BOOK REVIEWS

Topology of algebraic curves. An approach via dessins d'enfants, by Alex Degtyarev,
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Introductory courses of algebraic geometry usually explain that a smooth plane
complex algebraic curve is homeomorphic to a closed orientable surface of genus
$\frac{1}{2}(d-1)(d-2)$, where d is the degree of the defining equation of the curve. Home-
omorphism type of a *singular* curve in $\mathbf{P}^2$ also can be easily visualized. It is the
disjoint union of closed orientable surfaces, corresponding to irreducible factors of a
curve's equation, with points identified in a way dictated by a curve's singularities.
The subject of algebraic curves becomes very rich when one is interested in their
geometry and arithmetic. But what about the *topology* of a curve's *complement* in
$\mathbf{P}^2$? Here one runs very quickly into questions, which are easy to ask but hard to
answer. Degtyerev's book is about some such questions.
   Here is a sample of problems about fundamental groups of the complements,
some of which are open and answers to some are known, at least partially. For
which groups G does there exist an algebraic curve C such that $G = \pi_1(\mathbf{P}^2 \setminus C)$?
When is $\pi_1(\mathbf{P}^2 \setminus C)$ abelian, nilpotent, etc.? Can one relate $\pi_1(\mathbf{P}^2 \setminus C)$ to the data of
the degrees of irreducible complements and the type of singularities of C, and how
does the geometry of the pair $(\mathbf{P}^2, C)$ affect this fundamental group? How special
are fundamental groups of the complements to *special* plane curves, for example, for
curves all irreducible components of which have degree 1 (arrangements of lines)?
For curves with nonlinear components, torsion in fundamental groups often does
appear, but are fundamental groups of the complements to arrangements of lines
torsion free?
   One can view all this as a version of more general problems about fundamental
groups or topological types of quasi-projective varieties. General results in the
topology of the latter have a lot to say about topology of $\mathbf{P}^2 \setminus C$. However, some
part of the picture may be specific for plane curves. For instance, $\mathbf{P}^2 \setminus C$ is a
manifold of real dimension four, but is homotopy equivalent to a two-dimensional
complex. On the other hand, the class of groups which are fundamental groups of
the complements to hypersurfaces in $\mathbf{P}^N$ coincides with the class of fundamental
groups of the complements to plane curves (in fact $\pi_1(\mathbf{P}^N \setminus V) = \pi_1(H \setminus V \cap H)$

for any hypersurface V and generic plane H in $\mathbf{P}^N$). What else is special for this particular class of fundamental groups of quasi-projective manifolds?

The subject has a history that is more than hundred years long. Some effective approaches use only topological methods, but the most interesting results involve the relation of $\pi_1(\mathbf{P}^2 \setminus C)$ to algebraic geometry associated with pair $(\mathbf{P}^2, C)$, in particular, Hodge theory. The main highlights of the developments in the topology of the complements are as follows.

The Zariski–van Kampen description of $\pi_1(\mathbf{P}^2 \setminus C)$ (cf. [38]). Consider the equation $f(x, y) = 0$ of a curve C in sufficiently general affine coordinates. Denote by x_0 a sufficiently general point on the x-axis X, where $L_{x_0} = \{(x_0, y) \in \mathbf{C}^2 | y \in \mathbf{C}\}$ is the corresponding "vertical" line and $p = (x_0, y_0) \in L_{x_0}$ is a basepoint. Then generators $\kappa_i, i = 1, \ldots, \deg C$ of the free group $\pi_1(L_{x_0} \setminus L_{x_0} \cap C, p)$ also generate the group $\pi_1(\mathbf{P}^2 \setminus C, p)$, but satisfy in it certain relations. Let

$$(1) \qquad \mathrm{Cr}(C) = \{x \in X | \mathrm{Card}(L_x \cap C) < \deg(C)\}$$

be the critical set of projection of C onto X. For any loop γ in $X \setminus \mathrm{Cr}$ based at x_0, while one moves along γ, a continuous change of points of $C \cap L$ yields a diffeomorphism of the pair $(L_{x_0}, L_{x_0} \cap C)$. This diffeomorphism, in turn, induces the automorphism Φ_γ of the free group $\pi_1(L_{x_0} \setminus L_{x_0} \cap C, x_0)$. If γ_j are generators of the free group $\pi_1(X - \mathrm{Cr}(C), x_0)$, then the defining relations of $\pi_1(\mathbf{P}^2 \setminus C, p)$ are

$$(2) \quad \Phi_{\gamma_j}(\kappa_i) = \kappa_i, i = 1, \ldots, \deg C, j = 1, \ldots, \mathrm{Card}\,\mathrm{Cr}(C), \quad \kappa_1 \cdot \ldots \cdot \kappa_{\deg C} = \mathrm{id}.$$

A permutation of the finite set $L_{x_0} \cap C$ obtained by moving along γ is the monodromy transformation central to the Riemann description of multivalued functions in one variable: the inverse map of the projection of C onto the x-axis is such a multivalued function. In 1912 Enriques attempted to extend Riemann's theory to the multivariable case and considered the problem of finding which finite groups can appear as the monodromy groups of an algebraic function in two variables having a given C as its ramification locus. Relations (2) appeared in his work as conditions which should be satisfied by the generators of the monodromy groups of multivalued algebraic functions, such as solution $z(x, y)$ to an equation $z(x, y)^n + a_1(x, y)z(x, y)^{n-1} + \ldots + a_n(x, y) = 0$, where $a_i(x, y)$ are polynomials. Enriques' calculation yields a quotient of $\pi_1(\mathbf{P}^2 \setminus C)$ by the intersection of subgroups of finite index (as was pointed out already by Zariski in [42]) and bears a hint of the idea of an *algebraic* fundamental group which started being developed only half century later by Grothendieck, Serre, and Abhyankar. His calculations describe the group having the same profinite completion as $\pi_1(\mathbf{P}^2 \setminus C)$. In fact, comparison of Enriques and van Kampen's approaches leads to the question, Is the intersection of subgroups of finite index trivial, i.e., are the fundamental groups of the complements to plane curves residually finite? This is one of many "easy to ask" questions that are still wide open, but in 1993 Toledo [36] found an example of a projective variety with a nonresidually finite fundamental group (cf. [36]). On many occasions, one sees in the early period of the topology of plane algebraic curves the seeds of fundamental ideas which became fully developed much later.

Construction of presentation (2) allowed the carrying out of several important calculations, but it shed little light on most problems about fundamental groups mentioned above. Often presentation (2) is not the most useful one, and specific features in special situations may lead to different important presentations. For

example Brieskorn (cf. [5]) gives presentations of fundamental groups of complements to arrangements of hyperplanes corresponding to some Coxeter groups. It is transparently related to the presentation of the Coxeter group itself and is also a generalization of a standard presentation of Artin's braid group, leading to Deligne's determination of their homotopy type by showing the vanishing of higher homotopy groups of these complements, cf. [12].

Commutativity of $\pi_1(\mathbf{P}^2 \setminus C)$ in the case when C has nodes as the only singularities. Nodes, i.e., points near which the curve can be given by local equation $x^2 = y^2$, are the mildest singularities, and the above claim was made by Zariski in a 1929 paper (cf. [39]). His argument relied on the Severi assertion in [35, *Anhang F*, 1921] that the variety of plane curves of fixed degree and with fixed number of nodes is irreducible. It was meant to be applied to obtain several fundamental results in algebraic geometry of curves, but Severi's proof was incorrect. Another such application, besides Zariski's, was the irreducibility of the moduli space of smooth curves of a given genus. Alternative proofs for the latter were found very early (Deligne and Mumford's proof in [9] still uses some of Severi's ideas), but the problem of the irreducibility of families of nodal plane curves was solved by Harris only in 1986 in [20]. The question on commutativity of the fundamental groups of the complements to nodal curves became known as the "Zariski problem". Its solution, independent of Severi's assertion, actively sought over decades, was found in 1979 by Fulton (cf. [18]) for algebraic fundamental groups and by Deligne (cf. [11]) in the topological case. A few years later, Nori in [33] discovered a solution to a generalization of the Zariski problem for curves with arbitrary singularities on arbitrary projective surfaces implying commutativity of $\pi_1(\mathbf{P}^2 \setminus C)$ in the case when C is nodal. To date, this is the strongest tool for identifying cases in which the fundamental group of the complement is commutative.

Dependence of the fundamental group on the local type of singularities. Early studies of special classes of curves signaled dependence of $\pi_1(\mathbf{P}^2 \setminus C)$ on the complexity of the local type of singularities of C as well as their number. Evidence for this started coming in with Zariski's assertion about the fundamental group of the complements to nodal curves and continued to his calculations of nonabelian fundamental groups in several explicit examples of curves with mild singularities. Later calculation of fundamental groups of the complements to the curves $(x^p + y^p)^q + (y^q + z^q)^p = 0$ as a free product of two cyclic groups $\mathbf{Z}_p * \mathbf{Z}_q$, due to Oka, (cf. [34]) reinforced such expectation.

The first quantitative results in this direction were obtained by A. Libgober in [25] and [26]. It is convenient to state these results using the Alexander invariants of groups, first developed in the context of knot theory. Interestingly, Zariski used polynomial invariants in his study of covering space (cf. [41]), but the connection with Alexander polynomials associated with knots was not made at the time (cf. Mumford's question in [42, Ch. 8, Appendix II]).

Let G be a finitely generated and finitely presented group endowed with a surjection $\Pi : G \to \Sigma$ onto a cyclic group Σ with generator σ. Let $K_\Sigma = \operatorname{Ker} \Pi$. The exact sequence

$$(3) \qquad\qquad 0 \to K_\Sigma/K_\Sigma' \to G/K_\Sigma' \to \Sigma \to 0,$$

where $'$ denotes the commutator, induces the action of a generator σ of Σ on $K_\Sigma/K_\Sigma' \otimes \mathbf{Q}$. The characteristic polynomial of σ is called the *Alexander polynomial*

of G relative to surjection Π. For irreducible curves in $\mathbf{P}^2$ or $\mathbf{C}^2$, as well as for knots in the 3-sphere, the abelianization of the fundamental group G of the complement is cyclic, and its orientation gives the canonical choice of σ, i.e., the resulting polynomial is an invariant of the group. From its definition, it depends only on G/G'', where G'' is the commutator of G'. This construction is due to Alexander (cf. [1]), at least in the case of fundamental groups of links, and subsequently it was developed by Reidemeister, Seifert, Fox, and many others.

To a germ C_P of a plane curve at point $P \in \mathbf{P}^2$, one can associate a link in the 3-sphere as follows. Let B_P be a small ball about P (its real dimension is four). The link of C_P is the intersection of C_P with the boundary ∂B_P of B_P. Complexity of singularity of C_P at P is reflected by complexity of its link: if a germ C_P is smooth, then $C_P \cap \partial B_P$ is unknotted in ∂B_P; if C_P has a node at P, then $C_P \cap \partial B_P$ is a pair of linked circles; if C_P has local equation $x^2 = y^3$, then $C_P \cap \partial B_P$ is a trefoil knot, etc. The Alexander polynomials respectively are $1, t-1, t^2-t+1$. In general, the links associated with singularities are of a special kind: they are iterated torus links. This implies, in particular, that their Alexander polynomials are cyclotomic.

Consider an irreducible curve C in $\mathbf{P}^2$ with arbitrary singularities $P_1, \ldots, P_N$. Let $\Delta_C(t)$ be the Alexander polynomial of $\pi_1(\mathbf{P}^2 \setminus C)$, and let $\Delta_{C,P_i}(t)$ be the Alexander polynomial of the link of a germ of C at P_i. One of the main results of [25] is the divisibility relation,

$$(4) \qquad\qquad \Delta_C(t) | \Pi_{i=1,\ldots,N} \Delta_{C,P_i}(t).$$

For example, for irreducible curves C having nodes and cusps as the only singularities, the Alexander polynomial has the form $(t^2 - t + 1)^s$ for some integer $s \geq 0$. Another consequence of relation (4) is the cyclotomicity of the Alexander polynomials of arbitrary algebraic curves, which allows one to show the nonrealizability of many groups as the fundamental groups of complements to plane curves. For example, the roots of the Alexander polynomial of the fundamental group of the figure-eight knot *are not* roots of unity, and hence this group is not $\pi_1(\mathbf{P}^2 \setminus C)$ for any algebraic curve C.

$\pi_1(\mathbf{P}^2 \setminus C)$ **and algebraic surfaces.** The idea of studying properties of a plane curve using invariants of covering surfaces of $\mathbf{P}^2$ ramified along this curve appears in [40] in the special case of cyclic coverings and irreducible curves with nodes or cusps as the only singularities (the case of arbitrary singularities was studied later in [26]). The main result of [40] is that the dimension of space of holomorphic 1-forms (called *irregularity*) on such cyclic multiple planes may be used to detect noncommutativity of $\pi_1(\mathbf{P}^2 \setminus C)$: if this group is abelian, then irregularity of any cyclic covers ramified along C is zero. On the other hand, irregularity can be related to the dimension of the space of curves passing through the cusps of a given curve. In particular, a special position of cusps yields noncommutativity of the fundamental group of the complement. An archetypal example is the curve with equation $f_2(x, y, z)^3 + f_3(x, y, z)^2 = 0$, where f_n is a generic form of degree n. Here the cusps are the solutions of the system $f_2(x, y, z) = f_3(x, y, z) = 0$ (there are no other singularities if forms f_n are generic), and hence six cusps are situated on a curve of degree 2. Zariski showed in [40] that this implies that irregularity of a 6-fold cyclic cover of $\mathbf{P}^2$ ramified along this curve is nonzero and hence the fundamental group of its complement is nonabelian.

B. Moishezon (cf. [30]) attempted to reverse the above idea and obtain invariants of an embedded projective surface $X \subset \mathbf{P}^N$ using the branching curve C of a generic linear projection onto $\mathbf{P}^2$, and he viewed $\pi_1(\mathbf{P}^2 \setminus C)$ as an invariant of deformation type of the pair $(X, \mathbf{P}^N)$. In the case of a pluricanonically embedded surface of general type X, one expects to obtain an invariant of its deformation type only. In a series of papers (later ones jointly with Teicher, cf. [32]), explicit calculations for many concrete pairs $(X, \mathbf{P}^N)$ were carried out. Moishezon used the interpretation of automorphisms Φ_γ of the free group $\pi_1(L_{x_0} \setminus C \cap L_{x_0})$ in the Zariski–van Kampen presentation (2) as *braids* (as was done earlier by Chisini), viewing the latter as the isotopy classes of diffeomorphisms of a disk fixing a finite subset of points which are identity on the boundary. The crucial piece of data associated with the curve (and determining the fundamental group) became the homomorphism $\pi_1(\mathbf{P}^1 \setminus \mathrm{Cr}) \to B_d$, where B_d is the braid group on $d = \deg C$ strings, which he called the *braid monodromy*. Composition of the braid monodromy with the map associating to a braid the corresponding permutation, is the Riemann monodromy of a multivalued function associated with the projection of C on the x-axis. A special type of combinatorial analysis, partially motivated by Hurwitz's analysis of monodromy with values in a symmetric group, was developed for finding "normal forms" for the braid monodromy of singular curves. In particular, if $C_{d(n)}$ is the branching curve of the generic projection of a smooth surface of degree n in $\mathbf{P}^3$, then $\pi_1(\mathbf{P}^2 \setminus C_{d(n)}) = B_n/(\Delta_n^2)$, where Δ_n^2 is the generator of the center of the braid group. Later on, precise conjectures on the structure of $\pi_1(\mathbf{P}^2 \setminus C)$, where C is a branching curve of a generic projection, were put forward (cf. [37]). With the advent of symplectic geomerty, braid monodromy became a very effective tool for the study of symplectic 4-manifolds and embedded symplectic curves (cf. [3]). Early expectations that the diffeomorphism (or symplectomorphism) type of manifold underlying a projective surface would determine its deformation type and that $\pi_1(X \setminus C)$ would provide invariants of a diffeomorphism (or symplectomorphism) type different from Donaldson's invariants turned out to be incorrect (cf. [17], [6]).

One of the striking outcomes of the above combinatorial analysis of braid monodromy is a counterexample to the assertion by Chisini about the existence of plane curves of given degree and numbers of cusps and nodes. A braid monodromy associates with a curve $C \subset \mathbf{P}^2$ the collection of braids $\beta(\gamma_i)$, where $\gamma_i \in \pi_1(\mathbf{P}^1 \setminus \mathrm{Cr}(C))$, which must satisfy the relation $\Pi \beta(\gamma_i) = \Delta^2$. This is an analog of the Hurwitz relation $\Pi \sigma(\gamma_i) = \mathrm{id}$ for ordinary monodromy $\sigma : \pi_1(\mathbf{P}^1 - \mathrm{Cr}) \to S_d$ with the values in the symmetric group associated with the cover $C \to \mathbf{P}^1$. Moreover, the local type of critical point corresponding to each generator γ_i determines the conjugacy class of the braid $\beta(\gamma_i)$. In [7] Chisini asserted that for a given collection of braids $\Gamma_i \in B_d, \Pi \Gamma_i = \Delta^2$, each belonging to the conjugacy classes of braids appearing in the braid monodromies of curves with nodes and cusps, there exists an algebraic curve of degree d for which the braid monodromy satisfies $\beta(\gamma_i) = \Gamma_i$. Work [31] showed the existence of a counterexample to this claim, but later on it was realized that braid monodromy is very effective in describing singular *symplectic* curves (cf. [4]). Disproval of Chisini's claim was interpreted as showing the existence of symplectic curves not isotopic to algebraic ones. An interesting property (the so-called "Chisini problem") of homomorphisms onto a symmetric group of the fundamental groups of the complements to the branched curves of generic projections was proved by Kulikov (cf. [23]).

Characteristic varieties and calculation of abelianization of the commutator of $\pi_1(\mathbf{P}^2 \setminus C)$. Part of the fundamental group of the complement that is relatively well understood is the abelianization of its commutator $\pi_1(\mathbf{P}^2 \setminus C)'$. One of the problems leading to studying this abelianization is the calculation of the homology of abelian covers of $\mathbf{P}^2 \setminus C$ or covers of $\mathbf{P}^2$ ramified over C.

Homology groups of *cyclic* covers of $\mathbf{P}^2$, associated with a curve C with arbitrary singularities, were calculated by Libgober in 1981 (cf. [26]), extending the results of [40]. The answer was given in terms of the dimensions of linear systems of curves whose local equations near singular points of C belong to certain ideals. Such ideals, introduced under the name "the ideals of quasi-adjunction" in [26], represent the first appearance of *multiplier ideals*, which became a very active area of study since the late 1990s (cf. [24]). An upshot of [26], in the case when $H_1(\mathbf{P}^2 \setminus C, \mathbf{Z})$ is cyclic, is an algebro-geometric expression for the quotient

$$(5) \qquad\qquad \pi_1(\mathbf{P}^2 \setminus C)'/\pi_1(\mathbf{P}^2 \setminus C)'' \otimes \mathbf{Q}$$

in terms of dimensions of linear systems of curves associated with singularities of C: the geometry of singularities determines the topology, i.e., the vector space (5) and vice versa.

The case of an arbitrary $H_1(\mathbf{P}^2 \setminus C, \mathbf{Z})$ was treated in [27]. Similarly to the cyclic case, the first Betti number b_1 of abelian covers of $\mathbf{P}^2$ ramified along a curve C has an algebro-geometric expression in terms of the dimension of the linear system of curves depending on singularities of C. On the other hand, this b_1 can be described in terms of the quotient (5) endowed with the canonical structure of a finitely generated module over the group ring $\mathbf{C}[\pi_1(\mathbf{P}^2 \setminus C)/\pi_1(\mathbf{P}^2 \setminus C)']$ (this module structure is obtained from the action of an abelian group $\pi_1(\mathbf{P}^2 \setminus C)/\pi_1(\mathbf{P}^2 \setminus C)'$ on (5) coming from the exact sequence (3) with $\Sigma = \pi_1(\mathbf{P}^2 \setminus C)/\pi_1(\mathbf{P}^2 \setminus C)'$). In fact, the first Betti number of abelian covers depends only on *an invariant* of (5) that is a certain subvariety of the torus of characters Char $H_1(\mathbf{P}^2 \setminus C, \mathbf{Z})$ called the *characteristic variety of* $\pi_1(\mathbf{P}^2 \setminus C)$. It is defined as the (reduced) support of the module (5) considered as a subvariety of Spec $\mathbf{C}[\pi_1(\mathbf{P}^2 \setminus C)/\pi_1(\mathbf{P}^2 \setminus C)'] =$ Char $H_1(\mathbf{P}^2 \setminus C, \mathbf{Z}))$, cf. [27]. This definition works for any group G, and in the case of fundamental groups of links yields the classical multivariable Alexander polynomial, extending the Alexander construction mentioned earlier. As a corollary of the calculation of the first Betti number of abelian covers, one obtains an algebro-geometric expression for this topological invariant.

Characteristic varieties have an interpretation in terms of cohomology of local systems (cf. [21]), which in turn opens the way for using Hodge theoretical methods (cf. [10], [28]). Characteristic varieties attached to fundamental groups of general quasi-projective varieties are very special. For example, in many cases they are known to be finite unions of translated subgroups of the torus of characters. This fundamental property found numerous applications; cf. for example [14] or [15] where it was shown that the only fundamental groups of Kähler manifolds which are also fundamental groups of 3-manifolds are the finite groups.

Beautiful properties of abelian covers of $\mathbf{P}^2$ branched over an arrangement of lines were discovered by Hirzebruch ([22]) in the 1980s: for some arrangements and abelian covers the resulting complex surfaces are the quotients of ball in $\mathbf{C}^2$ by arithmetic subgroups of $PU(1,2)$. The theory of characteristic varieties allowed one to describe in a unified way the irregularity of these ball quotients (cf. [27]).

In another direction, characteristic varieties, in a special case of arrangements of lines, attracted much attention due to their relation with combinatorics. Positive-dimensional irreducible components of characteristic varieties containing $1 \in \mathrm{Char}(\pi_1(\mathbf{P}^2 \setminus \mathcal{A}))$, where $\mathcal{A}$ is an arrangement of lines, correspond to pencils (i.e., one-dimensional families of plane curves) with at least three special members. Each special member is a union of lines, and a union of lines in all such special members form the given arrangement ([29]). This correspondence uncovered several combinatorial aspects underlying the properties of $\pi_1(\mathbf{P}^2 \setminus \mathcal{A})$.

The relation between characteristic varieties and pencils of plane curves led to considering in [2] the pencils of curves with multiple fibers. Connection of the latter with orbifold fundamental groups became one of many currently active developing directions.

The book. Degtyarev's book is a welcome and timely addition to the literature on the topology of plane algebraic curves, which so far, besides being one of several subjects in the textbook [13], had included only research and survey articles. It combines a lot of introductory material, presenting a number of developments mentioned above (including introduction to the theory of Alexander polynomials and braid monodromy) and a detailed exposition of some of the author's research done over the last twenty years. The main guiding theme of the book is a description of approaches to the classification of the fundamental groups of plane curves of low degree, to which the author was the main contributor. But there are many related topics discussed in depth, spreading from algebraic curves over $\mathbf{R}$ and geometry of elliptic surfaces.

The text is separated into two parts. The first part deals with a description of technical tools, and the second contains concrete applications of these techniques to classification problems.

Classification of equisingular families of plane curves (sometimes called "families of rigidly isotopic curves"), for which the fundamental groups of the complements are the main invariants, in the case of curves of degree up to 4, is easy but becomes much harder already in the case of quintics and sextics. In fact, there are 221 equisingular families of quintics, and a list of equisingular families of sextics— should it be possible to compile—must be frighteningly long. Nevertheless, the list of fundamental groups of quintics was found by Degtyarev in 1989. Results on classification of curves of degree 6 continued emerging gradually over last two decades, and it has been completed for many classes of singular sextics. This book contains the most complete description of such classification results to date (though the author remarks that "topology of plane sextics is a vast subject deserving a separate monograph").

One of the central points, which makes the classifications of plane sextics so interesting, is the relation of the latter with the theory of K3 surfaces. A K3 surface is a simply connected complex surface admitting a nonvanishing holomorphic 2-form. There is a 20-dimensional family on complex surfaces satisfying this condition and containing countably many 19-dimensional families of projective ones. K3 surfaces are among the most beautiful and most studied objects in mathematics, which still continue to reveal new facets. Areas in which they play an important role spread from number theory to string theory and dynamics. A minimal resolution of singularities of double covers of $\mathbf{P}^2$ branched over a sextic curve having simple (in the sense of Arnold) singularities are K3 surfaces. This is the reason for the

appearance of K3 surfaces in connection with plane sextics. Another tool prominent in this book is the relation of sextics to trigonal curves, i.e., curves admitting a map of degree 3 onto $\mathbf{P}^1$, embedded into Hirzebruch surfaces (i.e., $\mathbf{P}^1$-fibrations over $\mathbf{P}^1$).

A way to make sense out of the enormous number of equisingular types of sextics is to relate them to a manageable combinatorial object. This is done in the book, based on author's works over the years, by relating them to special graphs embedded into closed orientable surfaces representing an overdecorated version (due to Orevkov) of "*dessins d'enfant*" that appeared in Grothendieck's "*l'Esquisse d'un Programme*" [19]. Detailed exposition of such graph theory is given in Chapters 1 and 4. Sextic curves having a point of multiplicity 3 admit interpretation as a subclass of trigonal curves, i.e., threefold covers of $\mathbf{P}^1$, in a blowup of $\mathbf{P}^2$ at a point (Hirzebruch surface). Indeed, a proper preimage of such a sextic in a blowup of $\mathbf{P}^2$ at its triple point, mapped by the natural map of such a blowup onto its exceptional curve, yields the presentation of the sextic as a trigonal curve. Moreover, the double cover of the Hirzebruch surface totally ramified along the union of a embedded trigonal curve and the exceptional curve yields an elliptic surface, i.e., a surface admitting dominant maps onto $\mathbf{P}^1$ and having as a generic fiber a smooth elliptic curve. This correspondence and a concise introduction to the theory of elliptic surfaces, including their role in real algebraic geometry, are presented in Chapter 3. As mentioned earlier in this review, braid monodromy of plane curves and its specialization to B_3-valued case, which is central in the isotopy classification of trigonal curves, are discussed in Chapters 3 and 5. This is preceded by an introductory Chapter 2 containing a discussion of the braid group B_3 together with the closely related modular group $PSL_2(\mathbf{Z})$ (i.e., the quotient of B_3 by its center). This chapter also includes properties of the Burau homomorphism of B_3 into $\mathrm{Aut}(\mathbf{Z}[t, t^{-1}]^2)$, used in the study of Alexander invariants later in the book. Braid monodromy in connection with *dessins d'enfant* is the subject of Chapter 5.

Part II contains a unified exposition of several classification results spread over a vast literature, as well as many new results. Discussion of classification of Alexander modules of trigonal curves is given in Chapter 6. Fundamental groups of sextics per se, including detailed tables for many classes of sextics, is the subject of Chapters 7 and 8. There is an extensive discussion of the so-called Oka conjecture characterizing sextics with nontrivial Alexander polynomials as the sextics of torus type, i.e., having an equation of the form $f_2^3 + f_3^2 = 0$, where f_d are possibly nongeneric forms of degree d. This chapter also includes the complete classification of classes of equisingular deformations and the fundamental groups of the complements to plane quintics already mentioned. Chapter 9 discusses the transcendental lattices of elliptic surfaces constructed using trigonal curves as well as Mordell–Weil groups and extremal elliptic surfaces. Factorization of monodromy, representing part of the earlier-mentioned combinatorial analysis of braid monodromies is discussed in Chapter 10. One of the main results here is the existence of the *exponentially growing in k* number of nonequivalent factorizations of length k of the elements of B_3. The problem of deciding if there are pairs of plane algebraic curves of the same degree with the same number of cusps and nodes but different fundamental groups is known as finding "Zariski pairs". The main result of the concluding chapter is the existence of topologically distinct Zariski k-plets (extending the notion of Zariski pairs to the case $k > 2$) in a variety of contexts: irreducible trigonal curves, real Lefschetz fibrations, extremal elliptic surfaces, etc.

The book contains several appendices that help the reader connect to the author's terminology and which help a novice understand the material in the main body of the book.

The text is clearly written, it contains a good and systematic exposition of known and new material. This book should be a must-have for anyone working in this area and also for everyone who runs into plane singular curves and wants to know how to work with them.

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