relation.

We also consider here other singularities which will define relations of the form:

$$h_1^p F_1 + h_2^q F_2 + h_3^r F_3 = 0, \tag{1.5}$$

where (p, q, r) is either $(3, 3, 3)$ or $(2, 4, 4)$. Such relations (1.5) in turn correspond to orbifold rational pencils with respective orbifold structures (cf. Theorem 5.16).

1) For an irreducible curve, there is only one choice of ε, up to sign, i.e. $\Delta_{\mathcal{C},\varepsilon}$ is independent of it; for other types of quasi-toric relations see Section 6.

While commonly the existence of irrational pencils on surfaces is obtained by an extension of the de Franchis method (from 1-form with vanishing wedge product, cf. [12, 49]), rational orbifold pencils are obtained rather differently: here the pencils are a byproduct of the splitting of Albanese varieties of cyclic multiple planes into a product of elliptic curves which we derive either from Roan's Decomposition Theorem for abelian varieties with an automorphism (cf. [9]) or directly, using a Hodge theoretical refinement of the argument used in the proof of divisibility theorem in [32]. For example for the cyclic multiple planes branched over a reduced curve with nodes and cusps as the only singularities, the Albanese variety splits as a product of elliptic curves with j-invariant equal to zero. One may contrast this with the case of Jacobians of cyclic covers of $\mathbb{P}^1$ where Jacobians are completely decomposable very rearly (cf. [21, 38] for a discussion of factorization of Jacobians of curves).

In particular, the relation between Alexander polynomials and Mordell–Weil groups allows us to give bounds (1.1) on the degree of Alexander polynomials. These follow from the bounds on the rank of Mordell–Weil groups obtained from the connection with the Mordell–Weil groups of certain elliptic surfaces and from the Shioda–Tate formula (cf. [48]). However, the correspondence between the Alexander polynomials and the ranks of the Mordell–Weil group should be of independent interest (cf. [27][2)]).

The curves with the largest known values of deg $\Delta_{\mathcal{C}}(t)$ are given in Section 6. The bound presented here is sharp for sextics, however, inequality (1.1) is apparently far from being sharp in general. Perhaps a better understanding of the Mordell–Weil rank of (1.4) can yield a better estimate. Also, note that Corollary 3.13 provides a partial answer to [36, Problem 2.1].

Another application of the results presented in this paper is an alternative argument to confirm Oka's conjecture of sextic curves having a non-trivial Alexander polynomial (i.e. that equations of such curves have the form $P^2 + Q^3$). The answer to Oka's conjecture was first obtained by A. Degtyarev (cf. [15, 16]).

For the sake of clarity we often start our discussions focusing on the case of irreducible curves having only nodes and cusps as singularities. The results, however, are obtained, as was already mentioned, for curves with a wider class of singularities, which we call δ-essential and δ-partial (cf. Definition 5.1). Moreover the results are applicable to reducible and non-reduced curves as well. From the point of view of fundamental groups, non-reduced curves correspond to homomorphisms ε that are more general than those given by the linking number of loops with $\mathcal{C}$.

The condition of being δ-essential is purely local, meaning that the local Alexander polynomial of the link of the singularity considered w.r.t. the restriction of ε on the local fundamental group is divisible by the cyclotomic polynomial of degree δ. As we shall see, this is the natural class of curves leading to the elliptic pencils.

1.1. Organization of the paper. The content of the paper is as follows. In Section 2 we give definitions for Alexander polynomials w.r.t. any homomorphism ε (as mentioned above), for orbifold surfaces and morphisms, and for quasi-toric relations. In Section 3 we relate the Alexander polynomial to the Mordell–Weil group of the threefold (1.4) associated with F. In Section 4, the $\mathbb{C}(x, y)$-points of the threefolds (1.3) are presented as quasi-toric relations of F, i.e. functional equations of the form $f^2 + g^3 + Fh^6 = 0$, over the ring $\mathbb{C}[x, y, z]$. The

[2)] Professor R. Kloosterman informed us that the described here correspondence between the ranks of Mordell–Weil threefolds and the Alexander polynomials can be used to improve the bound (1.1) on the degree of the Alexander polynomial of the cuspidal curves by a factor close to 2. This result appears in his paper [26]. Additional results related to this work appear in [6].

correspondence between the points of (1.3) and the quasi-toric relations where F fits in, is explained at the end of Section 3. In Section 5 we generalize the results to a larger class of curves. Finally, the Section 6 contains applications and a list of explicit examples. This provides a list of curves that fit into quasi-toric relations of all rational orbifolds of elliptic type. Also one of these examples provides the largest values known to date of the degree of the Alexander polynomial of an irreducible curve with nodes and cusps.

1.2. Notation. We shall use the following notation:

- F a (possibly reducible or even non-reduced) non-zero homogeneous polynomial in $\mathbb{C}[x,y,z]$ such that F is not a power, that is, $G^k = F$ for $G \in \mathbb{C}[x,y,z]$ implies $k = 1$.
- $\mathcal{C}$ the set of zeroes of F, hence a reduced curve.
- V_n a non-singular model of a cyclic multiple plane $z^n = F(x,y,1)$
- $\overline{V}_n$ a model of cyclic multiple plane in $\mathbb{P}^2 \times \mathbb{P}^1$.
- W_F° an affine model of the elliptic threefold corresponding to a curve $F = 0$.
- W_F a smooth projective birational model of W_F°.
- $\widetilde{W}_F$ singular projective model of W_F (cf. Proposition 3.9).
- E_0 the elliptic curve with j-invariant zero.
- $\overline{E}_Q$ a model of E_0 in $\mathbb{P}(2,3,1) \times \mathbb{P}^1$.
- W a split elliptic threefold $V_6 \times E_0$.
- $\overline{W} = \overline{V}_6 \times_{\mathbb{P}^1} \overline{E}_Q$ a birational model of split elliptic threefold $V_6 \times E_0$.

2. Preliminaries

In this section we will review several results on Alexander invariants which appear in the literature and extend them to the generality required in this paper.

2.1. Alexander polynomial relative to a surjection of the fundamental group. We shall consider reducible, not necessarily reduced curves in $\mathbb{P}^2$. Let $\mathcal{C}$ be a plane curve given as the set of zeroes of a homogeneous polynomial $F = F_1^{\varepsilon_1} \cdots F_r^{\varepsilon_r}$, which is not a power (see Notation 1.2), which means $\gcd(\varepsilon_1, \dots, \varepsilon_r) = 1$. Consider by $\mathcal{C} := \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$ its decomposition into irreducible components and let $(\varepsilon_1, \dots, \varepsilon_r)$, $\varepsilon_i \in \mathbb{Z}_+$, be the collection of multiplicities of each irreducible factor F_i in the equation of $\mathcal{C}$. So if $d_i = \deg F_i$ denotes the degree of $\mathcal{C}_i$, then the total degree of $\mathcal{C} := \{F = 0\}$ is given by $d := \sum \varepsilon_i d_i = \deg F$.

Let $\mathcal{C}_0$ be a line transversal to $\mathcal{C}$ which we shall view as the line at infinity and let $G := \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C})$. Recall that $H_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C})$ is a free abelian group with r generators having a canonical identification with $\mathbb{Z}^r$ (cf. [35]). The isomorphism is given by mapping the class of the boundary of a small holomorphic 2-disk transversal to the component $\mathcal{C}_i$ to $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^r$ (with 1 appearing as the i-th component). Let ε be the epimorphism $\varepsilon : G \to H_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}) \to \mathbb{Z}$ given by $\varepsilon(\gamma_i) := \varepsilon_i$. Let $\mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$ denote the group ring over $\mathbb{Q}$, $K_\varepsilon = \ker \varepsilon$, and $K'_\varepsilon = [K_\varepsilon, K_\varepsilon]$ be the commutator of K_ε. By the Hurewicz Theorem, $K_\varepsilon / K'_\varepsilon$ can be identified with the homology of infinite cyclic cover of $\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}$ corresponding to ε and hence can be viewed as a module over the group ring $\mathbb{Z}[\mathbb{Z}]$.

Homomorphisms (resp. epimorphism) $\varepsilon : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}) \to \mathbb{Z}$ such that $\varepsilon(\gamma_i) \geq 1$ for all i are in one-to-one correspondence with polynomials, (resp. non-power polynomials) considered up to a scalar factor, having $\mathcal{C}$ as the zero set. Indeed if F_i is an irreducible polynomial having $\mathcal{C}_i$ as its set of zeros, then one defines the polynomial corresponding to ε as

$$F_\varepsilon(x, y, 1) = \prod_i F_i(x, y, 1)^{\varepsilon(\gamma_i)}. \tag{2.1}$$

Vice versa, given a polynomial F having $\mathcal{C}$ as its zero set one defines the homomorphism $\varepsilon_F : H_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}) \to \mathbb{Z}$ using

$$\varepsilon_F(\gamma_i) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_i} \frac{dF}{F} \tag{2.2}$$

and extends it as the composition

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}) \to H_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}) \to \mathbb{Z}.$$

This homomorphism is in fact an epimorphism if F is not a power. It will also be denoted by ε_F.

Definition 2.1 (cf. [32, 45]). The Alexander polynomial $\Delta_{\mathcal{C},\varepsilon}(t)$ of $\mathcal{C}$ relative to a surjection $\varepsilon : G \to \mathbb{Z}$ is a generator of the order of the torsion of the $\mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$-module $K_\varepsilon / K'_\varepsilon \otimes \mathbb{Q}$ normalized in such a way that it is a polynomial in t satisfying $\Delta_{\mathcal{C},\varepsilon}(0) = 1$. By the discussion in the previous paragraph, a (not necessarily reduced) equation F of $\mathcal{C}$ defines both the set of zeroes $\mathcal{C}$ and a surjection ε_F. Hence $\Delta_F(t)$ will also denote $\Delta_{\mathcal{C},\varepsilon_F}(t)$.

The Alexander polynomial $\Delta_{\mathcal{C},\varepsilon}$ can be expressed in terms of characteristic varieties studied in [35] or in terms of G'/G'' viewed as a module over $\Lambda := \mathbb{Z}[H_1] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ as follows. The polynomial $\Delta_{\mathcal{C},\varepsilon}$ is the order of the torsion of

$$(G'/G'' \otimes \mathbb{Q}) \otimes_\Lambda \Lambda/(t_1 - t^{\varepsilon(\gamma_1)}, \dots, t_r - t^{\varepsilon(\gamma_r)})$$

up to a power of $(t-1)$, viewed as a $\mathbb{Q}[t, t^{-1}]$-module in the obvious way. The zeroes of $\Delta_{\mathcal{C},\varepsilon}$ can also be seen as the intersection of the characteristic variety $\Sigma_1(\mathcal{C})$ with the 1-dimensional torus of equation $L_\varepsilon := \{(t^{\varepsilon(\gamma_1)}, \dots, t^{\varepsilon(\gamma_r)})\} \subset (\mathbb{C}^*)^r$ (cf. [8, Theorem 2.26]).[3)]

We shall also need the local version of the polynomials $\Delta_{\mathcal{C},\varepsilon}$ defined similarly. Let P be a singular point of $\mathcal{C}$. The epimorphism $\varepsilon : G \to \mathbb{Z}$ induces a homomorphism ε_P of the local fundamental group of $\mathcal{C}$ to $\mathbb{Z}$ and hence the Alexander polynomial of the link of P w.r.t. the homomorphism ε_P.[4)] In other words, if $i : \mathbb{S}_P \setminus \mathcal{C} \hookrightarrow \mathbb{P}^2 \setminus \mathcal{C}$ is the inclusion from a sufficiently small sphere around P in the total space and γ is a meridian around a component of the link, then

$$\varepsilon_P(\gamma) := \varepsilon(i_*(\gamma)). \tag{2.3}$$

These polynomials will be denoted by $\Delta_{\mathcal{C},\varepsilon,P}(t)$. We have the following proposition.

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3) Though most often the Alexander polynomials are considered in the case when $\varepsilon_i = 1$, case $\varepsilon_i \neq 1$ was considered for example by Oka in [45, Section 4] as *θ-Alexander polynomials*.

4) If the image of ε_P has index k in $\mathbb{Z}$, then the Alexander polynomial $\Delta_{\mathcal{C},\varepsilon,P}(t)$ is $\Delta(t^k)$ where Δ is the Alexander polynomial relative to the surjection $G \xrightarrow{\varepsilon_P} \varepsilon_P(G) = \mathbb{Z}$. Recall that this represents the order of the 1-dimensional homology of the cyclic cover corresponding to ε which is this case has k connected components cf. [32, 35]

Proposition 2.2. *Let $\mathcal{C}$ be a plane curve and $(\varepsilon_1, \dots, \varepsilon_r)$ denote the multiplicities of its irreducible components. Then $\Delta_{\mathcal{C},\varepsilon}(t)$ divides the product of the local Alexander polynomials:*

$$\Delta_{\mathcal{C},\varepsilon}(t) \mid \prod_P \Delta_{\mathcal{C},\varepsilon,P}(t) \prod (t^{\varepsilon_i} - 1)^{k_i} \tag{2.4}$$

for some $k_i \in \mathbb{Z}^{\geq 0}$. In particular, if the local Alexander polynomials have only roots of unity of degree δ as their roots, then

$$\Delta_{\mathcal{C},\varepsilon}(t) = \prod_{\lambda \mid \delta} \varphi_\lambda(t)^{s_\lambda} \prod (t^{\varepsilon_i} - 1)^{k_i},$$

where $\varphi_\lambda(t)$ is the cyclotomic polynomial of the λ-th roots of unity. Moreover, if $s_\lambda > 0$, then

$$\lambda \mid d, \tag{2.5}$$

where $d := \sum d_i \varepsilon_i$ is the total degree.

Proof. Details of the arguments are similar to those used in the proof of the Divisibility Theorem (cf. [32, 34] and Lemma 3.6 below). The starting point is the surjection of the fundamental group of a regular neighborhood of $\mathcal{C}$ in $\mathbb{P}^2 \setminus \mathcal{C}_0$ onto $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C})$ (which is a consequence of the Lefschetz Hyperplane Section Theorem). On the other hand, one uses the Mayer–Vietoris sequence for the ε-cyclic cover of this neighborhood in order to split it into a union of cyclic covers: those of the local singularities and the neighborhood of the non-singular part of $\mathcal{C}$. This shows that the Alexander polynomial of the neighborhood of $\mathcal{C}$ is equal to the product of the Alexander polynomials of singularities and a divisor of second product in (2.4). These divisors come as contributions of H_0 and H_1 of terms of the Mayer–Vietoris sequence corresponding to the intersections of the complements to the links of the singularities with the mentioned $\mathbb{C}^*$-bundle of the non-singular part of $\mathcal{C}$. This yields the divisibility (2.4).

The second divisibility relation is a generalization of the divisibility at infinity (cf. [32]) and follows from the calculation of the Alexander polynomial of the link at infinity with multiplicities. It is equal to $(t^{\sum \varepsilon_i d_i} - 1)^{r-1}$ as a consequence of the Torres relation (cf. [55]) applied to the Hopf link with multivariable Alexander polynomial $(t_1 \cdots t_r - 1)^{r-1}$. □

2.2. Orbifold curves. Now we recall some basic definitions needed here referring for more details to (see [5]).

Definition 2.3. An *orbifold* curve $S_{\overline{m}}$ is (an open or closed) Riemann surface S with a function $\overline{m} : S \to \mathbb{N}$ whose value is 1 outside a finite number of points. A point $P \in S$ for which $\overline{m}(P) > 1$ is called an *orbifold point*.

One may think of a neighborhood of a point $P \in S_{\overline{m}}$ with $\overline{m}(P) = d$ as the quotient of a disk (centered at P) by a rotation of angle $\frac{2\pi}{d}$. A loop around P is considered to be trivial in $S_{\overline{m}}$ if its lifting bounds a disk. Following this idea, orbifold fundamental groups can be defined as follows.

Definition 2.4. For an orbifold $S_{\overline{m}}$, let $P_1, \dots, P_n$ be the orbifold points,

$$m_j := \overline{m}(P_j) > 1.$$

Then, the *orbifold fundamental group* of $S_{\overline{m}}$ is

$$\pi_1^{\text{orb}}(S_{\overline{m}}) := \pi_1(S \setminus \{P_1, \dots, P_n\})/\langle \mu_j^{m_j} = 1 \rangle, \tag{2.6}$$

where μ_j is a meridian of P_j. We will denote $S_{\overline{m}}$ simply by $S_{m_1,\dots,m_n}$.

Remark 2.5. In this paper, we will mostly consider orbifold groups of $\mathbb{P}^1$ with three orbifold points. The groups

$$\pi_1^{\text{orb}}(\mathbb{P}^1_{(p,q,r)}) = \langle x, y, z : x^p = y^q = z^r = xyz = 1 \rangle$$

are subgroups of index two of full triangle groups. In particular, they can be identified with the orientation-preserving isometries of a plane tiled with triangles with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$ (cf. [42, Corollary 2.5]).

Definition 2.6. A dominant algebraic morphism $\varphi : X \to S$ defines an *orbifold morphism $X \to S_{\overline{m}}$* if for all $P \in S$, the divisor $\varphi^*(P)$ is an $\overline{m}(P)$-multiple.

One has the following result regarding orbifold morphisms.

Proposition 2.7 ([5, Proposition 1.5]). *Let $\rho : X \to S$ define an* orbifold morphism *$X \to S_{\overline{m}}$. Then φ induces a morphism $\varphi_* : \pi_1(X) \to \pi_1^{\text{orb}}(S_{\overline{m}})$. Moreover, if the generic fiber is connected, then φ_* is surjective.*

Proposition 2.7 will be applied systematically throughout this paper. We will show a typical example of this. Suppose $F = F_1F_2F_3$ fits in a functional equation of type

$$h_1^3F_1 + h_2^2F_2 + h_3^6F_3 = 0, \tag{2.7}$$

where h_1, h_2, and h_3 are polynomials. Note that F_i are not necessarily irreducible or reduced. Also note that (2.7) induces a pencil map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by

$$\varphi([x : y : z]) = [h_1^3F_1 : h_2^2F_2].$$

Consider $\mathcal{C} := \{F = 0\}$. As $\varphi|_{\mathbb{P}^2\setminus\mathcal{C}}$ has three multiple fibers (over $P_3 = [0 : 1]$, $P_2 = [1 : 0]$, and $P_6 = [1 : -1]$), one has an orbifold morphism $\varphi_{2,3,6} : \mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1_{2,3,6}$. In particular, if the pencil φ is primitive (in the sense that it coincides with its Stein factorization), then by Proposition 2.7, there is an epimorphism

$$\varphi_{2,3,6} : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{2,3,6}) = \frac{\mu_2\mathbb{Z}_2 * \mu_3\mathbb{Z}_3}{(\mu_2\mu_3)^6},$$

where μ_i are as in Definition 2.4 and $\mu_2\mu_3 = \mu_6$ according to (2.6). Finally, note that

$$V := \text{Char}\left(\frac{\mu_2\mathbb{Z}_2 * \mu_3\mathbb{Z}_3}{(\mu_2\mu_3)^6}\right) = \{\omega_6, \omega_6^{-1}\},$$

and the elements $\{\omega_6^{\pm 1}, \omega_6^{\pm 2}, \omega_6^{\pm 3}\}$ are roots of $\Delta_{\mathcal{C},\varepsilon}(t)$, where

$$\varepsilon : H_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}) \to \frac{\pi_1^{\text{orb}}(\mathbb{P}^1_{2,3,6})}{[\pi_1^{\text{orb}}(\mathbb{P}^1_{2,3,6}), \pi_1^{\text{orb}}(\mathbb{P}^1_{2,3,6})]} = \mathbb{Z}_6 \tag{2.8}$$

is induced by $\varphi_{2,3,6}$ on the abelianizations of the groups. Therefore one has that $\Delta_{\mathcal{C},\varepsilon}(t)$ is of the form

$$(t-1)^{s_1}(t+1)^{s_2}(t^2+t+1)^{s_3}(t^2-t+1)^{s_6}p(t),$$

where $p(t)$ has no 6-th roots of unity as zeroes and s_i are non-negative integers.

Using this technique one can show the following result needed in the sequel and which we shall prove for completeness (cf. [23, Chapter 2, Theorem 2.3]).

Proposition 2.8. *The number of multiple members in a primitive pencil of plane curves (with no base components) is at most two.*

Proof. Let us assume that the pencil is generated by two multiple fibers, that is, we have $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, given by $\varphi([x:y:z]) = [f^p : g^q]$, where $(p,q) = 1$ (otherwise, the pencil is not primitive). Assume there is a third multiple member, that is, $f^p + g^q + h^r = 0$.

According to Proposition 2.7, one obtains an orbifold morphism with connected fibers, and thus an epimorphism $\varphi_* : \pi_1(\mathbb{P}^2 \setminus \{p_1,\dots,p_n\}) \to \pi_1^{\mathrm{orb}}(\mathbb{P}^1_{p,q,r})$, i.e. both groups are trivial. The groups $\pi_1^{\mathrm{orb}}(\mathbb{P}^1_{p,q,r})$ are virtually torsion free (cf. [42, Theorem 2.7]) and they are subgroups of order 2 of the Schwartz group, which is infinite (cf. [42, Corollary 2.4]). In particular, they are non-trivial, which contradicts the triviality of $\pi_1(\mathbb{P}^2 \setminus \{p_1,\dots,p_n\})$. □

Remark 2.9. For pencils other than pencils of plane curves, using logarithmic transforms, one can obtain elliptic fibrations with any number of multiple fibers (cf. [23, 28]).

In this paper we are interested in a particular type of orbifold morphisms.

Definition 2.10. We say a 2-dimensional orbifold $S_{\overline{m}}$ is a *rational orbifold curve of elliptic type* if $e_{\mathrm{top}}(S_{\overline{m}}) = 2$ (that is, $S_{\overline{m}}$ is a compact Riemann sphere) and

$$\sum \frac{1}{m_i} = n - 2,$$

where $n := \#\{P \in S \mid \overline{m}(P) > 1\}$ is the number of orbifold points.

Lemma 2.11. *The possible rational orbifold curves of elliptic type are:*

(1) $(2,3,6)$,

(2) $(3,3,3)$,

(3) $(2,4,4)$,

(4) $(2,2,2,2)$.

Remark 2.12. The reason to call such orbifolds of elliptic type is the following. Consider $\widehat{S}_{\overline{m}}$ the regular covering of order $\ell := \mathrm{lcm}(\overline{m})$ (the least common multiple of the orbifold multiplicities) ramified with index m_i at the $\frac{\ell}{m_i}$ preimages of P_i (the orbifold point of order m_i) of $S_{\overline{m}}$. In general, if $\sum \frac{1}{m_i} \in \mathbb{Z}$, then $\widehat{S}_{\overline{m}}$ is a Riemann surface of genus

$$1 + \frac{\ell}{2}\left(\sum (n-2) - \frac{1}{m_i}\right).$$

Thus, according to Lemma 2.11, for any rational orbifold curve $\widehat{S}_{\overline{m}}$ of elliptic type the associated covering $\widehat{S}_{\overline{m}}$ is an elliptic curve (that is, a complex compact curve of genus 1).

The results in this paper require that the least common multiple of the orbifold indices be > 2. Therefore type (4) from the list above will be disregarded.

2.3. Quasi-toric relations.

Definition 2.13. A *quasi-toric relation of type* (p,q,r) is a sextuple

$$\mathcal{R}_{\mathrm{qt}}^{(p,q,r)} := (F_1, F_2, F_3, h_1, h_2, h_3)$$

of non-zero homogeneous polynomials in $\mathbb{C}[x,y,z]$ satisfying the following functional relation:

$$h_1^p F_1 + h_2^q F_2 + h_3^r F_3 = 0. \tag{2.9}$$

The *support* of a quasi-toric relation $\mathcal{R}_{\mathrm{qt}}^{(p,q,r)}$ as above is the zero set

$$\mathcal{C} := \{F_1 F_2 F_3 = 0\}.$$

In this context, we may also refer to $\mathcal{C}$ as a curve that *satisfies (or supports) a quasi-toric relation* of type (p,q,r).

We will say a quasi-toric relation of type (p,q,r) is *of elliptic type* if (p,q,r) is of the form (1)–(3) in Lemma 2.11.

Remark 2.14. Given a quasi-toric relation $\mathcal{R}_{\mathrm{qt}}^{(p,q,r)} = (F_1, F_2, F_3, h_1, h_2, h_3)$ as above, note that

$$p \deg h_1 + \deg F_1 = q \deg h_2 + \deg F_2 = r \deg h_3 + \deg F_3 = \kappa \tag{2.10}$$

and hence

$$\sum \deg h_i + \left(\frac{\deg F_1}{p} + \frac{\deg F_2}{q} + \frac{\deg F_3}{r}\right) = \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)\kappa.$$

Therefore, if $\mathcal{R}_{\mathrm{qt}}^{(p,q,r)}$ is of elliptic type, then

$$\sum \deg h_i + \left(\frac{\deg F_1}{p} + \frac{\deg F_2}{q} + \frac{\deg F_3}{r}\right) = \kappa. \tag{2.11}$$

Remark 2.15. Special classes of curves satisfying quasi-toric relations have already been considered, namely, the class of *curves of torus type* $(2,3)$, i.e. quasi-toric relations of type $(2,3,r)$ of the form $(1,1,F,h_1,h_2,1)$ (cf. [43, 47]) and the class of *curves having quasi-toric decompositions* (p,q,pq), i.e. quasi-toric relations of type (p,q,pq) of the form $(1,1,F,h_1,h_2,h_3)$ (cf. [29]). Also, functional equations of a similar sort appear in [46, Section 5.3].

Recall that a *quasi-toric decomposition of* F is a collection of homogeneous polynomials $f, g, h \in \mathbb{C}[x,y,z]$ such that the following identity holds:

$$f^p + g^q + h^{pq} F = 0, \tag{2.12}$$

for two co-prime positive integers $p, q > 1$. Analogously, a curve is of torus type (p,q) if it admits a quasi-toric decomposition as in (2.12) where $h = 1$.

To each quasi-toric relation $\mathcal{R}_{\mathrm{qt}}^{(p,q,r)}$ of elliptic type satisfying (2.9) there corresponds a map from a certain cyclic multiple plane branched over the curve supporting $\mathcal{R}_{\mathrm{qt}}^{(p,q,r)}$ to the elliptic curve $\widehat{S}_{(p,q,r)}$ (see Remark 2.12) as follows. Let $\ell := \operatorname{lcm}(p,q,r)$ and

$$\omega = \frac{\deg F_1}{p} + \frac{\deg F_2}{q} + \frac{\deg F_3}{r}$$

(note that according to (2.11) one has $\omega = \kappa - \sum \deg h_i \in \mathbb{Z}$). Let V_ℓ be the following surface given in the weighted projective space $\mathbb{P}^3(\omega,1,1,1)$:

$$V_\ell := \left\{ (u,x,y,z) \in \mathbb{P}^3(\omega,1,1,1) \mid u^\ell = F_1^{\frac{\ell}{p}} F_2^{\frac{\ell}{q}} F_3^{\frac{\ell}{r}} \right\}. \tag{2.13}$$

Then let $\widehat{S}_{(p,q,r)}$ be the elliptic curve in $\mathbb{P}^2$ given by the equation $z^\ell = x^{\frac{\ell}{p}} y^{\frac{\ell}{q}} (-x-y)^{\frac{\ell}{r}}$. To each quasi-toric relation there corresponds the following map:

$$\begin{aligned} \mathbb{P}^3(\omega,1,1,1) \supset V_\ell &\to \widehat{S}_{(p,q,r)} \subset \mathbb{P}^2, \\ (u,x,y,z) &\mapsto (h_1^p F_1, h_2^q F_2, u h_1 h_2 h_3). \end{aligned} \tag{2.14}$$

(Note that the map is well defined by (2.10) and (2.11).)

Definition 2.16. We will say that two quasi-toric relations $(h_1,h_2,h_3,F_1,F_2,F_3)$ and $(\overline{h}_1,\overline{h}_2,\overline{h}_3,\overline{F}_1,\overline{F}_2,\overline{F}_3)$ of the same elliptic type (e_1,e_2,e_3) are *equivalent* iff the corresponding maps (2.14) coincide, i.e. there exists a non-trivial rational function $\lambda \in \mathbb{C}(x,y,z)^*$ such that

$$\overline{h}_i^{e_i} \overline{F}_i = \lambda h_i^{e_i} F_i \quad \text{and} \quad \overline{h}_1 \overline{h}_2 \overline{h}_3 = \lambda h_1 h_2 h_3. \tag{2.15}$$

Example 2.17. Note that any quasi-toric relation $(F_1,F_2,F_3,h_1,h_2,h_3)$ of elliptic type $(2,3,6)$ is equivalent to one of the form $(1,1,F_3F_2^2F_1^3,\overline{h}_1,\overline{h}_2,\overline{h}_3)$ since

$$\lambda := F_1^3 F_2^2, \quad \overline{h}_1 := F_1^2 F_2 h_1, \quad \overline{h}_2 := F_1 F_2 h_2 \quad \text{and} \quad \overline{h}_3 := h_3$$

satisfy (2.15).

In other words, any $(2,3,6)$ quasi-toric relation is equivalent to one of the form

$$\overline{h}_1^2 + \overline{h}_2^3 + F_1^3 F_2^2 F_3 \overline{h}_3^6 = 0.$$

This stresses the idea that non-reduced components are indeed unavoidable when one works with quasi-toric relations.

3. Mordell–Weil group of elliptic threefolds with fiber having $j = 0$

3.1. Elliptic pencils, rank of Mordell–Weil group, and the degree of Alexander polynomials. In this section, let us fix $F \in \mathbb{C}[x,y,z]$, an irreducible homogeneous polynomial of degree $d = 6k$, whose set of zeroes in $\mathbb{P}^2$ is a curve $\mathcal{C}$ that has only nodes and cusps as singularities (i.e. $\mathcal{C}$ has either $x^2 = y^2$ or $x^2 = y^3$ as a local equation around each singular point) and let, unless otherwise stated, the homomorphism $\varepsilon : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}) \to \mathbb{Z}$ be given by the total linking number with $\mathcal{C}$ in the affine plane $\mathbb{P}^2 \setminus \mathcal{C}_0$, i.e. satisfies $\varepsilon(\gamma) = 1$ for a meridian around $\mathcal{C}$.

Consider a threefold W_F containing, as a Zariski open subset, the affine threefold W_F° given in $\mathbb{C}^4$ by the following equation:

$$u^2 + v^3 = F(x, y, 1). \tag{3.1}$$

The projection onto the (x, y)-plane exhibits W_F as an elliptic threefold whose fibers over generic points have j-invariant equal to zero.[5)]

The main result of the present section is a calculation of the Mordell–Weil group of the $\mathbb{C}(x, y)$-points of W_F in terms of the classical Alexander polynomial $\Delta_{\mathcal{C}}(t)$ of $\mathcal{C}$ (that is, the Alexander polynomial w.r.t. the homomorphism ε described above).

Theorem 3.1. *The $\mathbb{Z}$-rank of the Mordell–Weil group of W_F over $\mathbb{C}(x, y)$ is equal to the degree of the Alexander polynomial $\Delta_{\mathcal{C}}(t)$ of $\mathcal{C}$.*

Let $V_6(F)$ (or simply V_6) denote a smooth model the 6-fold cyclic cover of $\mathbb{P}^2$ branched along $F = 0$ corresponding to the surjection $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}) \to \mathbb{Z}_6$ which is defined using the composition of the homomorphism ε as above and the reduction modulo 6. The surface $V_6(F)$ contains, as an open subset, the affine hypersurface in $\mathbb{C}^3$ given by the equation

$$z^6 = F(x, y, 1). \tag{3.2}$$

We shall assume that $V_6(F)$ supports a holomorphic action of the group μ_6 of roots of unity of degree 6, extending the action of this group affine surface given by $(x, y, z) \mapsto (x, y, \omega_6 z)$ where

$$\omega_6 = \exp\left(\frac{2\pi\sqrt{-1}}{6}\right) = \frac{1+\sqrt{-3}}{2}$$

is selected primitive root of unity of degree 6. Such a smooth model V_6 can be obtained for example using an equivariant resolution of singularities of projective closure of (3.2).

Recall that the degree of the Alexander polynomial $\Delta_{\mathcal{C}}(t)$ of $\mathcal{C}$ is equal to $2q$, where q is the irregularity $q := \dim H^0(V_6, \Omega^1_{V_6})$ of V_6 (cf. [32]). Let E_0 denote the elliptic curve with j-invariant equal to zero. As its biregular model one can take the projective closure of affine curve $u^2 + v^3 = 1$.

We shall start the proof of Theorem 3.1 by describing the Albanese varieties of 6-fold cyclic multiple planes V_6. Such a description will be based on the following Decomposition Theorem due to S. Roan (see [9, Theorem 3.2]).

Theorem 3.2. *Let X be an abelian variety of dimension g and α be an automorphism of X of order $d \geq 3$. Let Φ_α be the collection of the eigenvalues of the automorphism ρ_α induced by α on the universal cover of X and let $X_{\langle\alpha\rangle}$ be the union of fixed points of powers α^i for $1 \leq i < d$. Assume that $X_{\langle\alpha\rangle}$ is finite and $\#\Phi_\alpha = \frac{\varphi(d)}{2}$, where φ is the Euler function. Then $\varphi(d) \mid 2g$ and $(\mathbb{Q}(\xi_d), \Phi_\alpha)$ is a CM-field (cf.* [9, Section 3]*). Moreover, there are $k = \frac{2g}{\varphi(d)}$ abelian varieties $X_1, \dots, X_k$ of CM-type $(\mathbb{Q}(\xi_d), \Phi_\alpha)$ such that X factors as $X_1 \times \cdots \times X_k$ and α decomposes into a product of automorphisms of X_i induced by primitive d-th roots of unity.*

Recall (cf. [50]) that the Albanese variety of a smooth projective variety X is defined as

$$\mathrm{Alb}(X) = H^0(X, \Omega^1_X)^*/H_1(X, \mathbb{Z}) \tag{3.3}$$

5) Since we are interested in the Mordell–Weil group, which is a birational invariant of W_F, the actual choice of W_F is not important. Nevertheless, a particular biregular model of W_F will be given (cf. Lemma 3.9)

and the Albanese map $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is given by

$$P \mapsto \int_{P_0}^{P} \omega, \tag{3.4}$$

where P_0 is a base point and the integral in (3.4) is viewed as a linear function on $H^0(X, \Omega_X^1)$, well defined up to the periods (i.e. the integrals over loops representing classes in $H_1(X, \mathbb{Z})$). The map alb is well defined up to translation (i.e. a choice of P_0). A choice of a positive line bundle on X yields a polarization of $\mathrm{Alb}(X)$ making it into an abelian variety. In the case of the cyclic multiple plane V_6 we select a point P_0 in the locus with maximal ramification index of its projection onto $\mathbb{P}^2$. The map alb is universal in the sense that for any map $X \to A$ into an abelian variety A, there exists a factorization

$$X \to \mathrm{Alb}(X) \to A. \tag{3.5}$$

It follows from (3.3) that when X carries a biholomorphic action of a group G fixing the base point P_0, the action of G on $H^0(X, \Omega_X^1)$, given by

$$g(\omega) = (g^{-1})^* \omega,$$

induces the action on $\mathrm{Alb}(X)$. For this action the map alb is equivariant:

$$\mathrm{alb}(g \cdot P) = g_*^{-1}(\mathrm{alb}(P)). \tag{3.6}$$

For example, for $g \in G$ one has

$$g_*^{-1}\, \mathrm{alb}(P) = \int_{P_0}^{P} (g^{-1})^* \omega = \int_{gP_0}^{gP} \omega = \mathrm{alb}(g \cdot P).$$

We will need also a local version of the above construction corresponding to the mixed Hodge structure associated with a germ of plane curve singularity (with assumptions stated below):

$$f(x, y) = 0. \tag{3.7}$$

While there are several constructions of a mixed Hodge structure associated with a germ of singularity (3.7) (cf. [53]), we shall consider only the case when the monodromy action on the cohomology of Milnor fiber is semi-simple (e.g. the ordinary cusps, nodes and more generally the singularities appearing in Tables 1, 2 and 3 in Section 5). In this case one can identify the (co)homology of the Milnor fiber with the (co)homology of the link of the surface singularity

$$z^N = f(x, y), \tag{3.8}$$

where N is the order of the monodromy of the cohomology of Milnor fiber (cf. for example [37] for a similar discussion). More precisely we have the following:

Lemma 3.3. *Let* (3.7) *be a germ of a plane curve (possibly reducible and non-reduced) with semi-simple monodromy of order* $N > 1$ *and the Milnor fiber* F_f. *Let* $L_{f,N}$ *be link of the corresponding surface singularity* (3.8). *Then there is the isomorphism of the mixed Hodge structures*

$$(\mathrm{Gr}_3^W H^2(L_{f,N}))(1) = \mathrm{Gr}_1^W H^1(F_f), \tag{3.9}$$

where the mixed Hodge structure on the left is the Tate twist of the mixed Hodge structure constructed in [19] *or* [20] *and the one on the right is the mixed Hodge structure on vanishing cohomology constructed in* [52].[6)]

Recall that the construction of the mixed Hodge structure on the cohomology of a link of an isolated singularity is based on the identification of the latter with the cohomology of the punctured regular neighborhood of the exceptional set in a resolution for (3.8). Alternatively, one can use the mixed Hodge structure on the local cohomology of (3.8) supported at the singularity of (3.8). Dualizing one obtains the mixed Hodge structure on the homology as well.

Proof. Consider z as a holomorphic function on the germ $V_{N,f}$ of the surface singularity $z^N = f(x,y)$ (cf. (3.8)) and the Milnor fiber $F_{N,f}$ of z, i.e. the subset of $V_{N,f}$ given by the equation $z = t$ for fixed t. It has canonical identification with the Milnor fiber of $f(x,y)$ over t^N. Monodromy of the Milnor fiber of z coincides with T^N where T is the monodromy of $f = t$. Denote by Z_0 the subset of $V_{N,f}$ given by $z = 0$ and consider the Wang exact sequence of the mixed Hodge structures (cf. for example [18, Section 1.7]); below $\log T_u$ is the logarithm of the unipotent part of the monodromy which is trivial in our case and subscript 1 indicated the monodromy invariant subspace, i.e. the eigenspace with eigenvalue 1:

$$H^1(V_{N,f} - Z_0) \to H^1(F_{N,f})_1 \xrightarrow{\log T_u} H^1(F_{N,f})_1(-1) \to H^2(V_{N,f} - Z_0) \to 0.$$

The last term of this sequence is zero since the Milnor fiber has the homotopy type of a 1-complex. Since $\log T_u = 0$ by our assumptions and since $L^\circ_{N,f} = L_{N,f} \setminus Z_0$ has the mixed Hodge structure on its cohomology is constructed via the identification with the mixed Hodge structure on $V_{N,z} \setminus Z_0$, we obtain the isomorphisms

$$H^1(F_{N,f})_1(-1) = H^2(L^\circ_{N,f}), \quad \mathrm{Gr}^W_3 H^1(F_{N,f})_1(-1) = \mathrm{Gr}^W_3 H^2(L^\circ_{N,f}). \tag{3.10}$$

Identifying, as above, the Milnor fiber $F_{N,f}$ of z with the Milnor fiber of f, we obtain the identification of the cohomology of punctured neighborhood $L^\circ_{f,N}$ with the cohomology of Milnor fiber, i.e.

$$H^2(L^\circ_{f,N}) = H^1(F_f)(-1).$$

Moreover, the exact sequence of pair

$$H^2(L_{N,f}, L^\circ_{N,f}) \to H^2(L_{f,N}) \to H^2(L^\circ_{f,N}) \to H^3(L_{N,f}, L^\circ_{N,f})$$

yields the isomorphism

$$\mathrm{Gr}^W_3 H^2(L_{f,N}) = \mathrm{Gr}^W_3 H^2(L^\circ_{f,N}) \tag{3.11}$$

(in the case of irreducible germ the right homomorphism in (3.11) is trivial and Gr^W_3 can be omitted). This together with (3.10) yields the claim of the lemma. □

Example 3.4. Consider $f = x^3 + y^3$. For the order of the monodromy one has $N = 3$ and

$$\mathrm{rk}\, H^1(F_f, \mathbb{Q}) = 4, \quad \mathrm{rk}\, \mathrm{Gr}^W_2 H^1(F_f, \mathbb{Q}) = 2 \quad \text{and} \quad \mathrm{rk}\, \mathrm{Gr}^W_1 H^1(F_f, \mathbb{Q}) = 2.$$

6) Semi-simplicity assumption is yielded automatically if f is irreducible in which case $H^1(F_f)$ is pure of weight 1.

On the other hand $L_{f,3}$ can be viewed as the link of surface singularity $x^3+y^3+z^3=0$ which is the circle bundle over the elliptic curve in $\mathbb{P}^2$ given by the same equation. Gysin exact sequence yields that $\operatorname{rk} H^2(L_{f,3})(1)=2$. Both Hodge structure have natural identification with the Hodge structure of the elliptic curve with $j=0$.

Construction of the mixed Hodge structure on either side of (3.9) yields that the type of this mixed Hodge structure on corresponding *homology* is

$$(-1,-1),(-1,0),(0,-1),(0,0). \tag{3.12}$$

In fact with the assumption of semi-simplicity made in Lemma 3.3 the mixed Hodge structure on the homology of Milnor fiber is pure of type $(-1,0),(0,-1)$. The equivalence of categories of such mixed Hodge structures without torsion and the categories of 1-motifs constructed in [17, 10.1.3] allows to extract the abelian variety A which is the part of structure of 1-motif. We shall refer to this abelian variety as the local Albanese variety of the singularity (3.7). The polarization of the Hodge structure on Gr^W_{-1}, required for such an equivalence, is the standard polarization of such graded component associated with the Milnor fiber.

In the case of the cusp $x^2=y^3$ the assumptions of Lemma 3.3 are fulfilled. One can also describe the construction of corresponding local Albanese variety as taking the quotient of $(\operatorname{Gr}^0_F\operatorname{Gr}^W_1)^*$ by the homology lattice of the closed Riemann surface which is the compactified Milnor fiber. In particular, the Hodge structure on the cohomology of its Milnor fiber $x^2=y^3+t$ is the Hodge structure of the elliptic curve E_0 with $j=0$.

The above discussion yields:

Corollary 3.5. *The eigenvalues of the generator* $z\mapsto z\cdot\exp(\frac{2\pi\sqrt{-1}}{N})$ *of the group of covering transformations* (3.8) *acting on the cohomology of the singularity link* (3.8) *coincide with the eigenvalues of the action of the monodromy on the cohomology of the Milnor fiber* (3.7) *(cf.* [32] *or* [35, Section 1.3.1]*). The eigenvalues of the above generator of the group of deck transformations of the germ* (3.8) *acting on* $\operatorname{Gr}^0_F\operatorname{Gr}^W_1(H^1(L_{N,f}))$ *have the form* $e^{2\pi\sqrt{-1}\alpha}$ *where* α *runs through the elements the spectrum of the singularity* (3.7) *which belongs to the interval* $(0,1)$ *(cf.* [40]*).*

For example, in the case of the cusp $x^2=y^3$ the monodromy corresponding to the path $e^{2\pi\sqrt{-1}s}$ $(0\le s\le 1)$ in the positive direction given by the complex structure yields the monodromy of the Milnor fiber $F_{N,f}$ in Lemma 3.3 given by

$$(x,y,z)\mapsto\left(xe^{\frac{2\pi\sqrt{-1}s}{2}},ye^{\frac{2\pi\sqrt{-1}s}{3}}ze^{\frac{2\pi\sqrt{-1}s}{6}}\right).$$

In order to apply Theorem 3.2 to $\operatorname{Alb}(V_6)$, we shall need the following:

Lemma 3.6. *Let, as above, V_6 be a smooth $\mathbb{Z}_6$-equivariant model of a cyclic* 6*-fold covering space of $\mathbb{P}^2$ branched over a curve with only nodes and cusps as singularities. Let T be a generator of covering group of V_6. The automorphism of* $\operatorname{Alb}(V_6)$ *induced by T has only one eigenvalue (which is a primitive root of unity of degree 6).*

Proof. Note that it follows from [32] that the Alexander module of a cyclic multiple plane branched over a curve with nodes and cusps, up to summands $\mathbb{Q}[t,t^{-1}]/(t-1)$, is isomorphic to a direct sum

$$[\mathbb{Q}[t,t^{-1}]/(t^2-t+1)]^s. \tag{3.13}$$

More precisely, the proof of the Divisibility Theorem [32] shows that the Alexander module is a quotient of the direct sum of the Alexander modules of all singularities or, equivalently, the direct sum of the homology of Milnor fibers of singularities with module structure given by the action of the monodromy (in the present case of the cusp $x^2 = y^3$ the local Alexander module is just one summand in (3.13)).

Since the divisibility result is stated in [32] with assumption of irreducibility of the ramification locus $\mathcal{C}$ of (3.2), we shall review the argument. Denote by $T(\mathcal{C})$ a tubular neighborhood of $\mathcal{C}$ in $\mathbb{P}^2$ and consider the surjection

$$\pi_1(T(\mathcal{C}) \setminus \mathcal{C}) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \tag{3.14}$$

induced by embedding (the surjectivity is a consequence of the surjectivity, for $D \subset T(\mathcal{C})$, of the map $\pi_1(D \setminus D \cap \mathcal{C}) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{C})$, in turn, following from the weak Lefschetz theorem). The surjection (3.14) induces the surjection of the homology of 6-fold cyclic coverings:

$$H_1((T(\mathcal{C}) \setminus \mathcal{C})_6, \mathbb{Z}) \to H_1((\mathbb{P}^2 \setminus \mathcal{C})_6, \mathbb{Z}). \tag{3.15}$$

On the other hand, the covering space $(T(\mathcal{C}) \setminus \mathcal{C})_6$ decomposes as

$$(T(\mathcal{C}) \setminus \mathcal{C})_6 = \bigcup_P (B_P \setminus \mathcal{C})_6 \cup (U \setminus \mathcal{C})_6, \tag{3.16}$$

where B_i are small regular neighborhoods of all singular points P of $\mathcal{C}$, U is the regular neighborhood of the smooth locus of $\mathcal{C}$ and the subscript designates the 6-fold cyclic cover. The corresponding Mayer–Vietoris sequence yields a surjection:

$$\bigoplus_P H_1((B_P \setminus \mathcal{C})_6, \mathbb{Q}) \oplus H_1((U \setminus \mathcal{C})_6, \mathbb{Q}) \to H_1((T(\mathcal{C}) \setminus \mathcal{C})_6, \mathbb{Q}). \tag{3.17}$$

One can view the cohomology of the 6-fold cover of $(B_P \setminus \mathcal{C})_6$ as the cohomology of the punctured neighborhood of the part of exceptional curve in resolution of 6-fold cover of B_P branched over $B_P \cap \mathcal{C}$ outside of proper preimage of $\mathcal{C}$. For any cusp P the deck transformation acting on $\mathrm{Gr}^0_F \mathrm{Gr}^W_1 (H^1((B_P \setminus \mathcal{C})_6))$ has as eigenvalue the same primitive root of unity of degree 6 (corresponding to the element in the spectrum of $x^2 = y^3$ in the interval $(0, 1)$; in the case of the action on cohomology this part contains only $\frac{5}{6}$).

Each of the other spaces $(U \setminus \mathcal{C})_6, (T(\mathcal{C}) \setminus \mathcal{C})_6$ appearing in (3.17) can be viewed as a punctured neighborhood of a quasi-projective variety and as such also supports the canonical mixed Hodge structure (cf. [20]). Moreover, the sequence (3.17) is a sequence of mixed Hodge structures. It is shown in [32] that the map

$$H_1((T(\mathcal{C}) \setminus \mathcal{C})_6, \mathbb{Q}) \to H_1(V_6, \mathbb{Q}) \tag{3.18}$$

is surjective and that the composition of (3.17) and (3.18) takes $H_1((U \setminus \mathcal{C})_6, \mathbb{Q})$ to zero.

Both sequences (3.17) and (3.18) are equivariant with respect to the deck transformations. The sequence (3.18) yields that any eigenvalues of T acting on $\mathrm{Gr}^0_F H^1(V_6)$ must be an eigenvalue of T acting on $\mathrm{Gr}^0_F \mathrm{Gr}^W_1 H^1((T(\mathcal{C}) \setminus \mathcal{C})_6, \mathbb{Q})$ which is different from 1, i.e. is the eigenvalue of monodromy acting on Milnor fiber of cusp and corresponding to the part of spectrum in $(0, 1)$.

This implies that the action of the deck transformation on $\mathrm{Alb}(V_6)$ is the multiplication by the same root of unity of degree 6 as well (exponent of the only element of the spectrum belonging to $(0, 1)$). Also note that an i-th power of this automorphism $(1 \leq i < 6)$ has zero as the only fixed point, since the existence of a fixed point for such an i would yield an eigenspace of the monodromy corresponding to an eigenvalue which is not a primitive root of degree 6. □

Theorem 3.2, when applied to $X = \mathrm{Alb}(V_6)$ with $g = q = h^{1,0}(V_6)$, and $d = 6$, shows (since the condition on the fixed point sets follows from the explicit form of the action on $H^0(V_6, \Omega^1_{V_6})$) that $k = q$ and that each component $X_1, \ldots, X_q$ is the elliptic curve with an automorphism of order 6, i.e. the curve E_0 with $j = 0$.

Hence we obtain the following:

Proposition 3.7. *Let V_6 be a 6-fold cyclic multiple plane with branching curve having only nodes and cusps as singularities with irregularity q. Then*

$$\mathrm{Alb}(V_6) = E_0^q.$$

In particular, for the set of morphisms taking P_0 to the zero of E_0 one has

$$\mathrm{Mor}(V_6, E_0) = \mathrm{Hom}(E_0^q, E_0).$$

Next we shall reformulate this proposition in terms of the Mordell–Weil group of split elliptic threefold $W = V_6 \times E_0$ viewed as elliptic curve defined over $\mathbb{C}(V_6)$ (and which can be viewed as a cover of W_F cf. Section 3.2).

Recall that given an extension K/k and an abelian variety A over K, one has an abelian variety B over k (called the *Chow trace*) and homomorphism $\tau : B \to A$ defined over k such that for any extension E/k disjoint from K and abelian variety C over E and $\alpha : C \to A$ over KE exist $\alpha' : C \to B$ such that $\alpha = \tau \circ \alpha'$ (cf. [30, p. 97]).

In particular, to W over $\mathbb{C}(V_6)$, one can associate Chow trace which is the elliptic curve B over $\mathbb{C}$ such that quotient of the group of $\mathbb{C}(V_6)$-points of W by the subgroup of $\mathbb{C}$-points of B is a finitely generated abelian group (Mordell–Weil Theorem cf. [41] and [30, Theorem 1]). The Chow trace B of W is E_0 and the group $\tau B_{\mathbb{C}}$ is the subgroup in $\mathbb{C}(V_6)$ of points corresponding to constant maps $V_6 \to E_0$. We shall denote by $\mathrm{MW}(W)$ the quotient of the group of $\mathbb{C}(V_6)$-points by the subgroup of torsion point and the points of Chow trace. As already mentioned, $\mathrm{MW}(W)$ is a finitely generated abelian group. Proposition 3.7 can be reformulated in terms of the Mordell–Weil group as follows:

Corollary 3.8. *The Mordell–Weil group of $\mathbb{C}(V_6)$-points of $V_6 \times E_0$ is a free $\mathbb{Z}[\omega_6]$ module having rank q (where q is the irregularity of V_6).*

Proof. Non-zero elements of this Mordell–Weil group are represented by the classes of non-constant sections of $V_6 \times E_0 \to V_6$. Those corresponds to the maps $V_6 \to E_0$ up to translation. Hence the corollary follows from Propositions 3.7. □

Next we shall return to the threefold W_F which is a smooth birational model of (3.1). We shall view it as an elliptic curve over the field $K = \mathbb{C}(x, y)$. Note that it splits over the field $K(F^{\frac{1}{6}}) = \mathbb{C}(V_6)$.

3.2. Elliptic pencils on multiple planes and $\mathbb{P}^2$-points of elliptic threefolds. We want to have an explicit correspondence between the elliptic pencils on V_6 and $\mathbb{P}^2$ points of W_F.[7)] This is used in Theorem 3.10 below to obtain the relation between the Mordell–Weil

7) Following classical terminology, by elliptic (resp. rational) pencil we mean a morphism onto an elliptic (resp. rational) curve. An orbifold morphism onto a rational orbifold curve of elliptic type induces an elliptic pencil cf. Remark 2.12.

group of $\mathbb{P}^2$-points of W_F and the Mordell–Weil group of $\mathbb{C}(V_6)$-points of the split threefold $W = V_6 \times E_0$ and is based on Lemma 3.9 below. Before we state it we shall introduce several notations. Compactifications W_F of the threefold (3.1) have several useful biregular models in weighted projective spaces and their products which we shall derive. We can view E_0 as the curve given by the equation $u^3 + v^2 = w^6$ in weighted projective plane $\mathbb{P}(2,3,1)$. Let $\overline{E}_Q$ be a surface in the $\mathbb{P}(2,3,1) \times \mathbb{P}^1$ given by

$$A^6(u^3 + v^2) = w^6 B^6. \tag{3.19}$$

By $\mathrm{pr}^{\overline{E}_Q}_{\mathbb{P}(2,3,1)}$ (resp. $\mathrm{pr}^{\overline{E}_Q}_{\mathbb{P}^1}$) we shall denote the projections to the factors. Clearly, the map given by

$$(u, v, w, A, B) \mapsto (A^2 u, A^3 v, Bw, A, B)$$

takes $\overline{E}_Q \subset \mathbb{P}(2,3,1) \times \mathbb{P}^1$ to the surface in $\mathbb{P}(2,3,1) \times \mathbb{P}^1$ given by $u^2 + v^3 = w^6$ (no dependence on (A, B)), which is isomorphic to $E_0 \times \mathbb{P}^1$. The action of μ_6 on $\overline{E}_Q$ corresponding to the standard action on E_0 and trivial action on $\mathbb{P}^1$ is given by $(A, B) \to (A, \omega_6 B)$.

We denote by $\overline{V}_6$ the biregular model of the cyclic cover of $\mathbb{P}^2$ branched over $F = 0$ and $z = 0$, which is the surface in $\mathbb{P}^2 \times \mathbb{P}^1$ given by

$$\overline{V}_6 := \{([x : y : z], [M : N]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid z^{6k} M^6 = N^6 F(x, y, z)\}. \tag{3.20}$$

The action of the deck transformation is given by

$$(M, N) \mapsto (M, \omega_6 N).$$

The projection on the first (resp. the second) factor will be denoted by $\mathrm{pr}^{\overline{V}_6}_{\mathbb{P}^2}$ (resp. $\mathrm{pr}^{\overline{V}_6}_{\mathbb{P}^1}$).

We shall consider the threefold

$$\overline{W} = \overline{V}_6 \times_{\mathbb{P}^1} \overline{E}_Q \tag{3.21}$$

with the fibered product taken relative to the maps $\mathrm{pr}^{\overline{V}_6}_{\mathbb{P}^1}$ and $\mathrm{pr}^{\overline{E}_Q}_{\mathbb{P}^1}$ respectively with coordinates of respective copies of $\mathbb{P}^1$ identified using the relation

$$\frac{N}{M} = \frac{A}{B}. \tag{3.22}$$

The birational equivalence between $\overline{E}_Q$ and $E_0 \times \mathbb{P}^1$ yields the birational isomorphism

$$\overline{W} \to \overline{V}_6 \times E_0, \tag{3.23}$$

i.e. $\overline{W}$ is a projective model of W.

Lemma 3.9. *The threefold $\widetilde{W}_F$ in $\mathbb{P}(2,3,1,1,1)$ given by*

$$(u^3 + v^2) z^{6(k-1)} = F(x, y, z) \tag{3.24}$$

is birationally equivalent to W/μ_6 with the diagonal action of μ_6. It contains the hypersurface (3.1) *as an open set, i.e. is a model of W_F.*

Proof. The equations of $\overline{W}$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}(2,3,1) \times \mathbb{P}^1$ are (3.19), (3.20) and (3.22). Hence $\overline{W}$ is biregular to complete intersection in $\mathbb{P}^2 \times \mathbb{P}(2,3,1) \times \mathbb{P}^1$ given by

$$z^{6k} M^6 = N^6 F(x, y, z), \quad N^6(u^3 + v^2) = w^6 M^6. \tag{3.25}$$

Projection of this complete intersection on $\mathbb{P}^2 \times \mathbb{P}(3,2,1)$ has as its image the set of points (u, v, w, x, y, z) for which the determinant of the system (3.25) in M^6, N^6 is zero, i.e. is the hypersurface given by

$$z^{6k}(u^3 + v^2) = w^6 F(x, y, z). \tag{3.26}$$

Clearly this projection is a cyclic μ_6-covering of the hypersurface (3.26). Alternatively it is the quotient of (3.25) by the action of μ_6 given by $(M, N) \mapsto (M, \omega_6 N)$. Moreover it shows that (3.26) is the quotient of $\overline{W}$ by the *diagonal* (as follows from (3.22)) action of μ_6. Finally both the hypersurface in the statement of Lemma 3.9 and (3.26) have W_F as a Zariski open subset which yields the statement. □

Next we compare the $\mathbb{C}(V_6)$-points of E_0 and the $\mathbb{P}^2$-points of the elliptic threefold W_F. For V_6, which is a non-singular model of $\overline{V}_6$, the $\mathbb{C}$-split elliptic threefold $W = V_6 \times E_0$, as above, is the elliptic curve over $\mathbb{C}(V_6)$ obtained by the field extension $\mathbb{C}(V_6)/\mathbb{C}$. The $\mathbb{C}(V_6)$-points of W correspond to the rational maps $V_6 \to E_0$, which also can be viewed as sections of the projection $W \to V_6$ by associating with a map its graph in W and vice versa. The corresponding Mordell–Weil group was calculated in Corollary 3.8.

The group μ_6 acts (diagonally) on W and hence also on $\mathrm{MW}(W)$. We denote the invariant subgroup as $\mathrm{MW}(W)^{\mu_6}$. To a μ_6-invariant element $V_6 \to W$ of $\mathrm{MW}(W)$ corresponds a μ_6-invariant V_6-point $\phi : V_6 \to E_0$ in the sense that $\phi(\gamma(v)) = \gamma\phi(v), (\gamma \in \mu_6)$. Its graph $\Gamma_\phi \subset V_6 \times E_0$ is μ_6-invariant and taking the μ_6-quotient yields the map

$$\mathbb{P}^2 = V_6/\mu_6 = \Gamma/\mu_6 \to W/\mu_6 = W_F. \tag{3.27}$$

Hence we obtain a $\mathbb{P}^2$-point of W_F. Vice versa a section $\mathbb{P}^2 \to W_F$ lifts to a birational map of cyclic covers, i.e. the map $V_6 \to W = V_6 \times E_0$ (which follows from comparison of the complements to the branching loci of both coverings). This yields an equivariant elliptic pencil (i.e. commuting with the μ_6-action).

Theorem 3.10. *The correspondence $\phi \mapsto \Gamma_\phi/\mu_6$ induces an isomorphism*

$$\mathrm{MW}(W)^{\mu_6} \to \mathrm{MW}(W_F).$$

In particular, $\operatorname{rk}\mathrm{MW}(W_F) = 2q(V_6)$.

Proof. It is enough to check that for two equivariant maps ϕ_1, ϕ_2 in the same coset of the Chow trace the map $V \to E_0$ given by $v \mapsto \phi_1(v) - \phi_2(v)$ is constant with image $0 \in E_0$. Indeed this is a map to a point which due to equivariance should be the μ_6-fixed point of E_0, i.e. zero.

To see the second part, since $\mathrm{End}(E_0) = \mathbb{Z}[\omega_6]$, we infer from Proposition 3.7 or Corollary 3.8 that

$$\mathrm{Mor}(V_6, E_0) = \mathrm{Hom}(E_0^q, E_0) = \mathbb{Z}[\omega_6]^q.$$

The action of the group μ_6 on $\mathrm{Alb}(V_6)$ is via multiplication by ω_6 (i.e. as is in the case of local Albanese of the cusp). Since all elements in $\mathrm{End}(E_0)$ commute with complex multiplication, the action of μ_6 on $\mathrm{End}(E_0)$ is trivial and one obtains $2q$ as the $\mathbb{Z}$-rank of $\mathrm{MW}(W_F)$. □

Proof of Theorem 3.1. It follows immediately from Theorem 3.10 and the well-known relation between the degree of the Alexander polynomial and the irregularity of cyclic multiple planes (cf. [32]). □

Corollary 3.11 (cf. [27]). *For a degree* 6 *curve with six cusps, the* $\mathbb{Z}$*-rank of* $\mathrm{MW}(W_F)$ *is equal to* 2 *(resp.* 0*) if the cusps belong (resp. do not belong) to a conic. For a degree* 6 *curve with seven, eight and nine cusps, the* $\mathbb{Z}$*-ranks of* $\mathrm{MW}(W_F)$ *are equal to* 2*,* 4 *and* 6 *respectively.*

3.3. A bound of the rank of Mordell–Weil group of an elliptic threefold.

Theorem 3.12. *If* $d = 6k$ *is the degree of a homogeneous polynomial* $F \in \mathbb{C}[x, y, z]$*, for which the corresponding curve* $\mathcal{C} := \{F = 0\}$ *has only nodes and cusps as singularities, then the* $\mathbb{Z}$*-rank of the Mordell–Weil group of* W_F *satisfies*

$$\operatorname{rk} \mathrm{MW}(W_F) \leq \frac{5}{3}d - 2.$$

Proof. Let $\ell \subset \mathbb{P}^2(x, y, z)$ be a generic line in the base of the elliptic threefold

$$\pi : W_F \to \mathbb{P}^2.$$

Then $\mathrm{MW}(W_F) \to \mathrm{MW}(\pi^{-1}(\ell))$ is injective (cf. [27]) and a bound on $\operatorname{rk} \mathrm{MW}(\pi^{-1}(\ell))$ therefore yields a bound on the rank of $\mathrm{MW}(W_F)$.

On the other hand,

$$h^{1,1}(\pi^{-1}(\ell)) \geq \operatorname{rk} \mathrm{NS}(\pi^{-1}(\ell)) \geq \operatorname{rk} \mathrm{MW}(\pi^{-1}(\ell)).$$

The surface $\pi^{-1}(\ell)$ is a hypersurface in the weighted projective space $\mathbb{P}^3(3k, 2k, 1, 1)$, which is a quotient of the surface in $\mathbb{P}^3$ given by the equation

$$\mathcal{W}: \quad P^2 = Q^3 + F|_\ell$$

with the action of the group

$$G_k = \mathbb{Z}_{3k} \oplus \mathbb{Z}_{2k}, \quad (P, Q, a, b) \mapsto (\omega_{3k}^i P, \omega_{2k}^j Q, a, b)$$

(a, b are the coordinates of ℓ). The set of singularities of $\mathbb{P}^3/G_k$ corresponding to the fixed points of subgroups of G_k consists of two points if $k = 1$ (i.e. $(1, 0, 0, 0), (0, 1, 0, 0)$) and of the line $(P, Q, 0, 0)$ if $k > 1$. These points are outside of $\mathcal{W}$ if $k = 1$ and hence the surface $\pi^{-1}(\ell)$ is non-singular. In the case $k > 1$ the surface $\mathcal{W}$ has one singular point $p = (1, 1, 0, 0)$.

For all k, the elliptic surface $\widetilde{\pi^{-1}(\ell)}$ with blown up point p has an elliptic fibration with $6k = \deg F$ degenerate fibers each being isomorphic to a cubic curve with one cusp. Indeed, this elliptic fibration $\widetilde{\pi^{-1}(\ell)}$ for any singular fiber has *additive* reduction and the order of vanishing of the discriminant, i.e. the order of $F|_\ell^2$ is equal to 2 since ℓ is generic. Therefore all fibers of $\pi^{-1}(\ell)$ are irreducible (cf. [48, (13)]) and hence biregular to a cuspidal cubic.[8)]Moreover,

$$e_{\mathrm{top}}\left(\widetilde{\pi^{-1}(\ell)}\right) = 12k,$$

i.e. by the Noether formula

$$\chi\left(\widetilde{\pi^{-1}(\ell)}\right) = k = \frac{d}{6}.$$

Hence, using the calculation of the Hodge diamond of an elliptic surface (cf. [48, Section 6.9]) and the Tate–Shioda formula (cf. [48, Corollary 6.13]), we derive

$$\operatorname{rk} \mathrm{MW}(\pi^{-1}(\ell)) = \rho(\pi^{-1}(\ell)) - 2 \leq h^{1,1} - 2 = 10\chi - 2 = 10k - 2, \tag{3.28}$$

which gives the claim. □

8) The authors are grateful to referee for providing this argument.

This immediately yields:

Corollary 3.13. *The degree of the Alexander polynomial of an irreducible curve $\mathcal{C}$ of degree $d = 6k$ whose singularities are only nodes and cusps satisfies*

$$\deg \Delta_{\mathcal{C}} \leq \frac{5}{3}d - 2.$$

4. Quasi-toric relations corresponding to cuspidal curves

4.1. Quasi-toric relations and $\mathbb{P}^2$-points of elliptic threefolds. In this section we shall present an explicit relation between the quasi-toric relations introduced in Section 2.3 and the elements of the Mordell–Weil group of W_F. Such a relation is expected since it was shown earlier that quasi-toric relations correspond to elliptic pencils on cyclic multiple planes and orbifold pencils (cf. Section 2.3). On the other hand in the last section such pencils were related to the Mordell–Weil groups.

Let us consider the map from the threefold defined in (3.24) onto $\mathbb{P}^2$ induced by the projection centered at $x = y = z = 0$

$$\begin{array}{ccc} [u,v,x,y,z] & \in & \mathbb{P}(2,3,1,1,1) \setminus \{[u,v,0,0,0]\} \\ \downarrow & & \downarrow \\ [x,y,z] & & \mathbb{P}^2. \end{array} \tag{4.1}$$

Note that a rational section of this projection is given by

$$s(x,y,z) = [f(x,y,z), g(x,y,z), x\widetilde{h}(x,y,z), y\widetilde{h}(x,y,z), z\widetilde{h}(x,y,z)], \tag{4.2}$$

where $2(\deg \widetilde{h} + 1) = \deg f$ and $3(\deg \widetilde{h} + 1) = \deg g$, satisfying

$$(z\widetilde{h})^{6(d-1)}(f^3 + g^2) = (\widetilde{h}^d)^6 F$$

and hence

$$z^{6(d-1)}\left(f^3 + g^2\right) = \widetilde{h}^6 F.$$

As F can be chosen not to be divisible by z, by the unique factorization property of $\mathbb{C}[x,y,z]$, we get

$$h := \frac{\widetilde{h}}{z^{6(d-1)}} \in \mathbb{C}[x,y,z],$$

and hence

$$f^3 + g^2 = h^6 F$$

is a quasi-toric relation of F.

Conversely, given any quasi-toric relation $f^3 + g^2 = h^6 F$, for certain $f, g, h \in \mathbb{C}[x,y,z]$ such that $2(\deg h + d) = \deg f$ and $3(\deg h + d) = \deg g$, the map

$$s(x,y,z) = [f(x,y,z), g(x,y,z), xz^{d-1}h(x,y,z), yz^{d-1}h(x,y,z), z^d h(x,y,z)]$$

results in a section of the projection (4.1).

As a consequence, we obtain the following proposition.

Proposition 4.1. *The degree of the Alexander polynomial is equal to the number of equivalence classes of quasi-toric relations which correspond to independent elements (over* $\mathbb{Z}[\omega_6]$*) of the Mordell–Weil group of* W_F *given by equation* (3.24).

Also, using Theorem 3.10 and projection (4.1), one can give formulas for the additive structure in $\mathrm{MW}(W_F)$. By Corollary 3.8, it is enough just to give the action of $\mathbb{Z}[\omega_6]$ on the sections in $\mathrm{MW}(W_F)$ and the addition. Consider the sections

$$\sigma_1 := \left(\frac{f_2}{h_1^2}, \frac{f_3}{h_1^3}, x, y, z\right) \quad \text{and} \quad \sigma_2 := \left(\frac{g_2}{h_2^2}, \frac{g_3}{h_2^3}, x, y, z\right)$$

of W_F° and assume for simplicity that $h_1 = h_2 = 1$. Then one has the following.

Proposition 4.2. *Under the above conditions, we have*

$$\omega_6\sigma_1 = (\omega_6 f_2, -f_3, x, y, z), \tag{4.3}$$

also, if $\sigma_1 \neq \sigma_2$*, then*

$$\sigma_1 + \sigma_2 = \left(\frac{g_2 f_2^2 + g_2^2 f_2 + 2g_3 f_3 - 2F}{(f_2 - g_2)^2}, \frac{3f_2 g_2(f_3 g_2 - g_3 f_2) + (f_3 - g_3)(g_3 f_3 - 3F)}{(f_2 - g_2)^3}, x, y, z\right). \tag{4.4}$$

Otherwise, we have

$$2\sigma_1 = \left(-f_2\frac{9f_2^3 + 8f_3^2}{4f_3^2}, -\frac{27f_2^6 + 36f_2^3 f_3^2 + 8f_3^4}{8f_3^3}, x, y, z\right). \tag{4.5}$$

Proof. Since $\omega_6(u, v) = (\omega_6 u, -v)$, one obtains (4.3). The formulas can be easily obtained from the well-known formulas of the group law in E_0 (cf. [51, Chapter III.3]). □

Next we shall describe a geometric property of generators or the Mordell–Weil group viewed as elliptic pencils.

4.2. Primitive quasi-toric relations and orbifold maps. An alternative way to see that every quasi-toric relation of F contributes to the degree of its Alexander polynomial comes from the theory of orbifold surfaces and orbifold morphisms reviewed in Section 2.2.

As mentioned before, the elements of the Mordell–Weil group $\mathrm{MW}(W_F)$ are represented by μ_6-equivariant surjective maps $V_6 \to E_0$ (which we called elliptic pencils). To end this section, we will relate those which have irreducible generic members to generators of the Mordell–Weil group.

Definition 4.3. We call an elliptic pencil $V_6 \to E_0$ *primitive* if its generic fiber is irreducible.

One has the following result.

Proposition 4.4. *An elliptic pencil* $f : V_6 \to E_0$ *is non-primitive if and only if there exists an elliptic non-injective homomorphism* $\sigma \in \mathrm{Hom}(E_0, E_0)$ *such that* $f = \sigma \circ \varphi$.

Proof. It is enough to check that the Stein factorization of a non-primitive pencil

$$f : V_6 \to E_0$$

factorizes through an elliptic curve $f = \sigma \circ \varphi$, where $\sigma \in \operatorname{Hom}(E_0, E_0)$ and φ is a primitive elliptic pencil.

In order to see this, one can use the cyclic order-six action μ_6 and one obtains a pencil

$$\widetilde{f} : V_6/\mu_6 \to E_0/\mu_6 = \mathbb{P}^1_{2,3,6},$$

where $\mathbb{P}^1_{2,3,6}$ is $\mathbb{P}^1$ with an orbifold structure 2, 3, 6. Using the Stein factorization on $\widetilde{f}$, we get

$$V_6/\mu_6 \stackrel{\widetilde{\varphi}}{\to} S \stackrel{\widetilde{\sigma}}{\to} \mathbb{P}^1_{2,3,6},$$

where $\widetilde{f} = \widetilde{\sigma} \circ \widetilde{\varphi}$. Since V_6/μ_6 is a rational surface, one obtains that $S = \mathbb{P}^1$ with an orbifold structure given by at least three orbifold points of orders 2, 3, and 6. The orbifold points of order 2 and 3 correspond to double and triple fibers, whereas the order 6 point corresponds to a non-reduced (but not multiple) fiber of type $h^6 F$ where F is the branch locus of the 6-fold cover of $\mathbb{P}^2$. By Proposition 2.8, pencils of curves cannot have more than two multiple members, hence $S = \mathbb{P}^1_{2,3,6}$. The 6-fold cover E_0 of S ramified with orders 2, 3, and 6 on the three orbifold points of S allows for the existence of a factorization of f, say $V_6 \stackrel{\varphi}{\to} E_0 \stackrel{\sigma}{\to} E_0$ induced by $\widetilde{\varphi}$ and $\widetilde{\sigma}$. Since $\widetilde{\varphi}$ is primitive, the map φ is also primitive. □

Remark 4.5. Note that the elliptic pencil obtained by $2\sigma_1$ should be non-primitive, since it factors through $E_0 \to E_0$, given by the degree 4 map $x \mapsto 2x$ (see Proposition 4.4). In fact, $2\sigma_1$ produces the following quasi-toric relation:

$$k_2^3 + k_3^2 + 64 f_3^6 F = 0,$$

where $k_2 := f_2(9f_2^3 + 20f_3^2)$, $k_3 := (27f_2^6 + 36f_2^3 f_3^2 + 8f_3^4)$ (from (4.5)), and

$$\alpha k_2^3 + \beta k_3^2 = H_{(\alpha,\beta)}(f_2^3, f_3^2)$$

is non-irreducible since

$$H_{(\alpha,\beta)}(x, y) = 1728(\beta - \alpha)x^2 y^2 + (576\alpha - 512\beta)xy^3 + 729(\beta - \alpha)x^4 + 64\beta y^4 + 1944(\beta - \alpha)x^3 y$$

which decomposes into a product of four factors of type $(y - \lambda(\alpha, \beta)x)$, since $H_{(\alpha,\beta)}(x, y)$ is a homogeneous polynomial of degree 4 in x, y.

Note that k_2^3 and k_3^2 are the union of four members (counted with multiplicity) of the pencil $\alpha f_2^3 + \beta f_3^2$. Also, note that $\alpha k_2^3 + \beta k_3^2$ is a non-primitive pencil whose generic member also consists of four members of $\alpha f_2^3 + \beta f_3^2$.

In particular, the four-to-one map $\widetilde{\sigma} : \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$\widetilde{\sigma}([x : y : z]) := [x(9x - 20y)^3 : (27x^2 - 36xy + 8y^2)^2]$$

is such that $\widetilde{f} = \widetilde{\sigma} \circ \widetilde{\varphi}$ is the Stein factorization of $\widetilde{f}$, where

$$\widetilde{\varphi}([x : y : z]) = [f_2^3(x, y, z) : -f_3^2(x, y, z)]$$

and

$$\widetilde{f}([x : y : z]) = [k_2^3(x, y, z) : k_3^2(x, y, z)].$$

Summarizing the previous results, one obtains:

Corollary 4.6. *Let $2q$ be the degree of the Alexander polynomial of F, then there exist q primitive quasi-toric relations $\sigma_1, \dots, \sigma_q$ of F such that*

$$\mathrm{MW}(W_F) = \mathbb{Z}[\omega_6]\sigma_1 \oplus \cdots \oplus \mathbb{Z}[\omega_6]\sigma_q,$$

where the action of $\mathbb{Z}[\omega_6]$ on σ_i is described in (4.3)*,* (4.4) *and* (4.5)*.*

Proof. By Proposition 4.1 there exist q quasi-toric relations $\sigma_1, \dots, \sigma_q$ of F such that $\mathrm{MW}(W_F) = \mathbb{Z}[\omega_6]\sigma_1 \oplus \cdots \oplus \mathbb{Z}[\omega_6]\sigma_q$. The only question left to be proved is whether or not the quasi-toric relations σ_i can be chosen to be primitive. By Proposition 4.4, if σ_i was not primitive, then $\sigma_i = \sigma\widetilde{\sigma}_i$ for a certain $\sigma \in \mathbb{Z}[\omega_6] = \mathrm{Hom}(E_0, E_0)$ and $\widetilde{\sigma}_i$ primitive. Since σ cannot be a unit, it follows that $\widetilde{\sigma}_i \notin \mathbb{Z}[\omega_6]\sigma_1 \oplus \cdots \oplus \mathbb{Z}[\omega_6]\sigma_q$, which contradicts our assumption. □

Theorem 4.7. *For any irreducible plane curve $\mathcal{C} = \{F = 0\}$ whose only singularities are nodes and cusps the following statements are equivalent:*

(1) *$\mathcal{C}$ admits a quasi-toric relation of elliptic type* $(2, 3, 6)$*,*

(2) *$\mathcal{C}$ admits an infinite number of quasi-toric relations of elliptic type* $(2, 3, 6)$*,*

(3) *$\Delta_{\mathcal{C}}(t)$ is not trivial, i.e. $\Delta_{\mathcal{C}}(t) \neq 1$.*

Moreover, the set of quasi-toric relations of $\mathcal{C}$,

$$\{(f, g, h) \in \mathbb{C}[x, y, z]^3 \mid f^2 + g^3 + h^6 F = 0\},$$

has a group structure and it is isomorphic to $\mathbb{Z}^{2q}$, where $\Delta_{\mathcal{C}}(t) = (t^2 - t + 1)^q$. Also, $\mathcal{C}$ admits a finite number of primitive quasi-toric relations iff $q = 1$.

Proof. For the first part, (1) ⇔ (2) is an immediate consequence of the group structure of the set of quasi-toric relations, namely, once a quasi-toric relation σ is given, the set $\mathbb{Z}[\omega_6]\sigma$ provides an infinite number of such relations. Also (1) ⇔ (3) is a consequence of Proposition 4.1.

For the *moreover* part, the group structure and its rank is a consequence of the discussion in the previous subsection (see Proposition 4.1). The result about the cardinality of primitive quasi-toric relation can be shown as follows. As a consequence of Corollary 4.6, any two sets of q independent (over $\mathbb{Z}[\omega_6]$) primitive quasi-toric relations differ by a matrix in $\mathrm{GL}_q(\mathbb{Z}[\omega_6])$. Note that $\mathrm{GL}_1(\mathbb{Z}[\omega_6]) = \mathbb{Z}[\omega_6]^*$ is a finite group whereas $\mathrm{GL}_q(\mathbb{Z}[\omega_6])$ is infinite for $q \geq 2$ since it contains the subgroup generated by

$$\left(\begin{array}{c|c} A_n & 0 \\ \hline 0 & \mathrm{I}_{q-2} \end{array}\right) \quad \text{where } A_n := \begin{pmatrix} 1 & n \\ 1 & n+1 \end{pmatrix}$$

and I_{q-2} is the identity matrix of rank $q - 2$. Hence the result follows. □

Remark 4.8. As Theorem 4.7 states, the set of quasi-toric relations of such curves, if non-empty, should be infinite. This corrects the statement in [29, Theorem 1 (ii)].

Also this result has been recently noticed by Kawashima–Yoshizaki in [25, Proposition 3]. In this paper, a quasi-toric relation σ_0 is considered to build an infinite number of quasi-toric relations σ_n. One can check that in fact, using the group structure described above, $\sigma_n = (\omega_6 + 1)^n \sigma_0$. The resulting quasi-toric relations σ_n are not primitive and the generic member of the pencil associated with σ_n is the product of 3^n members of the original pencil associated with σ_0, which incidentally is the degree of the morphism $E_0 \to E_0$, $x \mapsto (\omega_6+1)^n x$.

5. Alexander polynomials of δ-curves

In this section we extend the results of previous sections to singularities more general than nodes and cusps.

Let us consider the general situation described in Section 2.1, that is, fix $F \in \mathbb{C}[x, y, z]$, a homogeneous polynomial of degree d which is not a power, whose set of zeroes in $\mathbb{P}^2$ is the curve $\mathcal{C}$. By (2.2) this is equivalent to fixing a projective plane curve $\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_r$ and a list of multiplicities $(\varepsilon_1, \dots, \varepsilon_r)$ (such that $\gcd(e_i) = 1$) or a surjection

$$\varepsilon_F : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}) \to \mathbb{Z}$$

(where $\mathcal{C}_0$ is a line at infinity transversal to $\mathcal{C}$) such that $\varepsilon_F(\gamma_i) = \varepsilon_i$, γ_i a meridian around $\mathcal{C}_i$. Recall that ε_i corresponds to the multiplicity of the i-th irreducible component of F.

5.1. Definition of δ-curves and classification of δ-essential singularities with respect to ε with $\delta = 3, 4, 6$.

Definition 5.1. Let $(\mathcal{C}, P)$ be a germ of a singular point of $\mathcal{C}$. We call $(\mathcal{C}, P)$ a *δ-essential singularity* (resp. *δ-co-prime singularity*) w.r.t. ε if and only if the roots of $\Delta_{\mathcal{C},\varepsilon,P}(t)$ are all δ-roots of unity (resp. no root of $\Delta_{\mathcal{C},\varepsilon,P}(t)$ is a δ-root of unity except for $t = 1$).

We say that a curve $\mathcal{C} \subset \mathbb{C}^2$ *has only δ-essential singularities* if there exists an epimorphism $\varepsilon : H_1(\mathbb{C}^2 \setminus \mathcal{C}) \to \mathbb{Z}$ such that $(\mathcal{C}, P)$ is a δ-essential singularity w.r.t. ε_P for all $P \in \operatorname{Sing}(\mathcal{C})$ (see (2.3) to recall the construction of ε_P).

A curve $\mathcal{C}$ is called *δ-partial* w.r.t. a homomorphism ε if any singularity P of $\mathcal{C}$ is either δ-essential or δ-co-prime.

We also call a curve *δ-total* w.r.t. to a homomorphism ε if it is δ-partial and all the roots of the global Alexander polynomial $\Delta_{\mathcal{C},\varepsilon}(t)$ w.r.t. ε are roots of unity of degree δ (not necessarily primitive).

Remark 5.2. The curves whose only singularities are (reduced) nodes and cusps necessarily have 6-essential singularities. As another example one can consider $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$, where $\mathcal{C}_1$ is the tricuspidal quartic and $\mathcal{C}_0$ is its bitangent and the epimorphism ε mapping the meridian of $\mathcal{C}_0$ to 2 and the meridian of $\mathcal{C}_1$ to 1 (i.e. the homomorphism of type $(2, 1)$. The local Alexander polynomial of $\mathcal{C}$ at a tacnode w.r.t. ε is given by $(t^3 + 1)$, whereas at a cusp it is simply $t^2 - t + 1$. Therefore all singularities of $\mathcal{C}$ w.r.t. homomorphism of type $(2, 1)$ are 6-essential. In particular, by Proposition 2.2, $\mathcal{C}$ is a 6-total curve.

Also note that tacnodes w.r.t. homomorphism of type $(1, 1)$ are 4-essential singularities as well as nodes w.r.t. the homomorphism of type $(1, 2)$.

As for 3-essential singularities, one has $\mathbb{A}_5$-singularities w.r.t. ε of type $(1, 1)$ and nodes w.r.t. ε of type $(1, 3)$.[9)]

[9)] Here and below we use standard ADE-notations for germs of simple plane curve singularities. In particular, the germs of $\mathbb{A}_n$-singularities (resp. $\mathbb{D}_n$-singularities) are locally equivalent to $x^{n+1} + y^2$ (resp. $x^2 y + y^{n-1}$).

Proposition 5.3. *A curve with only δ-essential singularities is δ'-total for some $\delta' \mid \delta$.*

Proof. The result is an immediate consequence of Proposition 2.2. □

Note that the converse of Proposition 5.3 is not true (see Section 6).

Also, Proposition 5.3 can be sharpened using [14], so that not all singularities are required to be δ-essential.

Remark 5.4. Note that there are examples of δ-partial curves that are not δ-total, for instance a union of two cuspidal cubics intersecting each other at three smooth points with multiplicity of intersection 3, that is, a sextic $\mathcal{C}$ with singularities $2\mathbb{A}_2 + 3\mathbb{A}_5$[10)] has Alexander polynomial $\Delta_{\mathcal{C}}(t) = (t^2 - t + 1)^2(t^2 + t + 1)$ (cf. [44]). Therefore $\mathcal{C}$ is 3-partial but not 3-total. This curve will be studied in more detail in Example 6.6.

Proposition 5.5. *Let $(\mathcal{C}, P)$ be a germ of curve singularity and let $\pi : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ be a birational morphism such that the support of the total pull-back $\pi^*(\mathcal{C})$ is a normal crossing divisor on $\widetilde{\mathbb{C}}^2$. Let $m_{i,j}$ be the multiplicity of $\pi^* f_j$ where f_j is the local equation of the j-th branch of $\mathcal{C}$ at P along the exceptional curve E_i of π. Let ε_j be the value of ε on the meridian corresponding to f_j. Then the local Alexander polynomial of $\mathcal{C}$ at P w.r.t. ε is given by*

$$\Delta_{\mathcal{C},\varepsilon,P} = (t-1)\prod_i (1 - t^{\sum_j m_{i,j}\varepsilon_j})^{-\chi(E_i^0)}, \tag{5.1}$$

where $E_i^0 = E_i \setminus \bigcup_{k \mid k \neq i} E_k$ and χ is the topological Euler characteristic.

Proof. Indeed, the infinite cyclic cover of the complement to the zero set of the germ of $\mathcal{C}$ at P corresponding to the homomorphism ε can be identified with the Milnor fiber of non-reduced singularity $f_1^{\varepsilon_1} \cdots f_r^{\varepsilon_r}$. Now the claim follows from A'Campo's formula (cf. [1]). □

Remark 5.6. Proposition 5.5 allows one to compile a complete list of δ-essential singularities with $\delta \leq k$.

In Table 1 a list of the possible 6-essential singularities is given. The first column shows the number of local branches of the singularity. The second column contains the reduced type of the singularity. The third column shows a list of all possible multiplicities for the branches. It is a consequence of the divisibility conditions on the multiplicities of each branch imposed by Proposition 5.5 which follows from the requirement to have a 6-essential singularity; see the proof below.

The forth column gives the list of multiplicities (s_1, s_2, s_3, s_6) of the irreducible factors of the Alexander polynomial in $\mathbb{Q}[t]$

$$\Delta_{P,\varepsilon}(t) = (t-1)^{s_1}(t+1)^{s_2}(t^2+t+1)^{s_3}(t^2-t+1)^{s_6}.$$

Finally the fifth column shows that the set of logarithms belonging to the interval $(0, 1)$ of the eigenvalues of (the semi-simple part of) the monodromy acting on $\mathrm{Gr}_F^0 \mathrm{Gr}_1^W$ of the cohomology of the Milnor fiber.

10) That is, the set of singularities consists of two points of type $\mathbb{A}_2$ and three points of type $\mathbb{A}_5$. Similar notations will be used in the rest of the paper.

r	reduced singularity type	possible types of ε	Alexander polynomials (s_1, s_2, s_3, s_6)	weight 1 part of spectrum in $(0, 1)$
1	$\mathbb{A}_2 \equiv y^2 - x^3$	(1)	$(0, 0, 0, 1)$	$\frac{5}{6}$
2	$\mathbb{A}_1 \equiv y^2 - x^2$	$(1, 1)$	$(1, 0, 0, 0)$	$\emptyset$
	$\mathbb{A}_3 \equiv y^2 - x^4$	$(2, 1)$	$(2, 0, 0, 1)$	$\frac{5}{6}$
	$\mathbb{A}_5 \equiv y^2 - x^6$	$(1, 1)$	$(1, 0, 1, 1)$	$\frac{2}{3}, \frac{5}{6}$
3	$\mathbb{D}_4 \equiv y^3 - x^3$	$(4, 1, 1)$, $(3, 2, 1)$,	$(2, 1, 1, 1)$	$\frac{2}{3}, \frac{5}{6}, \frac{2}{3}$
		$(2, 2, 2)$, $(1, 1, 1)$	$(2, 0, 1, 0)$	
	$\mathbb{D}_6 \equiv (x^2 - y^4)y$	$(1, 1, 2)$	$(2, 1, 1, 1)$	$\frac{2}{3}, \frac{5}{6}$
	$(x^3 - y^6)$	$(1, 1, 1)$	$(2, 2, 1, 2)$	$\frac{2}{3}, \frac{5}{6}$
4	$(x^4 - y^4)$	$(3, 1, 1, 1)$, $(2, 2, 1, 1)$	$(3, 2, 2, 2)$	$\frac{1}{3}, \frac{2}{3}, \frac{5}{6}$
	$(x^2 - y^4)(y^2 - x^4)$	$(1, 1, 1, 1)$	$(3, 2, 2, 2)$	$\frac{1}{3}, \frac{2}{3}, \frac{5}{6}$
5	$x^5 - y^5$	$(2, 1, 1, 1, 1)$	$(4, 3, 3, 3)$	$\frac{1}{3}, \frac{2}{3}, \frac{5}{6}$
6	$x^6 - y^6$	$(1, 1, 1, 1, 1, 1)$	$(5, 4, 4, 4)$	$\frac{1}{3}, \frac{2}{3}, \frac{5}{6}$

Table 1. 6-essential singularities.

Proposition 5.7. *Table* 1 *contains a complete list of* 6*-essential singularities.*

Proof. We will use the notation from Proposition 5.5. The contributing exceptional divisors (i.e. those with $\chi(E^\circ) \neq 0$) appearing at the end of a resolution of the singularity $(\mathcal{C}, P)$ have maximal multiplicity $\sum m_{i,j}\varepsilon_j$ among the components preceding it. Hence the primitive root corresponding to the component with maximal multiplicity will not cancel in (5.1). Moreover, the multiplicities of the contributing exceptional divisors have to divide 6.

In the case $r = 1$ this forces the singularity to be a cusp. Also, for $r > 1$ this forces the irreducible branches to be smooth. The rest of the list can be worked out just keeping in mind that since $\sum_i m_{i,j} \leq 6$, in particular removing any branch from a valid singularity with $r + 1$ branches, one should obtain a valid singularity with r branches. Finally, if Σ_r is a δ-essential singularity type of r branches whose only valid homomorphism is $(1, \dots, 1)$, then there is no δ-essential singularity Σ_{r+1} of $(r + 1)$ branches such that Σ_r results from removing a branch from Σ_{r+1}.

For example, for reduced singularity $\mathbb{A}_3$, Proposition 5.5 yields that a collection of multiplicities of branches $(\varepsilon_1, \varepsilon_2)$ yields a 6-essential non-reduced singularity if and only if $2(\varepsilon_1 + e_2) \,|\, 6$. This is satisfied only by the pair $(\varepsilon_1, \varepsilon_2) = (2, 1)$. Similarly, for singularity $\mathbb{D}_4$ the divisibility condition is $\varepsilon_1 + e_2 + e_3 \,|\, 6$ which is satisfied by three triplets indicated in the table, etc.

Note that the possible morphisms ε shown in the third column are all up to action of the permutation group except for the case $(x^2 - y^4)y$, where the last branch is not interchangeable with the others. So, whereas $(3, 2, 1)$ in the $\mathbb{D}_4$ case represents six possible morphisms, $(1, 1, 2)$ in the $(x^2 - y^4)y$ case only represents one possible morphism.

In order to check the last column, it is enough to compute the constants of quasi-adjunction (cf. [33]) different from $\frac{1}{2}$, since k is a constant of quasi-adjunction different from $\frac{1}{2}$ if and only if $1-k$ is an element of the spectrum of the singularity corresponding to the part of weight 1 (cf. [40]).

Constants of quasi-adjunction can be found in terms of a resolution of the singularity as follows.

If $f_1^{a_1}(x,y)\dots f_r^{a_r}(x,y)$ is an equation of the germ of plane curve singularity at the origin, $\pi : V \to \mathbb{C}^2$ is its resolution (i.e. the proper preimage of the pull-back is a normal-crossing divisor), E_k are the exceptional components of the resolution π, N_k (resp. c_k, resp. $e_k(\phi)$) is the multiplicity along E_k of $\pi^*(f_1^{a_1}\cdots f_r^{a_r})$ (resp. $\pi^*(dx\wedge dy)$, resp. $\pi^*(\phi)$, where $\phi(x,y)$ is a germ of a function at the origin), then for a fixed ϕ, κ_ϕ is the minimal solution to the system of inequalities (the minimum is taken over k)

$$N_k\kappa_\phi \geq N_k - e_k(\phi) - c_k - 1 \tag{5.2}$$

(cf. [35, (2.3.8)]).

For example, for $\mathbb{A}_2$-singularity (i.e. the cusp which can be resolved by three blow ups) one has for the last exceptional component

$$e(x)=2,\quad e(y)=3,\quad N=e(x^2-y^3)=6,\quad c=4$$

and hence $\kappa=\frac{1}{6}$. Therefore for the spectrum (i.e. the last column in Table 1) one obtains the well-known constant $\frac{5}{6}$.

Similarly, for the $\mathbb{D}_4$-singularity $x^3-y^3=0$ (which is resolved by the single blow up) with the multiplicities of the components $(4,1,1)$ one obtains only the inequality

$$6\kappa \geq 6 - e_1(\phi) - 2. \tag{5.3}$$

Hence for various choices of ϕ one obtains $\kappa=\frac{1}{2},\frac{1}{3},\frac{1}{6}$. Thus $\frac{2}{3},\frac{5}{6}$ are the only elements of the spectrum in $(0,1)$ corresponding to the part of weight 1. □

In the following tables the notations are the same as in the Table 1 and in Table 2 the factorization of the Alexander polynomial is the following:

$$\Delta_{P,\varepsilon}(t)=(t-1)^{s_1}(t+1)^{s_2}(t^2+1)^{s_4}.$$

r	reduced singularity type	possible types of ε	Alexander polynomials (s_1,s_2,s_4)	weight 1 part of spectrum in $(0,1)$
2	$\mathbb{A}_1\equiv y^2-x^2$	$(\varepsilon_1,\varepsilon_2)$	$(1,0,0)$	$\emptyset$
	$\mathbb{A}_3\equiv y^2-x^4$	$(1,1)$	$(1,0,1)$	$\frac{3}{4}$
3	$\mathbb{D}_4\equiv y^3-x^3$	$(2,1,1)$	$(2,1,1)$	$\frac{3}{4}$
4	(x^4-y^4)	$(1,1,1,1)$	$(3,2,2)$	$\frac{3}{4}$

Table 2. 4-essential singularities.

r	reduced singularity type	possible types of ε	Alexander polynomials (s_1, s_3)	weight 1 part of spectrum in $(0,1)$
2	$\mathbb{A}_1 \equiv y^2 - x^2$	$(\varepsilon_1, \varepsilon_2)$	$(1,0)$	$\emptyset$
3	$\mathbb{D}_4 \equiv y^3 - x^3$	$(1,1,1)$	$(2,1)$	$\frac{2}{3}$

Table 3. 3-essential singularities.

Proposition 5.8. *Table* 2 *(resp.* 3*) classifies the* 4*-essential (resp.* 3*-essential) singularities.*

The proof is same as the proof of Proposition 5.7.

Corollary 5.9. *For simple δ-essential singularities (i.e. of type $\mathbb{A}$ or $\mathbb{D}$) there is at most one eigenvalue of order $\delta' \mid \delta$ $(\delta' > 2)$ for the action of the monodromy on the Hodge component* $\mathrm{Gr}_F^0 \mathrm{Gr}_1^W$ *of the Milnor fiber.*

Proof. Inspection on Tables 1–3 shows that except for $t = 1$, the multiplicity of each root of the characteristic polynomial of the monodromy is equal to one. Since the conjugation of an eigenvalue of the monodromy acting on $\mathrm{Gr}_F^0 \mathrm{Gr}_1^W$ is an eigenvalue of its action on $\mathrm{Gr}_F^1 \mathrm{Gr}_1^W$, it follows that $t = -1$ is not an eigenvalue of the monodromy action on $\mathrm{Gr}_F^0 \mathrm{Gr}_1^W$. Again, by direct inspection of Tables 1–3 the result follows. □

5.2. Decomposition of the Albanese variety of cyclic covers branched over δ-total curves. The following auxiliary results will be useful in the rest of the arguments.

Lemma 5.10. *Let $\mathcal{C}$ be a δ-partial curve with r irreducible components. Then the polynomial*

$$\Delta^{\delta}_{\mathcal{C},\varepsilon} := (t-1)^{r-1} \prod_{k \mid \delta, k>1} \varphi_k(t)^{s_k}, \tag{5.4}$$

divides the Alexander polynomial $\Delta_{\mathcal{C},\varepsilon}$ of $\mathcal{C}$ w.r.t. ε, where s_i above denotes the multiplicity of the primitive root of unity of degree i in the Alexander polynomial of $\mathcal{C}$.

Moreover, let V_δ be the cyclic cover of $\mathbb{P}^2$ of degree δ branched over a curve $\mathcal{C}$ according to the multiplicities $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$. Then, the characteristic polynomial of the deck transformation acting on $H_1(V_\delta, \mathbb{C})$ equals

$$\frac{\Delta^{\delta}_{\mathcal{C},\varepsilon}}{(t-1)^{r-1}}.$$

If $\mathcal{C}$ is δ-total, then

$$\Delta^{\delta}_{\mathcal{C},\varepsilon} = \Delta_{\mathcal{C},\varepsilon}.$$

Proof. The first (resp. last) assertion about the Alexander polynomial follows directly from the definitions of δ-partial (resp. δ-total) curves and the well-known fact that the multiplicity of the root $t = 1$ for a curve with r irreducible components in the Alexander polynomial is equal to $r - 1$.

The *moreover* part follows from the relation between the homology of branched and unbranched covers (cf. [32]). □

Lemma 5.11. *For a plane curve singularity P, denote by* Alb_P *the local Jacobian (cf. Section* 3.1*). Let $\mathcal{C}$ be a δ-partial curve and let V_δ be cyclic cover of degree δ of $\mathbb{P}^2$ branched over $\mathcal{C}$. Then* $\mathrm{Alb}(V_\delta)$ *is isogenous to a quotient of the product of local Jacobians of singularities of $\mathcal{C}$.*

Proof. It follows from the proof of Lemma 3.6. □

Finally, we will give a description of the multiplicities s_i in terms of the Albanese variety of the ramified coverings of $\mathbb{P}^2$. Denote by E_0 (resp. E_{1728}) the elliptic curve with j-invariant zero (resp. 1728). Then one has the following.

Theorem 5.12. *Let $\mathcal{C}$ be a δ-partial curve ($\delta = 3, 4$ or 6) with δ-essential singularities of type $\mathbb{A}$ and $\mathbb{D}$. Then the Albanese variety* $\mathrm{Alb}(V_\delta)$ *corresponding to the curve $\mathcal{C}$ can be decomposed as follows:*

$$\begin{aligned} &\mathrm{Alb}(V_\delta) = E_0^{s_3} && \text{if } \delta = 3,\\ &\mathrm{Alb}(V_\delta) \sim A \times E_{1728}^{s_4} && \text{if } \delta = 4,\\ &\mathrm{Alb}(V_\delta) \sim A \times E_0^{s_3+s_6} && \text{if } \delta = 6, \end{aligned} \tag{5.5}$$

where s_i is the multiplicity of the i-th primitive root of unity in $\Delta_{\mathcal{C},\varepsilon}(t)$, A is an abelian variety of dimension s_2, "$=$" means isomorphic, and "$\sim$" means isogenous.

Proof. The deck transformation of V_δ induces the action of the cyclic group $\langle\alpha\rangle$ on the Albanese variety of V_δ. It follows from Roan's Decomposition Theorem (cf. [9, Theorem 2.1]) that $\mathrm{Alb}(V_\delta)$ is isogenous to a product $X_1 \times \cdots \times X_e$ where e is the number of orders of eigenvalues of α. We shall show that the abelian varieties X_i can be decomposed further to yields isogenies (5.5). By Lemma 5.10, one has that 1 is not an eigenvalue of α, and hence each component X_i supports an automorphism whose order is a non-trivial divisor of δ. We denote by A the component supporting an automorphism of order 2. In the cases of $\mathbb{A}$ or $\mathbb{D}$ singularities, Corollary 5.9 yields that the action of α on each component X_i has at most one eigenvalue. Indeed this is the eigenvalue of the monodromy on the component $\mathrm{Gr}_F^0\mathrm{Gr}_1^W$ of the Milnor fiber of the singularities whose local Alexander polynomial contains such an eigenvalue as a root. Each eigenvalue of the monodromy on $\mathrm{Gr}_F^0\mathrm{Gr}_1^W$ for a δ-essential singularity does not appear on $\mathrm{Gr}_F^1\mathrm{Gr}_1^W$ even if for different δ-essential singularities. In particular, even different δ-essential singularities contribute at most to the same root of the global Alexander polynomial as follows from the table in Corollary 5.9. This allows one to apply the second Decomposition Theorem (cf. [9, Theorem 3.2]) as was done above in the case of cuspidal curves. The result follows. □

Remark 5.13. (1) If $\mathcal{C}$ has singularities of type $x^3 - y^6$, then the above argument shows that $\mathrm{Alb}(V_6)$ has, up to isogeny, the factors which are the Jacobians of the projection model of $y^3 = x^6 - 1$ and the elliptic curves of E_{1728} and E_0.

(2) Whenever the number of local branches of a curve at singular point is greater than 3, the local Jacobians (cf. Section 3.1) depend on the moduli describing the local algebraic analytic type of singularities. For example, in the case of singularities $x^4 - y^4$ the local

Jacobian is the Jacobian of the 4-fold cyclic cover branched over four points. The quotient by an involution of such a cover may yield an arbitrary elliptic curve in the Legendre family of elliptic curves. Hence, the Albanese variety of the cyclic cover V_δ is isogenous to the quotient of abelian varieties whose moduli a priori depends on the analytic type of the singularities. It would be interesting to see if one can find examples showing that such variations can take place.

5.3. Elliptic threefolds corresponding to δ-curves. Now we shall relate the Mordell–Weil group of threefolds corresponding to $\mathcal{C}$ to $\Delta_{\mathcal{C},\varepsilon}$. Let W denote an elliptic threefold birational to the affine hypersurface given by the equation

$$\begin{aligned} u^2 + v^3 &= F(x, y, 1) && \text{if } \delta = 3, 6, \\ u^2 + v^3 &= F(x, y, 1)v && \text{if } \delta = 4. \end{aligned} \tag{5.6}$$

One has the following:

Theorem 5.14. *Let $\mathcal{C} = \{F = 0\}$ be a δ-partial curve as in Theorem* 5.12*. Then*

$$\begin{aligned} &\operatorname{rk} \mathrm{MW}(W_F) = s_3 && \textit{if } \delta = 3, \\ &\operatorname{rk} \mathrm{MW}(W_F) \geq s_4 && \textit{if } \delta = 4, \\ &\operatorname{rk} \mathrm{MW}(W_F) \geq s_3 + s_6 && \textit{if } \delta = 6. \end{aligned} \tag{5.7}$$

In addition, if $\mathcal{C}$ is a δ-total curve, then

$$\begin{aligned} &\operatorname{rk} \mathrm{MW}(W_F) = s_4 && \textit{if } \delta = 4 \textit{ and } \Delta_{\mathcal{C},\varepsilon}(-1) \neq 0, \\ &\operatorname{rk} \mathrm{MW}(W_F) = s_3 + s_6 && \textit{if } \delta = 6 \textit{ and } \Delta_{\mathcal{C},\varepsilon}(-1) \neq 0. \end{aligned}$$

Proof. As in the proof of Theorem 3.10 one has the identification of $\mathrm{MW}(W_F)$ and μ_i-invariant elements of the Mordell–Weil group of W_F over the extension $\mu_i(\mathbb{C}(x, y))$ of $\mathbb{C}(x, y)$ with the Galois group μ_i, where $i = 6$ for $\delta = 3, 6$ and $i = 4$ for $\delta = 4$. The threefold

$$u^2 + v^3 = F(x, y, 1)v$$

splits over the field $\mathbb{C}(x, y)(F^{\frac{1}{4}})$ since it is isomorphic to a direct product threefold using

$$v' = vF^{\frac{1}{2}}, \quad u' = F^{\frac{3}{4}}u.$$

The inequalities follow from

$$\operatorname{rk} \mathrm{MW}(W_F, \mathbb{C}(x, y)(F^{\frac{1}{\delta}})) \geq \operatorname{rk} \mathrm{Hom}(E_k^s, E_k),$$

where one has $k = 0, s = s_3$ for $\delta = 3$, $k = 0, s = s_3 + s_6$ for $\delta = 6$ and $k = 1728$, $j = s_4$ for $\delta = 4$. □

5.4. Statement and proof of the main theorem for δ-curves.

Theorem 5.15. *Let $\mathcal{C} = \{F = 0\}$ be a δ-partial curve for $\delta = 3, 4$ or 6. Then*

(1) *There is a one-to-one correspondence between quasi-toric relations corresponding to F and the $\mathbb{C}(x, y)$-points of the threefold W_F.*

If, in addition, the singularities of $\mathcal{C}$ *are as in Theorem* 5.12*, then:*

(2) *The multiplicities of the factors of the global Alexander polynomial of* δ*-partial curves satisfy the following inequalities:*

$$
\begin{aligned}
s_3 &\le \frac{5}{6}d - 1 \quad \textit{if } \delta = 3,\\
s_4 &\le \frac{5}{6}d - 1 \quad \textit{if } \delta = 4,\\
s_3 + s_6 &\le \frac{5}{6}d - 1 \quad \textit{if } \delta = 6.
\end{aligned}
$$

Proof. The argument is the same as in the proof of Theorem 3.12 since the rank of the Mordell–Weil group of the threefolds for each δ is bounded by the rank of Mordell–Weil group of the elliptic surface with d degenerate fibers each isomorphic to a cubic curve with a single cusp as follows from equations (5.6). For each δ we obtain a bound on the rank of the submodule of the Alexander module corresponding to the action of the deck transformation on the subspaces generated by the eigenvalues which are roots of unity of degrees 3, 4 and 6. □

As an immediate consequence of Theorem 5.15 (1) one has the following generalization of Theorem 4.7:

Theorem 5.16. *Let* $\mathcal{C} = \{F = 0\}$ *be a curve. Then the following statements are equivalent:*

(1) $\mathcal{C}$ *admits a quasi-toric relation of elliptic type* $(3, 3, 3)$ *(resp.* $(2, 4, 4)$ *or* $(2, 3, 6)$*),*

(2) $\mathcal{C}$ *admits an infinite number of quasi-toric relations of elliptic type* $(3, 3, 3)$ *(resp.* $(2, 4, 4)$ *or* $(2, 3, 6)$*).*

Also, (1) *and* (2) *above imply*

(3) $\mathcal{C}$ *is a* δ*-partial curve* $(\varphi_\delta(t) \mid \Delta_{\mathcal{C},\varepsilon}(t))$ *for* $\delta = 3$ *(resp.* 4 *or* 6*).*

Moreover, if the singularities of $\mathcal{C}$ *are as in Theorem* 5.12*, then* (1)*,* (2)*, and* (3) *are equivalent.*

Proof. First of all note that this result generalizes Theorem 4.7 since curves with only nodes and cusps as singularities and non-trivial Alexander polynomial automatically satisfy that $\Delta_{\mathcal{C},\varepsilon}(\omega_6) = 0$ for a primitive 6th-root of unity, and hence they are δ-partial curves (in fact, they are δ-total).

Statement (1) ⇔ (2) follows from the group structure of quasi-toric relations as exhibited in Theorem 5.15 (1). Also (1) ⇒ (3) is immediate since a quasi-toric relation of F induces an equivariant map $V_\delta \to E_\delta$, which induces a map $\mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1_{\overline{m}}$, where $\overline{m} = (3, 3, 3), (2, 4, 4)$, resp. $(2, 3, 6)$ according to $\delta = 3, 4$, resp. 6. This implies that $\Delta_{\mathcal{C},\varepsilon}(t) \neq 0$, where ε is defined as in (2.8).

Finally, (3) ⇒ (1) under the conditions of Theorem 5.12 is an immediate consequence of Theorem 5.14 since (3) implies $\operatorname{rk} \mathrm{MW}(W_F) > 0$. □

Remark 5.17. Note that (3) ⇒ (1) in Theorem 5.16 is not true for general singularities as the following example shows. Consider $\mathcal{C}$ the curve given by the product of six concurrent lines. Generically, $\mathcal{C}$ does not satisfy a quasi-toric relation of type $(2,3,6)$ or $(3,3,3)$ (cf. Remark 5.21), but its Alexander polynomial is not trivial, namely,

$$\Delta_{\mathcal{C}}(t) = (t-1)^5(t+1)^4(t^2+t+1)^4(t^2-t+1)^4.$$

5.5. Applications. Theorem 5.14 has implications for the structure of the characteristic variety $\Sigma(\mathcal{C})$ of a plane curve $\mathcal{C}$. Characteristic varieties of curves extend the notion of Alexander polynomials of curves to non-irreducible curves (for a definition see [35]). In [3, Theorem 1.6] (resp. [7, Theorem 1]), structure theorems for the irreducible components of $\Sigma(\mathcal{C})$ are given in terms of the existence of maps from $\mathbb{P}^2 \setminus \mathcal{C}$ onto Riemann surfaces (resp. orbifold surfaces). The following result sharpens [7, Theorem 1] for the special case of torsion points of order $\delta = 3, 4, 6$ on $\Sigma(\mathcal{C})$.

Corollary 5.18. *Let $\mathcal{C}$ be a curve whose singularities are as in Theorem* 5.12*, consider $X = \mathbb{P}^2 \setminus \mathcal{C}$, and consider $\Sigma_1(X)$ the first characteristic variety of X. For any $\rho \in \Sigma_1(X)$ torsion point ρ of order δ of $\Sigma_1(X)$ there exists an admissible orbifold map $f : \mathbb{P}^2 \to S_{\overline{m}}$ such that $\rho \in f^*\Sigma_1(S_{\overline{m}})$.*

Proof. Let $\rho = (\omega_\delta^{\varepsilon_1}, \dots, \omega_\delta^{\varepsilon_r}) \in \Sigma_1(X)$ be a torsion point of order δ in $\Sigma_1(X)$. Note that the homomorphism induced by $\varepsilon := (\varepsilon_1, \dots, \varepsilon_r)$ is such that the cyclotomic polynomial $\varphi_\delta(t)$ of the δ-roots of unity divides $\Delta_{\mathcal{C}_e,\varepsilon}(t)$. Hence the hypotheses of Theorem 5.16 (3) are satisfied and therefore there exists an orbifold Riemann surface $S_{\overline{m}}$ such that $\varphi_\delta(t)$ divides $\Delta_{\pi_1^{\text{orb}}(S_{\overline{m}})}(t)$ and a dominant orbifold morphism $f : X \to S_{\overline{m}}$. After performing a Stein factorization, we may assume that the induced homomorphism $f : X \to S_{\overline{m}}$ is surjective. Finally note that any such surjection induces an inclusion of characteristic varieties, that is,

$$f^*\Sigma_1(S_{\overline{m}}) \subset \Sigma_1(X).$$

Moreover, since $\varphi_\delta(t)$ divides $\Delta_{\pi_1^{\text{orb}}(S_{\overline{m}})}(t)$, it follows that $\rho \in f^*\Sigma_1(S_{\overline{m}})$. □

Remark 5.19. Theorem 5.16 cannot be generalized to δ-essential curves outside of the range $\delta = 3, 4, 6$. For instance, note that in [5] an example of an irreducible affine quintic $\mathcal{C}_5 \subset \mathbb{C}^2$ with $2\mathbb{A}_4$ as affine singularities is shown. The curve $\mathcal{C}_5$ is 10-total since

$$\Delta_{\mathcal{C}_5}(t) = t^5 - t^4 + t^3 - t^2 + t - 1.$$

However, there is no quasi-toric decomposition of C_5 of type $(2,5)$. According to [29, Theorem 1] this implies that the Albanese dimension of $\mathcal{C}_5$ is two and moreover, the Albanese image of any cyclic covering of $\mathbb{P}^2$ ramified along $\mathcal{C}_5$ must have either dimension zero or two.

Remark 5.20. Since no irreducible sextic has an Alexander polynomial of the form $\Delta_{\mathcal{C}}(t) = (t+1)^q$ (cf. [13]), Theorem 5.16 is another way to show that any irreducible sextic (with simple singularities) has a non-trivial Alexander polynomial if and only if it is of torus type (note that an irreducible sextic admits a quasi-toric decomposition if and only if it is of torus type as shown in Example 2.17). This is a weaker version of what is now known as *Oka's conjecture* (cf. [15, 16, 22]).

Remark 5.21. Consider the case when $\mathcal{C}$ is union of 6 concurrent lines. To these 6 lines $\ell_i = 0$ correspond 6 points L_i in $\mathbb{P}^1$ parameterizing lines in the pencil containing ℓ_i. The 3-fold cover $V_{\ell_i} : z^3 - \prod \ell_i$ has as compactification surface in the weighted projective space $\mathbb{P}(2,1,1,1)$, which after a weighted blow-up has a map onto the 3-fold cyclic cover C_{L_i} of $\mathbb{P}^1$ branched over the points L_i and having rational curves are fibers. Hence $\mathrm{Alb}(V_{\ell_i})$ is isomorphic to the Jacobian of C_{L_i}. If the L_i form a collection of roots of a polynomial over $\mathbb{Q}$ having as the Galois group over the later the symmetric (or alternating) group, then the results of [56] show that the above Jacobian is simple and hence does not have maps onto the elliptic curve E_0. On the other hand the Jacobian of the 3-fold cover of $\mathbb{P}^1$ branched over the roots of the polynomial $z^3 = x^6 - 1$ has E_0 as a factor, since E_0 is a quotient of the former by the involution. Note that the characteristic polynomial of the deck transformation on $H_1(V_{\ell_i})$ is $(t^2+t+1)^4(t-1)^5$.

Remark 5.22. Note that Oka's conjecture cannot be generalized to general reducible sextics as shown in Remark 5.17.

However, in light of Theorem 5.16 it seems that one can ask the following version of Oka's conjecture to non-reduced curves, namely:

Question 5.23. Is it true that a (possibly non-reduced) sextic $\mathcal{C} = \{F = 0\}$ (such that $\deg F = 6$) whose singularities are as in Theorem 5.12 has a non-trivial Alexander polynomial if and only if it admits a quasi-toric relation?

According to Theorem 5.16, one has the following rewriting of Question 5.23.

Proposition 5.24. *The answer to Question* 5.23 *is affirmative if and only if*

$$\Delta_{\mathcal{C},\varepsilon}(t) \neq (t-1)^{r-1}(t+1)^q$$

for any non-reduced curve $\mathcal{C}$ *of total degree* 6 *whose singularities are as in Theorem* 5.14.

6. Examples

The purpose of this section is to exhibit the different examples of elliptic quasi-toric relations. In Example 6.1 (resp. 6.2) we present quasi-toric relations of type $(2,3,6)$ and describe generators for the Mordell–Weil group of elliptic sections, which is of rank 3 (resp. 2). In Example 6.3 we present a cuspidal curve whose Alexander polynomial has the largest degree known to our knowledge. Finally, Examples 6.4 (resp. 6.5) show examples of quasi-toric relations of type $(2,4,4)$ (resp. $(3,3,3)$) and Example 6.6 shows an example of a curve with both $(2,3,6)$ and $(3,3,3)$ quasi-toric relations.

Example 6.1. Consider the sextic curve $\mathcal{C}_{6,9}$ with nine cusps. One easy way to obtain equations is as the preimage of the conic $\mathcal{C}_2 := \{x^2+y^2+z^2-2(xy+xz+yz)=0\}$ by the Kummer abelian cover $[x:y:z] \mapsto [x^3:y^3:z^3]$. It is well known that the Alexander polynomial of $\mathcal{C}_{6,9}$ is $\Delta_{\mathcal{C}_{6,9}}(t) = (t^2-t+1)^3$. Note that $\mathcal{C}_2$ belongs to the following pencils:

$$C_2 = (x+y-z)^2 + 4xz,$$
$$C_2 = (x+z-y)^2 + 4yx,$$
$$C_2 = (y+z+x)^2 - 4yz.$$

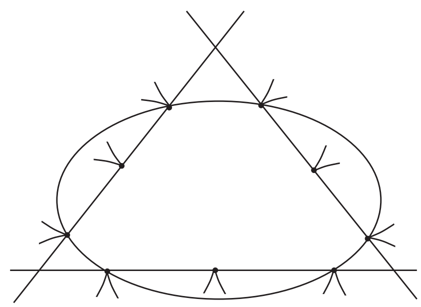
Therefore $\mathcal{C}_6$ has the following three quasi-toric relations:

$$\sigma_0 \equiv C_{6,9} = (x^3 + y^3 - z^3)^2 + 4(xz)^3,$$
$$\sigma_1 \equiv C_{6,9} = (x^3 + z^3 - y^3)^2 + 4(yx)^3,$$
$$\sigma_2 \equiv C_{6,9} = (y^3 + z^3 + x^3)^2 - 4(yz)^3.$$

In addition, note that

$$\sigma_3 \equiv C_{6,9} = 4(x^2 + y^2 + z^2 + xz + yz + xy)^3 - 3(x^3 + y^3 + z^3 + 2(xz^2 + x^2z + yz^2 + xyz + x^2y + y^2z + xy^2))^2$$

leads to another quasi-toric relation.



If we consider $\sigma_i = (g_{2,i}, g_{3,i})$ as elements of the elliptic curve

$$E_0 = \{u^3 + v^2 = F(x, y)\}$$

over $\mathbb{C}(x, y)$, then $\mathbb{Z}[\omega_6]\sigma_1 \oplus \mathbb{Z}[\omega_6]\sigma_2 \oplus \mathbb{Z}[\omega_6]\sigma_3$ is the group of all quasi-toric relations of $\mathcal{C}_{6,9}$. For instance,

$$\sigma_0 = -\sigma_1 - \sigma_2 + (2\omega_6 - 1)\sigma_3 = \sigma_{[1,1,1-2\omega_6]}. \tag{6.1}$$

One can obtain the above relation using (4.3), (4.5), and (4.4).

This example was first considered in this context by Tokunaga [54, Theorem 0.2]. Note that the author exhibits 12 primitive quasi-toric relations of $\mathcal{C}_{6,9}$ such that $h = 1$. Whether or not those decompositions are the only ones satisfying $h = 1$ remains open.

Example 6.2. Consider the tricuspidal quartic

$$\mathcal{C}_{4,3} := \{C_{4,3} = x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x + y + z) = 0\}.$$

Since $\mathcal{C}_{4,3}$ is the dual of a nodal cubic, it should contain a bitangent, which is the dual of the node. In our case one can check that $\ell_0 := \{L_0 = x + y + z = 0\}$ is the bitangent at the points $P := [1 : \omega_3 : \omega_3^2]$ and $Q := [1 : \omega_3^2 : \omega_3]$, where ω_3 is a root of $t^2 + t + 1$.

We shall note that $C_{4,3}L_0^2$ is a non-reduced sextic whose singularities are 6-essential (see Remark 5.2). Its twisted Alexander polynomial w.r.t. the multiplicities $(1, 2)$ is given by $(t^2 - t + 1)^2$.

By Theorem 5.14, the group of quasi-toric relations has $\mathbb{Z}[\omega_6]$-rank one. In fact, it is generated by the following:

$$\sigma_1 \equiv C_{4,3}L_0^2 = 4C_2^3 + C_3^2,$$
$$\sigma_2 \equiv C_{4,3}L_0^2 = 4\widetilde{C}_2^3 + \widetilde{C}_3^2,$$

where

$$\begin{aligned} C_2 &:= zx + \omega_3 yz - (1+\omega_3)xy, \\ C_3 &:= (x^2y - x^2z - y^2x - 3(1+2\omega_3)xyz + y^2z + z^2x - yz^2), \\ \widetilde{C}_2(x,y,z) &:= C_2(x,z,y), \\ \widetilde{C}_3(x,y,z) &:= C_3(x,z,y). \end{aligned}$$

In addition we show some interesting primitive quasi-toric relations coming from the combinations of the generators

$$\sigma_{[\omega_6^i,\omega_6^j]} := \omega_6^i \sigma_1 + \omega_6^j \sigma_2.$$

Such primitive quasi-toric relations have the following form:

$$\begin{aligned} \sigma_{[1,1]} &\equiv C_{4,3} L_0^2 x^6 = 4C_{4,[1,1]}^3 + C_{6,[1,1]}^2, \\ \sigma_{[1,\omega_6]} &\equiv C_{4,3} L_0^2 (x-z)^6 = 4C_{4,[1,\omega_6]}^3 + C_{6,[1,\omega_6]}^2, \\ \sigma_{[1,\omega_6^2]} &\equiv C_{4,3} L_0^2 z^6 = 4C_{4,[1,\omega_6^2]}^3 + C_{6,[1,\omega_6^2]}^2, \\ \sigma_{[1,\omega_6^3]} &\equiv C_{4,3} L_0^2 (z-y)^6 = 4C_{4,[1,\omega_6^3]}^3 + C_{6,[1,\omega_6^3]}^2, \\ \sigma_{[1,\omega_6^4]} &\equiv C_{4,3} L_0^2 y^6 = 4C_{4,[1,\omega_6^4]}^3 + C_{6,[1,\omega_6^4]}^2, \\ \sigma_{[1,\omega_6^5]} &\equiv C_{4,3} L_0^2 (x-y)^6 = 4C_{4,[1,\omega_6^5]}^3 + C_{6,[1,\omega_6^5]}^2, \end{aligned}$$

where $C_{k,[\omega_6^i,\omega_6^j]}$ denotes a homogeneous polynomial of degree k corresponding to the quasi-toric relation $\sigma_{[\omega_6^i,\omega_6^j]}$. For simplicity we only show the first pair of polynomials:

$$\begin{aligned} C_{4,[1,1]} &= \frac{\omega_6}{3}(x^4 + z^2y^2 + x^2z^2 - 2xyz^2 - 2xy^2z + x^3z - 2x^2yz + x^3y + x^2y^2), \\ C_{6,[1,1]} &= (x^4 + x^3y + x^3z - 2x^2z^2 + 4x^2yz - 2x^2y^2 + 4xy^2z + 4xyz^2 - 2z^2y^2) \\ &\quad \times (2x^2 + xy - yz + zx). \end{aligned}$$

The equations above can easily be obtained using (4.3), (4.5) and (4.4).

Example 6.3. Consider the tricuspidal quartic

$$\mathcal{C}_{4,3} := \{x^2y^2 + z^2y^2 + x^2z^2 - 2xyz(x+y+z)\}$$

as above, the bitangent

$$\ell_0 := \{l_0 = x + y + z = 0\},$$

and two tangent lines, say

$$\begin{aligned} \ell_1 &:= \{l_1 = -8x + y + z = 0\} \quad (\text{at } R = [1,4,4]), \\ \ell_2 &:= \{l_2 = x - 8y + z = 0\} \quad\quad (\text{at } S = [4,1,4]). \end{aligned}$$

The Kummer cover $[x:y:z] \mapsto [l_0^3 : l_1^3 : l_2^3]$ produces a curve $\mathcal{C}_{12,39}$ of degree 12 with 39 cusps.

In order to obtain equations for $\mathcal{C}_{12,39}$, one can proceed as follows. Consider the projective isomorphism

$$\psi([x:y:z]) = [l_0 : l_1 : l_2] = [u:v:w], \tag{6.2}$$

which corresponds to the change of coordinates (given by ψ^{-1})

$$\begin{cases} x = u + v, \\ y = u + w, \\ z = 7u - v - w. \end{cases} \tag{6.3}$$

Without explicitly mentioning ψ its use will be clear from using coordinates $[x : y : z]$ or $[u : v : w]$.

After the change of coordinates (6.3), the quartic $\mathcal{C}_{4,3}$ can be given by the following equation in terms of u, v, w:

$$\begin{aligned} f(u,v,w) := {} & 12uvw^2 - 138u^2vw + 12v^2uw + 2v^3w + w^4 + v^4 - 36u^3w + 42u^2w^2 \\ & + 3v^2w^2 - 27u^4 - 12uw^3 + 2vw^3 + 42v^2u^2 - 36u^3v - 12uv^3 \end{aligned}$$

and hence the lines ℓ_0, ℓ_1, and ℓ_2 are defined by the equations $u = l_0 = 0$, $v = l_1 = 0$, and $w = l_2 = 0$ respectively (see (6.2)).

Therefore, the map

$$[x : y : z] \mapsto [l_0^3 : l_1^3 : l_2^3]$$

can also be described as

$$[x : y : z] \mapsto [u^3 : v^3 : w^3].$$

Note that $u = 0$ is bitangent since

$$f(0, v, w) = (w^2 + vw + v^2)^2.$$

Also note that $v = 0$ is a simple tangent at $[1 : 0 : 3] = \psi(R)$ since

$$f(u, 0, w) = -(3u^2 + 6uw - w^2)(w - 3u)^2.$$

Analogously, $w = 0$ is a simple tangent at $[1 : 3 : 0] = \psi(S)$ since

$$f(u, v, 0) = (v^2 - 6uv - 3u^2)(v - 3u)^2.$$

Hence

$$\mathcal{C}_{12,39} := \{f(u^3, v^3, w^3) = f((x + y + z)^3, (-8x + y + z)^3, (x - 8y + z)^3) = 0\}$$

is a curve of degree 12. Moreover, since the degree of π is 9 and the preimage of each of the four tangencies of $\mathcal{C}_{4,3}$ with ℓ_i is three cusps, we can conclude that $\mathcal{C}_{12,39}$ contains exactly $39 = 3 \cdot 9 + 4 \cdot 3$ cusps and no other singularities. Here is an equation for $\mathcal{C}_{12,39}$:

$$\begin{aligned} f_{12}(u,v,w) := {} & 27u^{12} - v^{12} - w^{12} - 12u^3v^3w^6 + 36u^9w^3 - 42u^6w^6 + 36u^9v^3 \\ & - 42u^6v^6 - 3v^6w^6 + 12u^3w^9 - 2v^3w^9 + 12u^3v^9 - 2v^9w^3 \\ & + 138u^6v^3w^3 - 12u^3v^6w^3. \end{aligned}$$

We compute the superabundance of $\mathcal{C}_{12,39}$ as follows. Let $\mathcal{J}$ be the ideal sheaf supported on the 39 cusps and such that $\mathcal{J}_\kappa = \mathfrak{m}$ the maximal ideal at any κ cusp of $\mathcal{C}_{12,39}$. The superabundance of $\mathcal{C}_{12,39}$ is $h^1(\mathcal{J}(7))$ and it coincides with the multiplicity of the 6-th root of unity as a root of the Alexander polynomial $\Delta_{\mathcal{C}_{12,39}}(t)$ of $\mathcal{C}_{12,39}$.

Note that

$$\chi(\mathcal{O}(7)) - \chi(\mathcal{O}/\mathcal{J}) = \chi(\mathcal{J}(7)) = h^1(\mathcal{J}(7)) - h^0(\mathcal{J}(7)).$$

Since $\chi(\mathcal{O}(7)) = \binom{7+2}{2} = 36$, $\chi(\mathcal{O}/\mathcal{J}) = \#\kappa = 39$, and $h^0(\mathcal{J}(7)) = 1$, one has that

$$h^1(\mathcal{J}(7)) = 4.$$

Note that $h^0(\mathcal{J}(7)) = 1$ since the only curve of degree 7 passing through all cusps is $z\widetilde{f}_2$, where $\widetilde{f}_2$ is the preimage by the Kummer map of the conic passing through the three cusps, R, and S.

Therefore, $\Delta_{\mathcal{C}_{12,39}}(t) = (t^2 - t + 1)^4$. This is, to our knowledge, the first example of a cuspidal curve whose Alexander polynomial has a non-trivial root with multiplicity greater than 3. As was discussed already in Example 6.1 as a cuspidal curve for which the Alexander polynomial has factors of multiplicity 3 one can take the dual curve of a smooth cubic (its fundamental group was calculated by Zariski [57]). For other examples cf. [10].

Example 6.4. Consider the moduli space of sextics with three singular points P, Q, R of types $\mathbb{A}_{15}$, $\mathbb{A}_3$, and $\mathbb{A}_1$ respectively. Such a moduli space has been studied in [4] and it consists of two connected components $\mathcal{M}_1$, $\mathcal{M}_2$. Both have as representatives reducible sextics which are the product of a quartic $\mathcal{C}_4$ and a smooth conic $\mathcal{C}_2$ intersecting at the point P (of type $\mathbb{A}_{15}$) and hence $Q, R \in \mathcal{C}_4$. There is a geometrical difference between sextics in $\mathcal{M}_1$ and $\mathcal{M}_2$. For one kind of sextics, say $\mathcal{C}_6^{(1)} \in \mathcal{M}_1$, the tangent line at P also contains Q, whereas for the other sextics, say $\mathcal{C}_6^{(2)} \in \mathcal{M}_2$, it does not. The Alexander polynomial of both kinds is trivial; however, if we consider the homomorphism $\varepsilon = (\varepsilon_4, \varepsilon_2) = (1, 2)$, where ε_i is the image of a meridian around $\mathcal{C}_i$, then

$$\Delta_{\mathcal{C}_6^{(i)},\varepsilon}(t) = \begin{cases} (t^2+1) & \text{if } i = 1, \\ 1 & \text{if } i = 2. \end{cases}$$

Note that $\mathcal{C}_6^{(1)}$ is a 4-total curve, but not all of its singularities are 4-essential, since one can check that $\mathbb{A}_{15}$ has a local Alexander polynomial $\Delta_{\mathbb{A}_{15},\varepsilon}(t) = (1+t^{12})(1+t^6)(1+t^3)(1-t)$. One can also use Degtyarev's Divisibility Criterion [14] to prove that the factors coming from $\Delta_{\mathbb{A}_{15},\varepsilon}(t)$ do not contribute.

Therefore, Theorem 5.14 can be applied and hence $\mathcal{C}_6^{(1)}$ has a quasi-toric relation of elliptic type $(2, 4, 4)$, whereas $\mathcal{C}_6^{(2)}$ does not. In particular, these are the equations for the irreducible components of $\mathcal{C}_6^{(1)} = \mathcal{C}_4 \cup \mathcal{C}_2$,

$$C_4 := 2xy^3 + 3x^2y^2 + 108y^2z^2 - x^4,$$
$$C_2 := 3x^2 + 2xy + 108x^2,$$

which fit in the following quasi-toric relation:

$$C_2h_1^2 + h_2^4 - C_4h_3^4 = 0,$$

where $h_1 := y$, $h_2 := x$, and $h_3 := 1$.

Example 6.5. As an example of a quasi-toric relation of elliptic type $(3, 3, 3)$ we can present the classical example $F = (y^3-z^3)(z^3-x^3)(x^3-y^3)$. The classical Alexander polynomial of $\mathcal{C} := \{F = 0\}$ is $\Delta_{\mathcal{C}}(t) = (t^2+t+1)^2(t-1)^8$ and it is readily seen that it is a 3-total curve, since its singularities (besides the nodes) are ordinary triple points, which are 3-essential singularities. Hence one can apply Theorem 5.16 and derive that F fits in a quasi-toric relation of elliptic type $(3, 3, 3)$. For instance:

$$x^3(y^3 - z^3) + y^3(z^3 - x^3) + z^3(x^3 - y^3) = 0. \tag{6.4}$$

However, according to Theorem 5.12 there should exist another relation independent from equation (6.4), namely

$$\ell_1^3 F_1 + \ell_2^3 F_2 + \ell_3^3 F_3 = 0, \tag{6.5}$$

where

$$F_i = (y - \omega_3^i z)(z - \omega_3^{i+1} x)(x - \omega_3^{i+2} y), \quad i = 1, 2, 3,$$

ω_3 is a third-root of unity, and

$$\begin{aligned}\ell_1 &= (\omega_3 - \omega_3^2)x + (\omega_3 - \omega_3^2)y + (\omega_3^2 - 1)z,\\ \ell_2 &= (\omega_3 - \omega_3^2)z + (\omega_3 - \omega_3^2)x + (\omega_3^2 - 1)y,\\ \ell_3 &= (\omega_3 - \omega_3^2)y + (\omega_3 - \omega_3^2)z + (\omega_3^2 - 1)x.\end{aligned}$$

Example 6.6. In Remark 5.4 we have presented a 6-total sextic curve $\mathcal{C}$ which is also 3-partial. In particular, according to Theorem 5.16, $\mathcal{C}$ admits quasi-toric relations both of type $(2, 3, 6)$ and $(3, 3, 3)$. We will show this explicitly. Note that $\mathcal{C}$ is the union of two cuspidal cubics (cf. [44, Corollary 1.2]) $\mathcal{C}_1 := \{F_1 = 0\}$ and $\mathcal{C}_2 := \{F_2 = 0\}$ given by the following equations:

$$\begin{aligned}F_1 &= y^3 - z^3 + 3x^2 z + 2x^3,\\ F_2 &= y^3 - z^3 + 3x^2 z - 2x^3.\end{aligned}$$

One can check that the curve $\mathcal{C}$ satisfies the following relations:

$$\begin{aligned}3f_3^2 - 4f_2^3 + F_1 F_2 h^6 &= 0,\\ 4x^3 - F_1 h^3 + F_2 h^3 &= 0,\end{aligned}$$

where $f_2 := yz + y^2 + z^2 - x^2$, $f_3 := z^3 - x^2 z - 2yx^2 + 2yz^2 + 2y^2 z + y^3$, and $h = 1$.

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