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Motivic infinite cyclic covers

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We associate with an infinite cyclic cover of a punctured neighborhood of a simple normal crossing divisor on a complex quasi-projective manifold (under certain finiteness conditions) an element in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$, which we call *motivic infinite cyclic cover*, and show its birational invariance. Our construction provides a unifying approach for the Denef–Loeser motivic Milnor fiber of a complex hypersurface singularity germ, and the motivic Milnor fiber of a rational function, respectively.

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1. Introduction

Infinite cyclic covers are fundamental objects of study in topology (e.g., in knot theory [28], but see also [25]) and algebraic geometry (e.g., for the study of Alexander-type invariants of complex hypersurface complements, see [9,10,16,17,23]).

The Milnor fiber of a hypersurface singularity germ (cf. [24]), can be viewed as an example of an infinite cyclic cover, since it is a retract of the infinite cyclic cover of the complement to the germ in a small ball about the singular point. Moreover, in this interpretation, the monodromy of the Milnor fiber corresponds to the action of the generator of the group of deck transformations of the infinite cyclic cover (cf. Section 2 below; but see also [18], where such an identification was used to define an abelian version of the Milnor fiber, and [8] for a detailed discussion in the homogeneous case).

Motivated by connections between the Igusa zeta functions, Bernstein–Sato polynomials and the topology of hypersurface singularities, Denef and Loeser defined in [5–7] the motivic zeta function and the motivic Milnor fiber of a hypersurface singularity germ; the latter is a virtual variety endowed with an action of the group scheme of roots of unity, from which one can retrieve several invariants of the (topological) Milnor fiber, e.g., the Hodge–Steenbrink spectrum, Euler characteristic, etc. The motivic Milnor fiber has also appeared in the Soibelman–Kontsevich theory of motivic Donaldson–Thomas invariants.

In this paper, we attach to an infinite cyclic cover associated to a punctured neighborhood of a simple normal crossing divisor E on a complex quasi-projective manifold X, an element in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$ of algebraic $\mathbb{C}$-varieties endowed with a good action of the pro-finite group $\hat{\mu} = \lim \mu_n$ of roots of unity, which we call a *motivic infinite cyclic cover*; see Section 3 for details. (Our terminology is inspired by the standard notion of "motivic Milnor fiber", cf. [7].) Among other consequences, this construction allows us to define a motivic infinite cyclic cover of a hypersurface singularity germ complement, which as we show later on coincides (in the localization of $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$ at the class $\mathbb{L}$ of the affine line) with the above-mentioned Denef–Loeser motivic Milnor fiber. Our class of coverings guarantees certain finiteness conditions (see Definition 2.1) which are present in the case of Milnor fibers, but which are not satisfied in general. Note that while these infinite cyclic covers are complex manifolds, they are not algebraic varieties in general. This paper provides an algebro-geometric interpretation of such covering spaces.

Our construction of motivic infinite cyclic covers is topological in the sense that it does not make use of arc spaces as is the case in earlier constructions of motivic Milnor fibers. We rely instead on the weak factorization theorem [1,4]. One of our main results, Theorem 3.7, shows that our notion of motivic infinite cyclic cover is a birational invariant, or equivalently, it is an invariant of the punctured neighborhood of E in X. Moreover, in Section 4 we show that the Betti realization of the motivic infinite cyclic cover is given by the cohomology with compact support of the infinite cyclic cover of the punctured neighborhood, e.g., their Euler characteristics coincide.

Finally, in Sections 5 and 6, we explain how the present construction of a motivic infinite cyclic cover generalizes the above-mentioned notion of motivic Milnor fiber of a hypersurface singularity germ, as well as the notion of motivic Milnor fiber of a rational function (compare with [27]).

2. Infinite cyclic cover of finite type

Let X be a smooth complex quasi-projective variety and E an algebraic (reduced) simple normal crossing divisor on X which shall be called a *deletion* (or *deleted) divisor*. Assume that $E = \sum_{i \in J} E_i$ is a decomposition of E into irreducible components E_i, where we assume that all divisors E_i are smooth. We use the following natural stratification of X given by the intersections of the irreducible components of E: for each $I \subseteq J$ consider

$$E_I = \bigcap_{i \in I} E_i \quad \text{and} \quad E_I^\circ = E_I \setminus \bigcup_{j \notin I} E_j. \tag{1}$$

Clearly, $X = \bigcup_{I \subseteq J} E_I^\circ$, $X \setminus E = E_\emptyset^\circ$ and $E = \bigcup_{\emptyset \neq I \subseteq J} E_I^\circ$.

Let $T^*_{X,E}$ be a punctured neighborhood of E on X. Sometimes we omit the subscript X and just write T^*_E. We recall here the construction of such a punctured neighborhood. For each smooth irreducible component E_i of E ($i \in J$), we choose a tubular neighborhood $T_{E_i} \to E_i$, and define the corresponding neighborhood of E_I (with $\emptyset \neq I \subseteq J$) by:

$$T_{E_I} := \bigcap_{i \in I} T_{E_i}.$$

We set

$$T_{X,E} := \bigcup_{\emptyset \neq I \subseteq J} T_{E_I}.$$

Note that if the chosen tubular neighborhoods T_{E_i} of the components E_i are small enough, then $T_{E_I} \to E_I$ is also a tubular neighborhood for the submanifold E_I (for a suitable projection map), and $T_{X,E}$ is a regular neighborhood of E, i.e., E is a deformation retract of $T_{X,E}$. Moreover, the germs of all these neighborhoods (and projection maps) are independent of all choices by the corresponding uniqueness result for T_{E_i}. We define punctured tubular neighborhoods of the strata E_I° by:

$$T^*_{E_I^\circ} := (T_{E_I}|_{E_I^\circ}) \setminus \bigcup_{i \in I} E_i,$$

and the punctured tubular neighborhood of E in X is then given by:

$$T^*_{X,E} := \bigcup_{\emptyset \neq I \subseteq J} T^*_{E_I^\circ}.$$

By construction, the homotopy types of the (germs of the) punctured neighborhood $T^*_{X,E}$ and projection map $T^*_{E_I^\circ} \to E_I^\circ$ are well-defined (i.e., independent of all choices). Moreover, $T^*_{X,E}$ is a union of locally trivial topological fibrations $T^*_{E_I^\circ} \to E_I^\circ$ over the strata E_I° (with $\emptyset \neq I \subseteq J$) of E, the fiber of the latter fibration being homeomorphic to $(\mathbb{C}^*)^{|I|}$, where $|I|$ denotes the number of elements in the set I.

Note that the punctured neighborhood $T^*_{X,E}$ is homotopy equivalent to the boundary of the regular neighborhood $T_{X,E}$, which sometimes is called the *link of E in X*.

Definition 2.1 *(Infinite cyclic cover of finite type).* Let $\Delta : \pi_1(T^*_{X,E}) \twoheadrightarrow \mathbb{Z}$ be an epimorphism,[1] and denote by $\widetilde{T}^*_{X,E,\Delta}$ the corresponding infinite cyclic cover (with Galois group $\mathbb{Z}$) of the punctured neighborhood $T^*_{X,E}$ of a simple normal crossing divisor $E \subset X$. For any $i \in J$, let δ_i be the boundary of a small (oriented) disk transversal to the irreducible component E_i. We call the infinite cyclic cover $\widetilde{T}^*_{X,E,\Delta}$ *of finite type* if $m_i = \Delta(\delta_i) \neq 0$ for all $i \in J$ (see Proposition 2.4 below for a justification of the terminology).

Remark 2.2. The surjectivity of the restriction of Δ on the kernel of $\pi_1(T^*_E) \to \pi_1(T_E) = \pi_1(E)$ is equivalent to the condition $\gcd(m_i|i \in J) = 1$. Sometimes we omit Δ and X in the notation and write simply $\widetilde{T}^*_E$. The map Δ will also be referred to as the *holonomy* of this infinite cyclic cover. Note also that Δ factors through $H_1(T^*_E)$, so the infinite cyclic covering $\widetilde{T}^*_{X,E,\Delta}$ depends only on the epimorphism $H_1(T^*_E) \to \mathbb{Z}$. Therefore, in the following we can assume that E and T^*_E are connected, and the choice of the basepoint for $\pi_1(T^*_E)$ has no relevance.

The infinite cyclic cover $\widetilde{T}^*_{X,E,\Delta}$ has the structure of complex manifold, but it is not an algebraic variety. In the following section, we will give an algebro-geometric (motivic) realization of $\widetilde{T}^*_{X,E,\Delta}$. The type of algebraic structure we consider is specified further in the following definition.

Definition 2.3. Let $T_1 = T_{X_1,E_1}$ and $T_2 = T_{X_2,E_2}$ be two regular neighborhoods of normal crossing divisors, and $\Delta_i : \pi_1(T^*_i) \to \mathbb{Z}$, $i = 1, 2$, be surjections on the fundamental groups of the corresponding punctured neighborhoods. We say that (T^*_1, Δ_1) and (T^*_2, Δ_2) are equivalent if there exist a birational map $\Phi : X_1 \to X_2$, which is regular on $T^*_1 \subset X_1$ (and respectively, Φ^{-1} is regular on $T^*_2 \subset X_2$), and which moreover induces a map $\Phi|_{T^*_1} : T^*_1 \to T^*_2$ such that $\Phi(T^*_1)$ and T^*_2 are deformation retracts of each other and the diagram:

[1] The surjectivity assumption is made here solely for convenience (in which case the corresponding infinite cyclic cover is connected), all results in this paper being valid for arbitrary homomorphisms to $\mathbb{Z}$. The only instance when non-surjective homomorphisms are considered is in Section 5.

$$\begin{array}{ccc} \pi_1(T_1^*) & \xrightarrow{(\Phi|_{T_1^*})_*} & \pi_1(T_2^*) \\ & \Delta_1 \searrow \quad \swarrow \Delta_2 & \\ & \mathbb{Z} & \end{array} \tag{2}$$

is commutative. Here $(\Phi|_{T_1^*})_*$ is the homomorphism induced by $\Phi|_{T_1^*}$ on the fundamental groups.

The following result justifies the terminology used in Definition 2.1. We will use rational coefficients, unless otherwise stated.

Proposition 2.4. *Let $\widetilde{T}_E^*$ be an infinite cyclic cover of finite type (as in Definition 2.1). Then for any $i \in \mathbb{Z}$, the rational vector spaces $H_c^i(\widetilde{T}_E^*)$ and $H^i(\widetilde{T}_E^*)$ are finite dimensional. Moreover, these cohomology groups are trivial for $|i|$ large enough.*

Proof. We begin by discussing the case of $H_c^i(\widetilde{T}_E^*)$. First note that, under the action of the group $\mathbb{Z}$ of deck transformations, the cohomology groups $H_c^i(\widetilde{T}_E^*)$ become in a usual way $\mathbb{Q}[\mathbb{Z}] \simeq \mathbb{Q}[t, t^{-1}]$-modules. Then it suffices to show that $H_c^i(T_E^*; \mathscr{L})$ is a finite dimensional rational vector space, where $\mathscr{L}$ is the local coefficient system on T_E^* with stalk $\mathbb{Q}[t, t^{-1}]$ corresponding to the representation defined on the meridians δ_i by $\delta_i \mapsto t^{m_i}$, $i \in J$.

Recall that T_E^* is a union of locally trivial fibrations $T_{E_I^\circ}^* \to E_I^\circ$ over the strata E_I° (with $\emptyset \neq I \subseteq J$) of E, the fiber of the latter fibration being homeomorphic to $(\mathbb{C}^*)^{|I|}$, where $|I|$ denotes the number of elements in the set I. Moreover, T_E^* has an open cover consisting of the sets $\{T_{E_i^\circ}^*\}_{i \in J}$, with intersections given by $\bigcap_{i \in I} T_{E_i^\circ}^* = T_{E_I^\circ}^*$. So by the associated Mayer–Vietoris spectral sequence, it suffices to show that each vector space $H_c^i(T_{E_I^\circ}^*; \mathscr{L})$ (with the induced local coefficients) is finite dimensional.

The claim follows from the Leray spectral sequence for the fibration $T_{E_I^\circ}^* \to E_I^\circ$, i.e.,

$$E_2^{p,q} = H_c^p(E_I^\circ; \mathcal{H}_c^q((\mathbb{C}^*)^{|I|}; \mathscr{L})) \Longrightarrow H_c^{p+q}(T_{E_I^\circ}^*; \mathscr{L})$$

since the (stalk of the local) coefficients $H_c^q((\mathbb{C}^*)^{|I|}; \mathscr{L})$ appearing in the E_2-term are torsion $\mathbb{Q}[t, t^{-1}]$-modules, hence finite dimensional vector spaces. Indeed, the torsion property follows from the assumption that $m_i \neq 0$, for all $i \in J$.

The case of $H^i(\widetilde{T}_E^*)$ follows now by Poincaré duality. $\square$

Remark 2.5. The above proof shows in fact that the cohomology groups $H_c^i(\widetilde{T}_E^*)$ and $H^i(\widetilde{T}_E^*)$ are torsion $\mathbb{Q}[t, t^{-1}]$-modules of finite type. Since $\mathbb{Q}[t, t^{-1}]$ is a principal ideal domain, it follows that $H_c^i(\widetilde{T}_E^*)$ has a well-defined associated order $\Delta_i(t)$, called the *i-th Alexander polynomial* of E, see [25]. Note that $\Delta_i(t)$ can be identified with the characteristic polynomial $\det(t \cdot \mathrm{Id} - T_i^*)$ of the (monodromy) action induced by the generating deck transformation T on $H_c^i(\widetilde{T}_E^*)$. Then it follows from the arguments used in the proof of Proposition 2.4 that all roots of the Alexander polynomials $\Delta_i(t)$ are roots of unity, so in particular, the semi-simple part of T_i^* is a finite order automorphism.

Remark 2.6. Proposition 2.4 motivates our search for a "motive" (in the sense of Section 3), whose Betti realization is that of $\widetilde{T}^*_E$ (cf. Section 4). Note that if instead of the punctured neighborhood T^*_E of E in X we consider the complement $X \setminus E$, then the corresponding infinite cyclic cover is in general not of finite type, in the sense that some of its cohomology groups can be infinite dimensional. In the case of complements to projective hypersurfaces see [9,23] for such an example.

Let us now consider the local situation of the hypersurface singularity germ which, together with the work of Denef–Loeser about the motivic Milnor fiber, inspired our Definition 2.1 and the results of the following sections.

Let $f(x_1, \cdots, x_n) = 0$ define a hypersurface singularity germ at the origin in $\mathbb{C}^n$. Let B_ϵ be a small enough ball about the origin in $\mathbb{C}^n$ and D^*_δ a small punctured disc in $\mathbb{C}$, for $0 < \delta << \epsilon$. Set $B_{\epsilon,\delta} := B_\epsilon \cap f^{-1}(D^*_\delta)$ and let $F = \{f = 0\} \cap B_\epsilon$. By Milnor's fibration theorem [24], one has a locally trivial topological fibration $\pi : B_{\epsilon,\delta} \to D^*_\delta$, whose fiber M_f is called the *Milnor fiber* of f at the origin. If $\exp : \mathbb{R} \to S^1 \simeq D^*_\delta$ is the universal covering map, then the fiber product $B_{\epsilon,\delta} \times_{D^*_\delta} \mathbb{R}$ formed by using the above maps π and exp is the infinite cyclic cover of $B_{\epsilon,\delta}$ corresponding to the homomorphism $\pi_1(B_{\epsilon,\delta}) \to \pi_1(D^*_\delta) = \mathbb{Z}$ given by the linking number with F. The covering map is just the projection of the fiber product on the first factor. Note that if $f = \prod_i f_i^{m_i}$ is the decomposition of the germ f as a product of distinct irreducible factors, the linking number homomorphism is defined by mapping the meridian generators δ_i of $\pi_1(B_{\epsilon,\delta})$ to $m_i \in \mathbb{Z}$, with $m_i \neq 0$ for each i. Moreover, if $m = \gcd(m_i)_i$, this infinite cyclic cover has exactly m connected components. On the other hand, the second projection of $B_{\epsilon,\delta} \times_{D^*_\delta} \mathbb{R} \to \mathbb{R}$ has the same fiber over $r \in \mathbb{R}$ as the Milnor fibration has over $\exp(r)$. Since $\mathbb{R}$ is contractible, we obtain the homotopy equivalence between the infinite cyclic cover of $B_{\epsilon,\delta}$ and the Milnor fiber M_f, hence this (local) infinite cyclic cover is of finite type (since M_f is so). Note that under this identification the monodromy of π corresponds to the deck transformation of the infinite cyclic cover, see also [8, pp. 106–107], and [18].

3. Motivic infinite cyclic covers

Most of our calculations will be done in the Grothendieck ring $K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{C}})$ of the category $\mathrm{Var}^{\hat{\mu}}_{\mathbb{C}}$ of complex algebraic varieties endowed with good $\hat{\mu}$-actions. Let us briefly recall the relevant definitions, e.g., see [7].

For a positive integer n, we denote by μ_n the group of all n-th roots of unity (in $\mathbb{C}$). The groups μ_n form a projective system with respect to the maps $\mu_{d\cdot n} \to \mu_n$ defined by $\alpha \mapsto \alpha^d$, and we denote by $\hat{\mu} := \lim \mu_n$ the projective limit of the μ_n.

Let X be a complex algebraic variety. A *good μ_n-action* on X is an algebraic action $\mu_n \times X \to X$, such that each orbit is contained in an affine subvariety of X. (This last condition is automatically satisfied if X is quasi-projective.) A *good $\hat{\mu}$-action* on X is a $\hat{\mu}$-action which factors through a good μ_n-action, for some n.

The Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$ of the category $\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}}$ of complex algebraic varieties endowed with a good $\hat{\mu}$-action is generated by classes $[Y, \sigma]$ of isomorphic varieties endowed with good $\hat{\mu}$-actions, modulo the following relations:

(i) *scissor relation:*

$$[Y, \sigma] = [Y \setminus Y', \sigma_{|_{Y \setminus Y'}}] + [Y', \sigma_{|_{Y'}}], \tag{3}$$

if Y' is a closed σ-invariant subset of Y.

(ii) *product relation:*

$$[Y \times Y', (\sigma, \sigma')] = [Y, \sigma][Y', \sigma']. \tag{4}$$

(iii)

$$[Y \times \mathbb{A}^1_{\mathbb{C}}, \sigma] = [Y \times \mathbb{A}^1_{\mathbb{C}}, \sigma'], \tag{5}$$

if σ and σ' are two affine liftings of the same $\mathbb{C}^*$-action on Y.

The third relation above is included for completeness, though it is not needed in this paper. We denote by $\mathbb{L}$ the class in $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$ of $\mathbb{A}^1_{\mathbb{C}}$, with the trivial $\hat{\mu}$-action.

The following topological lemma provides the crucial ingredients for our definition of motivic infinite cyclic covers.

Lemma 3.1. *Let A, B, C be connected topological spaces and let $A \to B$ be a locally trivial topological fibration with fiber C, so we have an exact sequence*

$$\pi_1(C) \to \pi_1(A) \to \pi_1(B) \to 0.$$

Let G be a group and let $\widetilde{A}$ be the covering space of A with Galois group G and holonomy map $\alpha : \pi_1(A) \to G$. Then $\widetilde{A}$ is a disjoint union of $[G : \mathrm{Im}\alpha]$ homeomorphic connected components. Let H be the image of composition $\pi_1(C) \to \pi_1(A) \xrightarrow{\alpha} G$ and denote by $\widetilde{C}$ the corresponding covering of C with Galois group H. Then there is a locally trivial topological fibration $\widetilde{A} \to \widetilde{B}$ with connected fiber $\widetilde{C}$, where $\widetilde{B}$ is the cover of B corresponding to the map $\pi_1(B) = \pi_1(A)/\mathrm{Im}\pi_1(C) \to G/H$. The number of (homeomorphic) connected components of $\widetilde{B}$ is equal to the index $[G : \mathrm{Im}\alpha]$.

If $A \cong B \times C \to B$ is a trivial fibration, then also $\widetilde{A} \cong \widetilde{B} \times \widetilde{C} \to \widetilde{B}$ is the projection of a trivial fibration (by using the group isomorphism $\pi_1(A) \cong \pi_1(B) \times \pi_1(C)$).

Proof. Clearly the map $A \to B$ induces the map of covering spaces $\widetilde{A} \to \widetilde{B}$. Indeed, one can view $\widetilde{A}$ as $\widetilde{A}' \times_{\mathrm{Im}\alpha} G$, where $\widetilde{A}'$ is the space of paths with initial point at the base point of A modulo the equivalence relation that identifies paths with the same end point such that the corresponding loop belongs to $\mathrm{Ker}(\alpha)$. The action of $\mathrm{Im}\alpha$ on $\widetilde{A}'$ follows from this description of $\widetilde{A}'$ and the action of Imα on G is via left multiplication. The space $\widetilde{A}$ can be viewed as the set of equivalences classes of pairs $(a, g), a \in \widetilde{A'}, g \in G$, such that (a_1, g_1) and (a_2, g_2) are equivalent iff there exists $h \in \mathrm{Im}\alpha$ such that $a_1 = ha_2, g_1 = hg_2$. The group G acts freely on $\widetilde{A}$ via action on the second factor and one has the canonical identification $\widetilde{A}/G = A$.

Next, we apply the same construction to the homomorphism

$$\pi_1(B) = \pi_1(A)/\mathrm{Im}\pi_1(C) \to G/H$$

to obtain the covering space $\widetilde{B}$ of B with covering group G/H. Writing $\widetilde{B}$ as a fiber product of a path space as above, one sees that the map from the space of paths of A to the space of paths of B starting at the respective base points induced by the map $A \to B$, is compatible with the above mentioned equivalences. Thus, we have a map $\widetilde{A} \to \widetilde{B}$.

The stabilizer of the fiber $\widetilde{C}$ of $\widetilde{A} \to \widetilde{B}$ is H. Finally the G-orbit of any point in $\widetilde{C}$ intersects $\widetilde{C}$ in its H-orbit. Hence $\widetilde{C}/H = C$. □

We shall apply the constructions of the Lemma 3.1 to describe certain covering spaces associated with the punctured neighborhoods of strata of normal crossings divisors.

Lemma 3.2. *Let (X, E) be as in the beginning of Section 2 and let $I \subseteq J$ such that $|I| = r$. The projection of the punctured neighborhood $T^*_{E_I^\circ}$ onto the stratum E_I° induces an exact sequence*

$$\mathbb{Z}^r \to \pi_1(T^*_{E_I^\circ}) \to \pi_1(E_I^\circ) \to 0. \tag{6}$$

Let $\Delta : \pi_1(T^*_E) \to \mathbb{Z}$ be a homomorphism onto an infinite cyclic group as in Definition 2.1. Let N be the index of the image subgroup $\Delta(\pi_1(T^*_{E_I^\circ}))$ in $\mathbb{Z}$ and, similarly, let M be the index of the image of $\mathbb{Z}^r$ in $\mathbb{Z}$. Let

$$\Delta_I : \pi_1(E_I^\circ) \to \mathbb{Z}/M\mathbb{Z} \tag{7}$$

be the map induced by Δ according to Lemma 3.1. Then the corresponding covering $\widetilde{E_I^\circ} \to E_I^\circ$ induced by Δ_I has N connected components, each being the cyclic cover of E_I° with the covering group $N\mathbb{Z}/M\mathbb{Z}$. Moreover, the infinite cyclic cover $\widetilde{T}^*_{E_I^\circ}$ (defined by $\ker(\Delta)$) fibers over $\widetilde{E}_I^\circ$, with fiber $(\mathbb{C}^*)^{r-1}$.

Proof. The exact sequence (6) is derived from the long exact sequence of homotopy groups associated to a locally trivial topological fibration, by using the connectivity of the fiber. Indeed, the fiber of

$$T^*_{E_I^\circ} \to E_I^\circ$$

is diffeomorphic to $(\mathbb{C}^*)^r$, hence $\pi_1((\mathbb{C}^*)^r) = \mathbb{Z}^r$ and $\pi_0((\mathbb{C}^*)^r) = 0$.

Let us discuss the second statement. First note that the image $\mathrm{Im}(\mathbb{Z}^r \to \pi_1(T^*_{E_I^\circ}))$ is the subgroup generated by the meridians δ_i with $i \in I$. Moreover,

$$\Delta(\mathrm{Im}(\mathbb{Z}^r \to \pi_1(T^*_{E_I^\circ}))) = M\mathbb{Z},$$

with

$$M = m_I := \gcd(m_i \mid i \in I).$$

Since the homomorphism $\pi_1(T^*_{E_I^\circ}) \to \pi_1(E_I^\circ)$ is surjective, it follows that the homomorphism $\Delta : \pi_1(T^*_{E_I^\circ}) \to \mathbb{Z}$ factors through $\pi_1(E_I^\circ)$. Hence we get a well-defined map

$$\Delta_I : \pi_1(E_I^\circ) \twoheadrightarrow N\mathbb{Z}/M\mathbb{Z}$$

given by $\Delta_I(\epsilon) = \Delta(\delta)$ for any $\epsilon \in \pi_1(E_I^\circ)$ and any $\delta \in \pi_1(T^*_{E_I^\circ})$ such that $\delta \mapsto \epsilon$.

Finally, by Lemma 3.1, the long exact sequence of homotopy groups associated to the $(\mathbb{C}^*)^r$-fibration $T^*_{E_I^\circ} \to E_I^\circ$ induces a locally trivial topological fibration

$$\widetilde{T}^*_{E_I^\circ} \to \widetilde{E}_I^\circ, \tag{8}$$

with connected fiber $\widetilde{(\mathbb{C}^*)^r} \simeq (\mathbb{C}^*)^{r-1}$, the infinite cyclic cover of $(\mathbb{C}^*)^r$ defined by the kernel of the epimorphism $\mathbb{Z}^r \twoheadrightarrow m_I\mathbb{Z}$ induced by the holonomy map Δ.

We conclude the proof by noting that the definition of the map (7) and the finite (algebraic) covering $\widetilde{E}_I^\circ$ depend on the (homotopy class of the) projection $T^*_{E_I^\circ} \to E_I^\circ$, but they are nevertheless intrinsic objects associated to our context. □

Definition 3.3. We denote by $\widetilde{E_I^\circ}$ the unramified cover of E_I° with Galois group $\mathbb{Z}/M\mathbb{Z}$ defined by the map (7), and with $M = m_I := \gcd(m_i \mid i \in I)$. The cover $\widetilde{E_I^\circ}$ is an algebraic variety with a good μ_M-action σ_I such that $E_I^\circ = \widetilde{E_I^\circ}/\mu_M$. It has N connected components. The fundamental group $\pi_1(\widetilde{E_I^\circ})$ is isomorphic to $\mathrm{Ker}(\Delta_I)$.

Remark 3.4. The proof of Lemma 3.2 applies to the following more general situation. Let $\mathcal{F} \to W$ be a vector bundle on a quasi-projective manifold W, and let $\{E_i \subset \mathcal{F} \mid i \in I\}$ be a collection of $|I| \geq 1$ independent sub-bundles of $\mathcal{F}$ of corank 1 (in particular, the collection $\{E_i \mid i \in I\}$ forms a normal crossing divisor in $\mathcal{F}$). Then one has a locally trivial topological fibration $\mathcal{F}^* := \mathcal{F} \setminus \cup_{i \in I} E_i \to W$ with fiber F homotopy equivalent to $(\mathbb{C}^*)^{|I|}$. Moreover, a homomorphism $\pi_1(\mathcal{F}^*) \to \mathbb{Z}$ with image $N\mathbb{Z}$ and so that the image of $\pi_1(F)$ is a subgroup of finite index M in $N\mathbb{Z}$, defines an infinite cyclic cover $\widetilde{\mathcal{F}}^*$ of $\mathcal{F}^*$ having N connected components, each of which is a locally trivial topological fibration with fiber equivalent to $(\mathbb{C}^*)^{|I|-1}$ and base $\widetilde{W}$ being an M-fold cyclic cover of W.

We now have all the ingredients for defining the main object of the paper:

Definition 3.5 *(Motivic infinite cyclic cover).* Let T_E^* be a punctured neighborhood of a normal crossing divisor in a quasi-projective manifold X as in Section 2, and let $\Delta : \pi_1(T_E^*) \to \mathbb{Z}$ be a surjection such that the corresponding infinite cyclic cover $\widetilde{T}^*_{X,E,\Delta}$ has a finite type. For each fixed subset $A \subseteq J$, we define the corresponding motivic infinite cyclic cover (of finite type) of T_E^* as

$$S^A_{X,E,\Delta} := \sum_{\substack{\emptyset \neq I \subseteq J \\ A \cap I \neq \emptyset}} (-1)^{|I|-1} [\widetilde{E_I^\circ}, \sigma_I](\mathbb{L}-1)^{|I|-1} \in K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}}), \tag{9}$$

where $\widetilde{E_I^\circ}$ are the covering spaces corresponding to Δ_I in Definition 3.3.

When $A = J$, we use the notation $S_{X,E,\Delta}$ or $S_{X,E}$.

Remark 3.6. Recall from Lemma 3.2 that the infinite cyclic cover $\widetilde{T}^*_{E_I^\circ}$ of $T^*_{E_I^\circ}$ is a $(\mathbb{C}^*)^{|I|-1}$-fibration over $\widetilde{E}_I^\circ$. Therefore, one can regard the product $[\widetilde{E_I^\circ}, \sigma_I](\mathbb{L}-1)^{|I|-1}$ appearing in (9) as a "motive" of the infinite cyclic cover $\widetilde{T}^*_{E_I^\circ}$, while the alternating sum on the right-hand side of (9) can be interpreted as the inclusion-exclusion principle for the cover $T_E^* = \bigcup_{\emptyset \neq I \subseteq J} T^*_{E_I^\circ}$.

The main result of this section is the following.

Theorem 3.7. *The above notion of motivic infinite cyclic cover is invariant under the equivalence relation described in Definition 2.3.*

Since any birational map $X_1 \to X_2$ providing an equivalence between punctured neighborhoods (cf. Definition 2.3) is, by the Weak Factorization Theorem [1] (see also [4] for the non-complete case), a composition of blow-ups and blow-downs, each inducing an equivalence between the corresponding punctured neighborhoods, it suffices to show that the above expression (9) is invariant under blowing up along a smooth center in E. Let us consider

$$p : X' := Bl_Z X \to X$$

the blow-up of X along the smooth center $Z \subset E$ of codimension ≥ 2 in X. Let us denote by E_* the exceptional divisor of the blow-up p, which is isomorphic to the projectivized normal bundle over Z, i.e., $E_* \cong \mathbb{P}(\nu_Z)$. We may also assume that the center Z of the blow-up is contained in E and has normal crossings with the components of E (cf. [1, Theorem 0.3.1, (6)]). Let us denote the preimage of the divisor E_i in X' by E_i'. Denote by E' the normal crossing divisor in X' formed by the E_i' together with E_*. Denote by $J' = J \cup \{*\}$ the family of indices of the divisor E'. For $I \subseteq J$ we denote by $I' \subseteq J'$ the family $I \cup \{*\}$. Finally, let $A' = A \cup \{*\}$.

By the above reduction to the normal crossing situation, we may assume that there is $I \subseteq J$ such that Z is contained in E_I. We consider the (surjective) homomorphism given by the composition

$$\Delta' : \pi_1(T^*_{X',E'}) \to \pi_1(T^*_{X,E}) \xrightarrow{\Delta} \mathbb{Z}$$

resulting from the identification $T^*_{X'E'} \xrightarrow{\cong} T^*_{X,E}$ induced by the blow-down map. We have $\Delta'(\delta'_i) = \Delta(\delta_i) = m_i$ $(i \in I)$ and $m_* := \Delta'(\delta_*) = \sum_{i\in I} m_i$, where δ'_i and δ_* are the meridians about the components E'_i and E_* of E'. Indeed, the blow-down map takes the 2-disk transversal to E_* (at a generic point) and bounded by δ_*, to the disk in X transversal to the components $E_i, i \in I$ containing Z and disjoint from the remaining components of E, i.e., one has the relation $\delta_* = \sum_{i\in I} \delta_i$ in $H_1(T^*_E)$. Note that $\widetilde{T}^*_{X',E',\Delta'}$ is of finite type since $\widetilde{T}^*_{X,E,\Delta}$ is so and $T^*_{X'E'} \cong T^*_{X,E}$. Moreover, if $m_* \neq 0$, then by Lemma 3.2 and Definition 3.3 applied to (X', E', Δ') we can define the corresponding motivic infinite cyclic cover by:

$$S^{A'}_{X',E',\Delta'} := \sum_{\substack{\emptyset\neq K\subset J' \\ K\cap A'\neq\emptyset}} (-1)^{|K|-1}[\widetilde{E^\circ_K}, \sigma_K](\mathbb{L}-1)^{|K|-1}. \tag{10}$$

If $m_* = 0$, then Lemma 3.2 cannot be applied directly for defining a finite cover (as in Definition 3.3) of the dense open stratum E°_* of the exceptional divisor E_*. However, as already pointed out in Remark 3.6, the main ingredient needed at this point for the definition of (10) is the "motive" of the infinite cyclic cover $\widetilde{T}^*_{E^\circ_*}$ of the punctured tubular neighborhood of E°_*. Such a "motive" can be defined by making use of Remark 3.4 as follows. First note that $T^*_{E^\circ_*}$ is a $\mathbb{C}^*$-fibration over E°_*, which in turn is a (Zariski) locally trivial fibration over the open dense stratum $Z^\circ := Z \cap E^\circ_I$ of Z, with fiber $\mathbb{C}^{s-|I|+1}\times(\mathbb{C}^*)^{|I|-1}$, where s is the codimension of Z in E (see the proof of Proposition 3.8 below). Hence $T^*_{E^\circ_*}$ is a $\mathbb{C}^{s-|I|+1} \times (\mathbb{C}^*)^{|I|}$-fibration over Z°, and Remark 3.4 together with Lemma 3.1 can now be used to show that the infinite cyclic cover $\widetilde{T}^*_{E^\circ_*}$ is a $\mathbb{C}^{s-|I|+1}\times (\mathbb{C}^*)^{|I|-1}$-fibration over the m_I-fold cover $\widetilde{Z^\circ}$ of Z° defined as in Lemma 3.2. So, in this case, we can replace the term $[\widetilde{E^\circ_*}]$ of (10) (which would correspond to the "motive" of $\widetilde{T}^*_{E^\circ_*}$) by the product $[\widetilde{Z^\circ}]\mathbb{L}^{s-|I|+1}(\mathbb{L}-1)^{|I|-1}$.

Finally, note that since punctured neighborhoods remain unchanged under blow-ups, it is easy to see by using Lemma 3.1 that, if $m_* \neq 0$, the product $[\widetilde{Z^\circ}]\mathbb{L}^{s-|I|+1}(\mathbb{L}-1)^{|I|-1}$ coincides in fact with the motive $[\widetilde{E^\circ_*}]$, so as it will become clear from the proof of our main theorem it suffices to assume from now on that $m_* \neq 0$.

Theorem 3.7 follows now from the following proposition.

Proposition 3.8. *With the above notations, we have the following equality of motives:*

$$S^A_{X,E,\Delta} = S^{A'}_{X',E',\Delta'} \in K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{C}}). \tag{11}$$

Note that we can always restrict the comparison of motives in Proposition 3.8 to strata in the center of blowup and in the exceptional divisor, respectively. Indeed, the blow-up map induces an isomorphism outside the center Z, so the strata in $E \setminus Z$ and $E' \setminus E_*$ are in one-to-one isomorphic correspondence; moreover, these isomorphisms can be lifted (e.g., by Lemma 3.2) to the corresponding unramified covers. It also suffices to prove the above result only in the case $A = J$.

The proof of Proposition 3.8 is by induction on the dimension of the center of blow-up.

3.1. Beginning of induction

Let us consider the following examples in relation to the starting case of induction, i.e., when the center Z is a point.

Example 3.9. Let X be a surface and let E_1 and E_2 be two smooth curves intersecting transversally at a point P. Let us consider the blow-up $X' = Bl_Z X$ of X at the center $Z = P$. The exceptional divisor is $E_* \cong \mathbb{P}^1$ and we have $E_*^\circ \cong \mathbb{C}^*$. Let $\delta_i \in H_1(T^*_{E_i^\circ}, \mathbb{Z})$ $(i = 1, 2)$ be the class of the fiber of the projection of punctured neighborhood $T^*_{E_i^\circ}$ onto the stratum E_i°. If $\Delta(\delta_1) = m_1$ and $\Delta(\delta_2) = m_2$, the contribution of P to $S_{X,E}$ is $-[\mu_{\gcd(m_1,m_2)}](\mathbb{L} - 1)$ and the contributions of the exceptional divisor E_* to $S_{X',E'}$ are[2]

$$[\widetilde{E_*^\circ}, \sigma_{\Delta'}] - \left([\widetilde{E_1' \cap E_*}, \sigma_{\Delta'}] + [\widetilde{E_2' \cap E_*}, \sigma_{\Delta'}]\right)(\mathbb{L} - 1).$$

Because $\gcd(m_1, m_2) = \gcd(m_1, m_1 + m_2) = \gcd(m_2, m_1 + m_2)$, we get: $[\widetilde{E_1' \cap E_*}, \sigma_{\Delta'}] = [\widetilde{E_2' \cap E_*}, \sigma_{\Delta'}] = [\mu_{\gcd(m_1,m_2)}]$. Finally, Lemma 3.2 asserts that $[\widetilde{E_*^\circ}, \sigma_{\Delta'}] = [\mu_{\gcd(m_1,m_2)}] \times (\mathbb{L}-1)$. To see this directly, let us describe explicitly the covering space of E_*° according to Lemma 3.2. In the notations of the above-mentioned lemma, we have that $M = m_1 + m_2$ and $N = \gcd(m_1, m_2)$. Indeed, denoting by δ_* the homology class of the meridian about E_*°, the homomorphism defining the infinite cyclic cover of the punctured neighborhood of E_*° is given by $\delta_i \mapsto m_i$ $(i = 1, 2)$ and $\delta_* \mapsto m_1 + m_2$. So Lemma 3.2 yields that the Galois group of the cover $\widetilde{E_*^\circ} \to E_*^\circ$ is $\mathbb{Z}/(m_1 + m_2)$ and, moreover, $\widetilde{E_*^\circ}$ has $\gcd(m_1, m_2)$ connected components, each being a connected cyclic cover of $\mathbb{C}^*$ (of degree $\frac{m_1+m_2}{\gcd(m_1,m_2)}$). Such a connected cover is biregular to $\mathbb{C}^*$, i.e., its motive is $\mathbb{L} - 1$, hence the motive of $\widetilde{E_*^\circ}$ is $[\mu_{\gcd(m_1,m_2)}](\mathbb{L} - 1)$. It follows that both contributions to the motivic infinite cyclic cover coincide.

Note that in the case when P belongs to only one irreducible component, say E_1, we have $[\widetilde{E_*^\circ}, \sigma_\Delta] = [\mu_{m_1}]\mathbb{L}$. In this case, the contribution to $S_{X',E'}$ is $[\widetilde{E_*^\circ}, \sigma_{\Delta'}] - [\widetilde{E_1' \cap E_*}, \sigma_{\Delta'}](\mathbb{L} - 1) = [\mu_{m_1}]\mathbb{L} - [\mu_{m_1}](\mathbb{L} - 1) = [\mu_{m_1}]$. This coincides with the contribution of P to $S_{X,E}$ which is $[\widetilde{E_1^\circ}|_P] = [\mu_{m_1}]$. □

[2] For simplicity, here and in the sequel we denote by $\sigma_{\Delta'}$ the good $\hat{\mu}$-action on the corresponding finite cover (defined by using the holonomy Δ') of a stratum in the exceptional divisor, cf. Definition 3.3.

Example 3.10. Let X be a threefold and E_1, E_2, E_3 be three divisors intersecting transversally at a point P. Consider the blow-up X' of X at the center $Z = P$. The divisor $E = \sum_i E_i$ of X transforms into the divisor E' in X' consisting of the proper transforms E'_i of the irreducible components E_i of E ($i = 1, 2, 3$), together with the exceptional component $E_* \cong \mathbb{P}^2$. As already mentioned, it suffices to restrict the comparison of motives $S_{X,E}$ and $S_{X',E'}$ only to contributions coming from the strata in the center of blowup and the exceptional divisor, respectively.

The exceptional divisor E_* acquires seven strata induced from the stratification of E'. These strata are:

- $L_{\{i,j\}} = E_* \cap E'_i \cap E'_j$, for $i, j \in \{1, 2, 3\}$ with $i \neq j$,
- $L_{\{i\}} = (E_* \cap E'_i) \setminus (L_{\{i,j\}} \cup L_{\{i,k\}})$, with $\{i, j, k\} = \{1, 2, 3\}$,
- $E_*^\circ = E_* \setminus \bigcup_{i=1}^3 E'_i$.

Note that the strata E_*° and $L_{\{i\}}$ are complex tori of dimension 2 and 1, respectively, while the strata $L_{\{i,j\}}$ are points.

Let T_S^* denote the punctured neighborhood (in X') of a stratum S in E_*, and let N_S denote the fiber of the projection $T_S^* \to S$. The fibers $N_{E_*^\circ}$, $N_{L_{\{i\}}}$, $N_{L_{\{i,j\}}}$ corresponding to the punctured neighborhoods of the strata in E_* are homotopy equivalent to real tori of dimensions 1, 2 and 3, respectively. The first homology group $H_1(T_S^*, \mathbb{Z})$ of the punctured neighborhood of a stratum S is generated by the image of $H_1(N_S, \mathbb{Z})$ under the homomorphism Δ', together with the classes of boundaries of normal disks (i.e., meridians) to components E'_i which intersect the closure of S. This observation allows us to calculate the image subgroup $\Delta'(\pi_1(T_S^*))$, which for each of the seven strata of E_* results in

$$\Delta'(\pi_1(T_S^*)) = \gcd(m_1, m_2, m_3)\mathbb{Z}. \tag{12}$$

Indeed, the images of homomorphisms

$$\Delta'_{N_S} : H_1(N_S, \mathbb{Z}) \to H_1(T_S^*, \mathbb{Z}) \to \mathbb{Z}$$

for each of the strata in E_* are given as follows:

- $\mathrm{Im}\Delta'_{N_{E_*^\circ}} = (m_1 + m_2 + m_3)\mathbb{Z}$, by the definition of the homomorphism Δ' on the meridian δ_* about E_*.
- $\mathrm{Im}\Delta'_{N_{L_{\{i\}}}} = \gcd(m_i, m_1 + m_2 + m_3)\mathbb{Z}$. Indeed, $H_1(N_{L_i}, \mathbb{Z})$ is generated by the meridian about the exceptional component E_* and the meridian about E'_i (which also can be viewed as a meridian of E_i).
- Similarly, $\mathrm{Im}\Delta'_{N_{L_{\{i,j\}}}} = \gcd(m_i, m_j, m_1 + m_2 + m_3)\mathbb{Z}$, for $i, j \in \{1, 2, 3\}$ with $i \neq j$.

So, to verify (12) for a stratum $L_{\{i\}}$, we use the fact that the homomorphism Δ' factors through the abelianization map, and the fact mentioned above that the image $\Delta'(H_1(T^*_{L_{\{i\}}}, \mathbb{Z}))$ is generated by $\gcd(m_i, \sum_{j=1}^3 m_j)$ and by the integers m_j, $j \in \{1,2,3\} \setminus \{i\}$ (which are the values of the holonomy on the boundaries of normal disks to components E'_j, $j \neq i$, which intersect the closure of $L_{\{i\}}$). Similarly, the image $\Delta'(H_1(T^*_{E^\circ_*}, \mathbb{Z}))$ is generated by $\sum_{j=1}^3 m_j$ and by the integers m_i, $i = 1,2,3$, corresponding to the values of the holonomy on the meridians to the components E'_i, $i = 1,2,3$, all of which intersect the closure E_* of E°_*. Finally, for a stratum $L_{\{i,j\}}$, the image $\Delta'(H_1(T^*_{L_{\{i,j\}}}, \mathbb{Z}))$ is generated by $\gcd(m_i, m_j, m_1+m_2+m_3) = \gcd(m_1, m_2, m_3)$.

It follows from Lemma 3.2 that for each of seven strata of E_*, the corresponding unbranched covers of Definition 3.3 have $\gcd(m_1, m_2, m_3)$ components, each of which is biregular to the stratum itself (since all these strata are tori). Hence the contribution of E_* to $S_{X',E'}$ is:

$$[\mu_{\gcd(m_1,m_2,m_3)}]\left((\mathbb{L}-1)^2 - 3(\mathbb{L}-1)(\mathbb{L}-1) + 3(\mathbb{L}-1)^2\right)$$

which equals the contribution of P to $S_{X,E}$, given by $[\mu_{\gcd(m_1,m_2,m_3)}](\mathbb{L}-1)^2$. $\square$

Example 3.11. Let X be a threefold, and $E = E_1 + E_2$ be a simple normal crossing divisor on X, with holonomy values m_1 and resp. m_2 on the meridians about its irreducible components. Let $m = \gcd(m_1, m_2)$. Choose a point Z contained in the (one-dimensional) intersection $E_J := E_1 \cap E_2$, for $J = \{1,2\}$, and consider the blow-up $X' = Bl_Z X$ of X along the center Z. We denote the exceptional divisor $\mathbb{P}(\nu_Z)$ by E_*. The divisor E is transformed under the blow-up into the divisor E' in X' consisting of the proper transforms E'_i $(i \in J)$ of the irreducible components E_i of E, together with the exceptional divisor $E_* \cong \mathbb{P}^2$.

Let us explicitly describe the contribution of the center Z and that of the exceptional divisor E_* to the motives $S_{X,E}$ and $S_{X',E'}$, respectively. Clearly, the class $[\widetilde{E_J}|_Z, \sigma_\Delta]$ equals $[\mu_m]$. So the contribution of Z to $S_{X,E}$ consists of $-[\mu_m](\mathbb{L}-1)$. On the other hand, the exceptional divisor E_* acquires four strata induced from the stratification of E', namely,

- $L_J = E'_1 \cap E'_2 \cap E_*$, which is just a point.
- $L_{\{i\}} = E_* \cap E'_i \setminus L_J \cong \mathbb{C}$, for $i \in I$.
- $E^\circ_* = E_* \setminus (E'_1 \cup E'_2) \cong \mathbb{C} \times \mathbb{C}^*$.

So the contribution of E_* to $S_{X',E'}$ is given by:

$$[\widetilde{E^\circ_*}, \sigma_{\Delta'}] - \left([\widetilde{L_{\{1\}}}, \sigma_{\Delta'}] + [\widetilde{L_{\{2\}}}, \sigma_{\Delta'}]\right)(\mathbb{L}-1) + [\widetilde{L_J}, \sigma_{\Delta'}](\mathbb{L}-1)^2. \tag{13}$$

Note that, since any of the four strata in E_* is either simply-connected or a product of a simply-connected space with a torus, any finite connected (unbranched) cover of such a stratum is biregular to the stratum itself. So in order to understand the motives of covering spaces appearing in (13), it suffices to compute the number of connected components of each cover. This can be done easily by using Lemma 3.2 as follows. First, recall that for a given stratum S of E_*, the number of connected components of the corresponding unbranched cover $\widetilde{S}$ (as in Definition 3.3) equals the index (in $\mathbb{Z}$) of the image (under the homomorphism Δ') of the fundamental group $\pi_1(T_S^*)$ of a punctured neighborhood of S in X'. Moreover, since Δ' factorizes through the abelianization map, it suffices to compute the index $[\mathbb{Z} : \mathrm{Im} H_1(T_S^*)]$. Finally, $H_1(T_S^*)$ is generated by $H_1(N_S)$ together with the (classes of) meridians to components E_i' intersecting the closure of S, where N_S denotes as before the (normal) fiber of the projection $T_S^* \to S$. In our situation, for each of the above strata in E_*, it can be easily seen that the image of $H_1(T_S^*)$ is generated by m_1+m_2, m_1 and m_2, i.e., each of the covering space appearing in (13) has exactly $\gcd(m_1+m_2, m_1, m_2) = m$ connected components. It follows that (13) can be computed as:

$$[\mu_m]\mathbb{L}(\mathbb{L}-1) - 2[\mu_m]\mathbb{L}(\mathbb{L}-1) + [\mu_m](\mathbb{L}-1)^2 = -[\mu_m](\mathbb{L}-1),$$

which equals the contribution of Z to $S_{X,E}$. □

Let us now prove the beginning case of induction for Proposition 3.8.

Proposition 3.12. *The assertion of Proposition 3.8 holds in the case when the center of blow-up Z is zero-dimensional.*

It suffices to prove Proposition 3.12 in the case when the center of blow-up is a single point. Indeed, the blow-up at a finite number of points can be regarded as a finite number of single point blow-ups.

We can thus assume that Z is a point. Let $r+1 = \mathrm{codim}_X Z$, which, by our assumption, equals $\dim X$. Then the exceptional divisor is $E_* \cong \mathbb{P}^r$. The divisor $E = \sum_i E_i$ of X transforms under the blow-up into the divisor E' in X' consisting of the proper transforms E_i' of the irreducible components E_i of E, together with the exceptional component E_*. It suffices to restrict the comparison of motives $S_{X,E}$ and $S_{X',E'}$ only to contributions coming from the strata in the center of blow-up and the exceptional divisor, respectively.

As in the above examples, we need to describe the stratification of $E_* \cong \mathbb{P}^r$ induced from that of E' (see (1) for the latter). Assume that $Z \subseteq \bigcap_{i=1}^k E_i$. We have the following result:

Lemma 3.13. *For each k with $1 \leq k \leq r+1$ we have the following identity in $K_0(\mathrm{Var}_{\mathbb{C}})$:*

$$[\mathbb{P}^r] = \sum_{l=0}^{k-1} \binom{k}{l} \mathbb{L}^{r-k+1}(\mathbb{L}-1)^{k-l-1} + [\mathbb{P}^{r-k}]. \tag{14}$$

The right-hand side describes the stratification of the exceptional divisor $E_* \cong \mathbb{P}^r$ induced by the divisor $\sum_{i=1}^k E'_i$ consisting of the proper transforms of components of E containing the center of blow-up. More precisely, by setting $K := \{1, \cdots, k\}$, the strata of E_* are:

- $L_K := (\bigcap_{i=1}^k E'_i) \cap E_*$, *which is isomorphic to* $\mathbb{P}^{r-k}$.
- $\binom{k}{l}$ *strata of dimension* $r - l$ *and of the form*

$$L_I := (\bigcap_{i \in I} E'_i) \cap E_* \setminus \bigcup_{i \in K \setminus I} E'_i,$$

  with $I \subset K$ *and* $1 \leq |I| = l \leq k - 1$, *each of which is isomorphic to* $\mathbb{C}^{r-k+1} \times (\mathbb{C}^*)^{k-l-1}$. *The class of each such stratum in* $K_0(\mathrm{Var}_{\mathbb{C}})$ *is equal to* $\mathbb{L}^{r-k+1}(\mathbb{L} - 1)^{k-l-1}$.
- $E_*^\circ := E_* \setminus \bigcup_{i=1}^k E'_i$, *of dimension* r, *which is isomorphic to* $\mathbb{C}^{r-k+1} \times (\mathbb{C}^*)^{k-1}$, *and whose class in* $K_0(\mathrm{Var}_{\mathbb{C}})$ *is* $\mathbb{L}^{r-k+1}(\mathbb{L} - 1)^{k-1}$.

Proof. Note that the stratum L_K is just the projectivization of the normal bundle of Z in $\bigcap_{i=1}^k E_i$, i.e., the exceptional divisor of the blow-up of Z inside $\bigcap_{i=1}^k E_i$. Also, the stratum E_*° can be regarded as $L_\emptyset$ (i.e., for $l = 0$), so all strata can be treated uniformly (see below).

We prove the identity (14) by induction on k. For $k = 1$ the equality (14) becomes $[\mathbb{P}^r] = \mathbb{L}^r + [\mathbb{P}^{r-1}]$, which corresponds to the stratification of the projective space consisting of the affine part and the (projective) hyperplane at infinity. Clearly $[L_K] = [\mathbb{P}^{r-1}]$ and $[E_*^\circ] = \mathbb{L}^r$. There are no strata of type L_I with $I \subset K$.

For $k = 2$ we are considering a new irreducible component E_2 of E containing Z. The class $[E_*^\circ]$ transforms from $\mathbb{L}^r$ (for $k = 1$) to $\mathbb{L}^{r-1}(\mathbb{L} - 1)$. Moreover, we have two strata $L_{\{1\}}$ and $L_{\{2\}}$ whose class is $\mathbb{L}^{r-1}$. In this case the equality (14) becomes $[\mathbb{P}^r] = \mathbb{L}^{r-1}(\mathbb{L} - 1) + 2\mathbb{L}^{r-1} + [\mathbb{P}^{r-2}]$.

For the general case, when moving from $k - 1$ to k, the stratum of minimal dimension $r - k + 1$ and with class $[\mathbb{P}^{r-k+1}]$ is subdivided into an affine piece $\mathbb{L}^{r-k+1}$ and (the class of) a hyperplane at infinity $[\mathbb{P}^{r-k}]$. Furthermore, each of the $\binom{k-1}{l}$ strata of dimension $r - l$ and with class $\mathbb{L}^{r-k+2}(\mathbb{L} - 1)^{k-l-2}$ is subdivided into a piece of dimension $r - l$ and type $\mathbb{L}^{r-k+1}(\mathbb{L} - 1)^{k-l-1}$ and a piece of dimension $r - l - 1$ and type $\mathbb{L}^{r-k+1}(\mathbb{L} - 1)^{k-l-2}$. Therefore, for the index k, the number of strata of dimension $r - l$ and type $\mathbb{L}^{r-k+1}(\mathbb{L} - 1)^{k-l-1}$ is the sum of the $\binom{k-1}{l-1}$ strata coming from strata of dimension $r - l + 1$ for the index $k - 1$, and the $\binom{k-1}{l}$ strata of dimension $r - l$ coming from strata of dimension $r - l$ for the index $k - 1$. Therefore, there are $\binom{k}{l} = \binom{k-1}{l-1} + \binom{k-1}{l}$ such strata of dimension $r - l$ for the index k. This proves the lemma. □

Proof of Proposition 3.12. As already pointed out, it suffices to check the invariance (11) of motivic infinite cyclic cover under blow-up in the case when Z is a single point. Assume $Z \subseteq \bigcap_{i=1}^k E_i$. Let $K = \{1, 2, \ldots, k\}$ and set $m = \gcd(m_1, \cdots, m_k)$, where the

m_i are the values of the holonomy on the meridians δ_i about the components E_i. Clearly, the class $[\widetilde{E}^\circ_{K|_Z}, \sigma_\Delta]$ equals $[\mu_m]$. Therefore, the corresponding contribution of Z to the left hand-side of (11) is $(-1)^{k-1}[\mu_m](\mathbb{L}-1)^{k-1}$. On the other hand, for any stratum S of the exceptional divisor E_* (as described in Lemma 3.13), the motive of the corresponding unbranched cover of Definition 3.3 can be computed by:

$$[\widetilde{S}, \sigma_{\Delta'}] = [\mu_m][S, \sigma_\Delta]. \tag{15}$$

In order to see this, we note that, since any such stratum S in E_* is (by Lemma 3.13) either simply-connected or a product of a simply-connected space with a torus, any finite connected (unbranched) cover of S is biregular to S. So in order to prove (15), it suffices to show that the unbranched cover $\widetilde{S}$ of Definition 3.3 has exactly m connected components. This can be done by using Lemma 3.2 as follows. First, recall that for a given stratum S of E_*, the number of connected components of the corresponding unbranched cover $\widetilde{S}$ equals the index in $\mathbb{Z}$ of the image (under the homomorphism Δ') of the fundamental group $\pi_1(T^*_S)$ of a punctured neighborhood of S in X'. Moreover, since Δ' factorizes through the abelianization map, it suffices to compute the index $[\mathbb{Z} : \mathrm{Im} H_1(T^*_S)]$. Finally, $H_1(T^*_S)$ is generated by $H_1(N_S)$ together with the (classes of) meridians to components E'_i intersecting the closure of S, where N_S denotes as before the (normal) fiber of the projection $T^*_S \to S$. In our situation, it is easy to see that, for any stratum S in E_*, we have:

$$[\mathbb{Z} : \mathrm{Im} H_1(T^*_S)] = \gcd(\sum_{i\in K} m_i, m_1, \cdots, m_k) = m. \tag{16}$$

Indeed, for any stratum L_I of E_* (where we also include the extremal cases when $I = \emptyset$ or $I = K$), the image of $H_1(N_{L_I}, \mathbb{Z})$ is generated by the integers $\sum_{i\in K} m_i$ and $\{m_i, i \in I\}$, while the remaining integers $\{m_i, i \in K \setminus I\}$ correspond to the values of holonomy on the meridians about the components E'_i $(i \in K \setminus I)$, all of which intersect the closure of L_I.

Taking into account the description of the stratification in Lemma 3.13 we have now to check that $(-1)^{k-1}[\mu_m](\mathbb{L}-1)^{k-1}$ equals

$$\left(\sum_{l=0}^{k-1}(-1)^l\binom{k}{l}[\mu_m]\mathbb{L}^{r-k+1}(\mathbb{L}-1)^{k-1}\right) + (-1)^k[\mu_m][\mathbb{P}^{r-k}](\mathbb{L}-1)^k.$$

Factoring out $[\mu_m](\mathbb{L}-1)^{k-1}$ and using that $[\mathbb{P}^{r-k}](\mathbb{L}-1) = \mathbb{L}^{r-k+1} - 1$, it remains to show the equality:

$$(-1)^{k-1} = \sum_{l=0}^{k}(-1)^l\binom{k}{l}\mathbb{L}^{r-k+1} + (-1)^{k-1}. \tag{17}$$

And (17) holds because $\sum_{l=0}^{k}(-1)^l\binom{k}{l} = (1-1)^k = 0$. This finishes the proof of Proposition 3.12. $\square$

3.2. Invariance of motivic infinite cyclic covers under blowups: general case

We begin with a few examples.

Example 3.14. Let X be a threefold and $E = \sum_{i\in J} E_i$, with $J = \{1,2,3\}$, be a simple normal crossing divisor. Let Z be the intersection of E_1 and E_2. Set $I = \{1,2\}$, so in the notations from the introduction, we have that $Z = E_I$. The component E_3 is transversal to Z. Let us consider the blow-up $X' = Bl_Z X$ of X along the center Z. As before we denote the exceptional divisor $\mathbb{P}(\nu_Z)$ by E_*.

The strata in Z are $E_I^\circ = E_1 \cap E_2 \setminus E_3$ and the point $E_J = \cap_{i\in J} E_i$, so the contribution of the center Z to the motivic infinite cyclic cover $S_{X,E}$ is:

$$-[\widetilde{E_I^\circ}, \sigma_\Delta](\mathbb{L}-1) + [\widetilde{E_J}, \sigma_\Delta](\mathbb{L}-1)^2$$

The exceptional divisor E_* acquires a stratification with strata of the form:

$$L_I := (\bigcap_{i\in I} E_i') \cap E_* \setminus \bigcup_{i\in J\setminus I} E_i',$$

with $I \subseteq J$, where the dense open stratum E_*° in E_* is identified with $L_\emptyset$. More precisely, the strata of E_* are in this case the following:

- $L_{\{1,3\}} = E_* \cap E_1' \cap E_3'$, $L_{\{2,3\}} = E_* \cap E_2' \cap E_3'$.
- $L_{\{1\}} = (E_* \cap E_1') \setminus E_3'$, $L_{\{2\}} = (E_* \cap E_2') \setminus E_3'$, $L_{\{3\}} = (E_* \cap E_3') \setminus (E_1' \cup E_2')$.
- $E_*^\circ = E_* \setminus \bigcup_{i=1}^{3} E_i'$.

So the contribution of the exceptional divisor E_* to the motive $S_{X',E'}$ is:

$$\begin{aligned}[\widetilde{E_*^\circ}, \sigma_{\Delta'}] &- \left([\widetilde{L_{\{1\}}}, \sigma_{\Delta'}] + [\widetilde{L_{\{2\}}}, \sigma_{\Delta'}] + [\widetilde{L_{\{3\}}}, \sigma_{\Delta'}]\right)(\mathbb{L}-1) \\ &+ \left([\widetilde{L_{\{1,3\}}}, \sigma_{\Delta'}] + [\widetilde{L_{\{2,3\}}}, \sigma_{\Delta'}]\right)(\mathbb{L}-1)^2.\end{aligned}$$

Note that by Example 3.9, applied to the blow-up of the point E_J of intersection of transversal curves $E_1 \cap E_3$ and $E_2 \cap E_3$ inside the surface E_3, we have that:

$$-[\widetilde{E_J}, \sigma_\Delta](\mathbb{L}-1) = [\widetilde{L_{\{3\}}}, \sigma_{\Delta'}] - \left([\widetilde{L_{\{1,3\}}}, \sigma_{\Delta'}] + [\widetilde{L_{\{2,3\}}}, \sigma_{\Delta'}]\right)(\mathbb{L}-1).$$

So in order to show that the contributions of Z and E_* to the motives $S_{X,E}$ and respectively $S_{X',E'}$ coincide, it suffices to prove the equality of motives:

$$-[\widetilde{E_I^\circ}, \sigma_\Delta](\mathbb{L}-1) = [\widetilde{E_*^\circ}, \sigma_{\Delta'}] - \left([\widetilde{L_{\{1\}}}, \sigma_{\Delta'}] + [\widetilde{L_{\{2\}}}, \sigma_{\Delta'}]\right)(\mathbb{L}-1). \tag{18}$$

Next, note that by the definition of blow-up, we have isomorphisms

$$L_{\{1\}} \cong E_I^\circ \cong L_{\{2\}}$$

which, moreover, extend (by Lemma 3.2) to the corresponding unbranched covers of Definition 3.3. Also, by Lemma 3.1, the (Zariski) locally trivial fibration $E_*^\circ \to E_I^\circ$ (with fiber $\mathbb{P}^1 \setminus \{2 \text{ points}\} = \mathbb{C}^*$) can be lifted to a $\mathbb{C}^*$-fibration $\widetilde{E_*^\circ} \to \widetilde{E_I^\circ}$. Finally, the Zariski triviality implies that $[\widetilde{E_*^\circ}, \sigma_{\Delta'}] = [\widetilde{E_I^\circ}, \sigma_\Delta](\mathbb{L}-1)$, which proves the claim. □

Example 3.15. Let X be a fourfold and $E = E_1 + E_2 + E_3 + E_4$ be a simple normal crossing divisor on X. Let the center Z be the intersection $E_1 \cap E_2$ (i.e., $Z = E_{\{1,2\}}$), hence the components E_3 and E_4 of E intersect Z transversally.

The center $Z = E_{\{1,2\}}$ is stratified by $E^\circ_{\{1,2\}}$ (open dense stratum), $E^\circ_{\{1,2,3\}}$, $E^\circ_{\{1,2,4\}}$ and $E_{\{1,2,3,4\}} = E^\circ_{\{1,2,3,4\}}$. In the notations of Example 3.14, the exceptional divisor E_* is stratified by the open dense stratum E_*°, the codimension one strata $L_{\{1\}}$, $L_{\{2\}}$, $L_{\{3\}}$ and $L_{\{4\}}$, the codimension two strata $L_{\{1,3\}}$, $L_{\{1,4\}}$, $L_{\{2,3\}}$, $L_{\{2,4\}}$ and $L_{\{3,4\}}$, and by the points $L_{\{1,3,4\}}$ and $L_{\{2,3,4\}}$.

Therefore the invariance under blowup of the motivic infinite cyclic cover is equivalent to the equality of the motives (where, for lack of space, we omit the reference to actions from our notation)

$$-[\widetilde{E}^\circ_{\{1,2\}}](\mathbb{L}-1) + ([\widetilde{E}^\circ_{\{1,2,3\}}] + [\widetilde{E}^\circ_{\{1,2,4\}}])(\mathbb{L}-1)^2 - [\widetilde{E}^\circ_{\{1,2,3,4\}}](\mathbb{L}-1)^3$$

and

$$\begin{aligned}&[\widetilde{E}^\circ_*] - ([\widetilde{L}_{\{1\}}] + [\widetilde{L}_{\{2\}}])(\mathbb{L}-1)\\ &\quad + ([\widetilde{L}_{\{1,3\}}] + [\widetilde{L}_{\{1,4\}}] + [\widetilde{L}_{\{2,3\}}] + [\widetilde{L}_{\{2,4\}}] + [\widetilde{L}_{\{3,4\}}])(\mathbb{L}-1)^2\\ &\quad - ([\widetilde{L}_{\{1,3,4\}}] + [\widetilde{L}_{\{2,3,4\}}])(\mathbb{L}-1)^3\end{aligned}$$

respectively. Note that by transversality, the dimension of the intersection of Z with the components E_3 and E_4 (and $E_{\{3,4\}}$) is less than the dimension of Z. So, by induction on the dimension of the center, we can assume for the blowup of $E_{\{3,4\}}$ along $E_{\{1,2,3,4\}} = Z \cap E_{\{3,4\}}$, with corresponding deleted divisor $E_{\{1,3,4\}} + E_{\{2,3,4\}}$, that

$$-[\widetilde{E}^\circ_{\{1,2,3,4\}}](\mathbb{L}-1) = [\widetilde{L}_{\{3,4\}}] - ([\widetilde{L}_{\{1,3,4\}}] + [\widetilde{L}_{\{2,3,4\}}])(\mathbb{L}-1). \tag{19}$$

Similarly, for the blowup of E_3 along the center $E_{\{1,2,3\}} = Z \cap E_3$, with deleted divisor $E_{\{1,3\}} + E_{\{2,3\}} + E_{\{3,4\}}$, we have that

$$\begin{aligned} &- [\widetilde{E}^{\circ}_{\{1,2,3\}}](\mathbb{L}-1) + [\widetilde{E}^{\circ}_{\{1,2,3,4\}}](\mathbb{L}-1)^2 \\ &\quad = [\widetilde{L}_{\{3\}}] - ([\widetilde{L}_{\{1,3\}}] + [\widetilde{L}_{\{2,3\}}] + [\widetilde{L}_{\{3,4\}}])(\mathbb{L}-1) \\ &\qquad + ([\widetilde{L}_{\{1,3,4\}}] + [\widetilde{L}_{\{2,3,4\}}])(\mathbb{L}-1)^2. \end{aligned} \tag{20}$$

Finally, for the blowup of E_4 along the center $E_{\{1,2,4\}} = Z \cap E_4$, with deleted divisor $E_{\{1,4\}} + E_{\{2,4\}} + E_{\{3,4\}}$, we have that

$$\begin{aligned} &- [\widetilde{E}^{\circ}_{\{1,2,4\}}](\mathbb{L}-1) + [\widetilde{E}^{\circ}_{\{1,2,3,4\}}](\mathbb{L}-1)^2 \\ &\quad = [\widetilde{L}_{\{4\}}] - ([\widetilde{L}_{\{1,4\}}] + [\widetilde{L}_{\{2,4\}}] + [\widetilde{L}_{\{3,4\}}])(\mathbb{L}-1) \\ &\qquad + ([\widetilde{L}_{\{1,2,4\}}] + [\widetilde{L}_{\{2,3,4\}}])(\mathbb{L}-1)^2. \end{aligned} \tag{21}$$

Note that in the two motives of infinite cyclic covers we can now cancel

$$(20) \cdot (\mathbb{L}-1) + (21) \cdot (\mathbb{L}-1) + (19) \cdot (\mathbb{L}-1)^2.$$

This is a reflection of the inclusion-exclusion principle, showing that strata of the center Z which are contained in the transversal components E_3, E_4 and their intersection $E_{\{3,4\}}$, give equal contributions to the two motives of infinite cyclic covers.

So it remains to show that the contribution of the dense open stratum of Z to the motivic infinite cyclic cover coincides with the contribution of strata in the exceptional divisor E_* which are not contained in the proper transforms of E_3 and E_4. That is, the invariance under blowup of the motivic infinite cyclic cover reduces to checking that

$$-[\widetilde{E}^{\circ}_{\{1,2\}}](\mathbb{L}-1) = [\widetilde{E}^{\circ}_{*}] - ([\widetilde{L}_{\{1\}}] + [\widetilde{L}_{\{2\}}])(\mathbb{L}-1).$$

Note that since $L_{\{1\}}$ is contained in the intersection $E_* \cap E'_1$, we have that $L_{\{1\}}$ is contained in the exceptional divisor of the blow up of E_1 along $E_1 \cap Z$, which is isomorphic to $E_1 \cap Z = E_1 \cap E_2$ (since the codimension of the center is one). Thus we have that $E^{\circ}_{\{1,2\}}$ is isomorphic to $L_{\{1\}}$ and, after lifting this isomorphism to the corresponding covers of Definition 3.3, we obtain: $\widetilde{E}^{\circ}_{\{1,2\}} \cong \widetilde{L}_{\{1\}}$. Analogously, we have that $\widetilde{E}^{\circ}_{\{1,2\}} \cong \widetilde{L}_{\{2\}}$. Furthermore, E_* is a Zariski locally trivial fibration over $E_{\{1,2\}}$ with fiber $\mathbb{P}^1$. Therefore, when we restrict E_* over $E^{\circ}_{\{1,2\}}$, we get a Zariski locally trivial fibration E°_* with fiber $\mathbb{C}^*$, because we delete the two different points in each fiber corresponding to the intersections with $L_{\{1\}}$ and $L_{\{2\}}$. Hence, as explained at the end of Example 3.14, we have a similar $\mathbb{C}^*$-fibration for the corresponding covering spaces, and the claim follows by multiplicativity of motives in a Zariski locally trivial fibration. □

Proof of Proposition 3.8. The proof is by induction on the dimension of the center of blowup. The beginning of induction (i.e., the case of one point) is proved in Proposition 3.12. Note that, in general, the center of blowup Z is either contained in a component E_i of E, or it is transversal to it, or it doesn't intersect it at all. We refer to components of the second kind as transversal components of E (with respect to Z). By collecting all indices i of components of E containing Z, we note that the center Z is contained in a set E_I (for some $I \subseteq J$) given by intersections of components of the deleted divisor. In particular, Z gets an induced stratification from that of E_I. So, there is a dense open stratum $Z \cap E_I^\circ$ in Z, together with positive codimension strata obtained by intersecting Z with collections of transversal components.

We begin the proof by first studying the case when the center of blowup is of type E_I, for some $I \subseteq J$. Let X' be the blowup of X along the center Z defined as the intersection $E_I := \bigcap_{i=1}^k E_i$ of some of the irreducible components of the deleted divisor E, and also assume that the irreducible components E_j for $j = k+1, \ldots, \ell$ of E intersect the center Z transversally, and no other components of E intersect Z. In this case, Z is stratified by a top dimensional open dense stratum E_I°, and by positive codimension strata obtained by intersecting Z with some of the transversal components E_j (with $j = k+1, \ldots, \ell$), i.e., strata of the form $E_{I\cup K}^\circ$, where $K \neq \emptyset$ and $K \subseteq \{k+1, \ldots, \ell\}$. Therefore, the contributions to the motivic infinite cyclic cover $S_{X,E}$ supported on Z are

$$(-1)^{k-1}[\widetilde{E_I^\circ}](\mathbb{L}-1)^{k-1} + \sum_{\emptyset \neq K \subseteq \{k+1,\ldots,\ell\}} (-1)^{k+|K|-1}[\widetilde{E_{I\cup K}^\circ}](\mathbb{L}-1)^{k+|K|-1}. \tag{22}$$

After blowing up X along Z, we get the deleted divisor $E' = (\bigcup_{j\in J} E'_j) \cup E_*$ of $X' = Bl_Z X$, where E_* is the exceptional locus of the blowup and E'_j is the proper transform of E_j (for $j \in J$). Note that, by the choice of the center Z of blowup, the k-fold intersection of the proper transforms of components E_i with $i = 1, \ldots, k$ is empty, i.e., $\bigcap_{i=1}^k E'_i = \emptyset$. The exceptional divisor E_* is stratified by the top dimensional open stratum $L_\emptyset = E_*^\circ$, by the codimension s (for $s < k$) strata obtained by intersecting E_* with s-fold intersections of the components $E'_1, \ldots, E'_k$ of E', i.e., by the strata L_G with $G \subset I$ a proper subset, and by strata contained in intersections of the proper transforms E'_j for $j = k+1, \ldots, \ell$ of the transversal components, i.e., strata of the type $L_{G\cup K}$ where $G \subset I$ is a proper subset of I and K is a nonempty subset of $\{k+1, \ldots, \ell\}$. Therefore the contributions to the motivic infinite cyclic cover $S_{X',E'}$ supported on E_* are:

$$[\widetilde{E_*^\circ}] + \sum_{\substack{G\subset I,\\ G\neq\emptyset, I}} (-1)^{|G|}(\mathbb{L}-1)^{|G|}\Big([\widetilde{L_G}] + \sum_{\substack{K\subseteq\{k+1,\ldots,\ell\}\\ K\neq\emptyset}} (-1)^{|K|}[\widetilde{L_{G\cup K}}](\mathbb{L}-1)^{|K|}\Big). \tag{23}$$

We can apply induction on the dimension of the center of blowup, and the exclusion-inclusion principle, to show that strata of the center Z which are contained in intersections of the transversal components E_j, for $j = k+1, \cdots, \ell$, give equal contributions to the motives $S_{X,E}$ and $S_{X',E'}$ of the corresponding infinite cyclic covers. More precisely, for each positive codimension stratum $E_{I\cup K}^\circ$ of Z, we get by induction for the blowup of E_K along the center $Z \cap E_K = E_{I\cup K}$, and with deletion divisor $E_K \cap (\sum_{i=1}^k E_i)$, a relation of the type

$$(-1)^{|K|-1}[\widetilde{E^\circ_{I\cup K}}](\mathbb{L}-1)^{|K|-1} + \sum_{K\subset K'\subseteq\{k+1,\dots,\ell\}} (-1)^{|K'|-1}[\widetilde{E^\circ_{I\cup K'}}](\mathbb{L}-1)^{|K'|-1}$$
$$= [\widetilde{L_K}] + \sum_{\substack{G\subset I,\\ G\neq\emptyset,I}} \sum_{K\subseteq K'\subset\{k+1,\dots,\ell\}} (-1)^{|G\cup(K'\setminus K)|}[\widetilde{L_{G\cup K'}}](\mathbb{L}-1)^{|G\cup(K'\setminus K)|}. \qquad (*_K)$$

By summing up all the products $(*_K)\cdot(\mathbb{L}-1)^{|K|}$ for the positive codimension strata $E^\circ_{I\cup K}$ of Z, we reduce the comparison of (22) and (23) to proving the identity:

$$(-1)^{k-1}[\widetilde{E^\circ_I}](\mathbb{L}-1)^{k-1} = [\widetilde{E^\circ_*}] + \sum_{\substack{G\subset I,\\ G\neq\emptyset,I}} (-1)^{|G|}[\widetilde{L_G}](\mathbb{L}-1)^{|G|}, \qquad (24)$$

i.e., it remains to show that the contribution of the dense open stratum of the center Z to the motivic infinite cyclic cover $S_{X,E}$ coincides with the contribution to $S_{X',E'}$ of any of the strata supported on the exceptional divisor E_* which are not contained in the proper transforms of the components of E which are transversal to Z.

Note that, for any subset $G \subsetneq I = \{1,\dots,k\}$ (including the empty set corresponding to $L_\emptyset = E^\circ_*$), we have that L_G is a Zariski locally trivial fibration over E°_I with fiber $(\mathbb{C}^*)^{k-|G|-1}$. Indeed, the closure $\bar{L}_G$ of L_G is the exceptional divisor of the blowup of E_G along Z. Therefore $\bar{L}_G$ is a Zariski locally trivial fibration over Z with fiber isomorphic to $\mathbb{P}^{k-|G|-1}$. When we restrict the fibration $\bar{L}_G \to Z$ over the open dense stratum E°_I of Z, we remove the fibers lying above the intersections of Z with the transversal components $E_{k+1},\dots,E_\ell$. To obtain L_G, we need to further subtract the intersections of the total space of the fibration $(\bar{L}_G)_{|E^\circ_I}$ with the components E'_i with $i \in I\setminus G$. Fiberwise, the effect of the latter operation is that we remove $k-|G|$ hyperplanes in general position, hence the fiber of $L_G \to E^\circ_I$ is isomorphic to a complex torus $(\mathbb{C}^*)^{k-|G|-1}$ of dimension $k-|G|-1$.

By Lemma 3.1, the (Zariski) locally trivial fibration $L_G \to E^\circ_I$ with fiber isomorphic to $(\mathbb{C}^*)^{k-|G|-1}$ can be lifted to a $(\mathbb{C}^*)^{k-|G|-1}$-fibration $\widetilde{L_G} \to \widetilde{E^\circ_I}$. Thus, the Zariski triviality implies that

$$[\widetilde{L_G}] = [\widetilde{E^\circ_I}](\mathbb{L}-1)^{k-|G|-1}.$$

Finally, the equality (24) follows from the Pascal triangle because the number of subsets G of I of given size $|G|$ equals the binomial coefficient $\binom{k}{|G|}$.

Let us now explain the proof in the general case, i.e., when the center Z is strictly contained in some set E_I, for $I \subseteq J$, and let $I = \{1,\cdots,k\}$. Assume that the codimension of Z in X is $r+1 \geq k$. Again, by induction, it suffices to show that the contribution of the dense open stratum $Z^\circ := Z \cap E^\circ_I$ of the center Z to the motivic infinite cyclic cover $S_{X,E}$ coincides with the contribution to $S_{X',E'}$ of the strata supported on the exceptional divisor $E_* = \mathbb{P}(\nu_Z)$ which are not contained in the proper transforms of the transversal components of E (with respect to Z), that is,

$$(-1)^{k-1}[\widetilde{Z}^{\circ}](\mathbb{L}-1)^{k-1} = [\widetilde{E^{\circ}_{*}}] + \sum_{\substack{G\subsetneq I,\\ G\neq\emptyset}} (-1)^{|G|}[\widetilde{L_G}](\mathbb{L}-1)^{|G|}. \tag{25}$$

On the right hand side of (25), we use the same notation as before for the stratification of the exceptional divisor E_*. Note that in this case we have to also allow $G = I$ in the sum of the right hand side term of (25) because $Z \subsetneq E_I$ and therefore $\bigcap_{i=1}^{k} E'_i \neq \emptyset$.

Note that, for any subset $G \subsetneq I = \{1, \ldots, k\}$ (including the empty set corresponding to $L_\emptyset = E^\circ_*$), we have that L_G is a Zariski locally trivial fibration over Z° with fiber $\mathbb{C}^{r-k+1} \times (\mathbb{C}^*)^{k-|G|-1}$. Indeed, the closure $\bar{L}_G$ of L_G is the exceptional divisor of the blowup of E_G along Z. Therefore $\bar{L}_G$ is a Zariski locally trivial fibration over Z with fiber isomorphic to $\mathbb{P}^{r-|G|}$. When we restrict the fibration $\bar{L}_G \to Z$ over the open dense stratum Z° of Z, we remove the fibers lying above the intersections of Z with the transversal components $E_{k+1}, \ldots, E_\ell$. To obtain L_G, we need to further subtract the intersections of the total space of the fibration $(\bar{L}_G)_{|Z^\circ}$ with the components E'_i with $i \in I\backslash G$. Fiberwise, the effect of the latter operation is that we remove $k-|G|$ hyperplanes in general position from $\mathbb{P}^{r-|G|}$, hence the fiber of $L_G \to Z^\circ$ is isomorphic to the cartesian product $\mathbb{C}^{r+1-k} \times (\mathbb{C}^*)^{k-|G|-1}$ of a complex affine space of dimension $r+1-k$ and a complex torus of dimension $k-|G|-1$. In the case $G = I$, we get that L_I is a Zariski locally trivial fibration over Z° with fiber the projective space $\mathbb{P}^{r-k}$.

By Lemma 3.1, the (Zariski) locally trivial fibration $L_G \to Z^\circ$ with fiber isomorphic to $\mathbb{C}^{r-k+1} \times (\mathbb{C}^*)^{k-|G|-1}$ can be lifted to a $\mathbb{C}^{r-k+1} \times (\mathbb{C}^*)^{k-|G|-1}$-fibration $\widetilde{L_G} \to \widetilde{Z^\circ}$. Thus, the Zariski triviality implies that, for $G \subsetneq I = \{1, \ldots, k\}$ (including the empty set corresponding to $L_\emptyset = E^\circ_*$), we have:

$$[\widetilde{L_G}] = [\widetilde{Z^\circ}]\mathbb{L}^{r-k+1}(\mathbb{L}-1)^{k-|G|-1}. \tag{26}$$

For $G = I$, the fiber $\mathbb{P}^{r-k}$ of $L_I \to Z^\circ$ is simply connected (as $r - k \geq 1$), hence the covering $L_I \to Z^\circ$ can be lifted to a $\mathbb{P}^{r-k}$-fibration $\widetilde{L_I} \to \widetilde{Z^\circ}$. Thus, Zariski locally triviality yields that

$$[\widetilde{L_I}] = [\widetilde{Z^\circ}](\mathbb{L}^{r-k} + \mathbb{L}^{r-k-1} + \cdots + \mathbb{L} + 1). \tag{27}$$

By substituting the equalities (26) and (27) into (25), and factoring out $[\widetilde{Z}^\circ](\mathbb{L}-1)^{k-1}$, it remains to show that:

$$(-1)^{k-1} = \mathbb{L}^{r-k+1} + \sum_{\substack{G\subsetneq I,\\ G\neq\emptyset}} (-1)^{|G|}\mathbb{L}^{r-k+1} + (-1)^k(\mathbb{L}^{r-k} + \mathbb{L}^{r-k-1} + \cdots + \mathbb{L} + 1)(\mathbb{L}-1). \tag{28}$$

Note that the right hand side of (28) can be written as:

$$\left[\sum_{G\subseteq I}(-1)^{|G|}\mathbb{L}^{r-k+1}\right]+(-1)^{k-1}.$$

So after canceling $(-1)^{k-1}$ from both sides of (28), it remains to show that:

$$\sum_{G\subseteq I}(-1)^{|G|}=0. \tag{29}$$

Since the number of subsets G of I of given size i equals the binomial coefficient $\binom{k}{i}$, it follows that (29) is equivalent to the following well-known identity:

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}=0.$$

Thus equation (25) holds. □

4. Betti realization

Let $V_{\mathbb{Q}}^{\text{end}}$ be the category of finite dimensional $\mathbb{Q}$-vector spaces endowed with an endomorphism. Remark that $V_{\mathbb{Q}}^{\text{end}}$ is equivalent to the category of torsion $\mathbb{Q}[t]$-modules, e.g., see [15, Section 3]. There exists a $\mathbb{Q}$-linear homomorphism

$$\xi : V_{\mathbb{Q}}^{\text{end}} \to \mathbb{Q}(t)$$

defined by

$$(V,M)\mapsto \exp\left(\sum_n \frac{\text{Trace}(M^n)}{n}t^n\right)=\frac{1}{\det(Id-tM)},$$

which satisfies

$$\xi\left((V,M)\right)=\xi\left((V_1,M_1)\right)\cdot\xi\left((V_2,M_2)\right)$$

for each exact sequence $0\to V_1\to V\to V_2\to 0$ such that V_1 is M-invariant, $M|_{V_1}=M_1$ and the map induced by M on $V_2=V/V_1$ coincides with M_2.

Remark 4.1. Note that if M^s denotes the semi-simple part of the endomorphism M, then $\xi\left((V,M)\right)=\xi\left((V,M^s)\right)$. So for the definition of ξ it suffices to take into consideration only the semi-simple part of M.

By [2, p. 377], there is a monomorphism

$$(\xi,\ \text{for}) : K_0(V_{\mathbb{Q}}^{\text{end}})\to \mathbb{Q}(t)^*\times K_0(V_{\mathbb{Q}}),$$

with $V_{\mathbb{Q}}$ the abelian category of finite dimensional rational vector spaces, and

$$\text{for} : K_0(V_{\mathbb{Q}}^{\text{end}}) \to K_0(V_{\mathbb{Q}}), \quad [(V,M)] \mapsto [V]$$

the corresponding forgetful functor. Hence, by Remark 4.1, this identification yields that

$$[(V,M)] = [(V,M^s)] \in K_0(V_{\mathbb{Q}}^{\text{end}}). \tag{30}$$

Denote by $V_{\mathbb{Q}}^{\text{aut}}$ the category of finite dimensional $\mathbb{Q}$-vector spaces endowed with a finite order automorphism. Then there exists an additive map (called the *Betti realization*)

$$\chi_b : K_0(\text{Var}_{\mathbb{C}}^{\hat{\mu}}) \to K_0(V_{\mathbb{Q}}^{\text{aut}}) \to K_0(V_{\mathbb{Q}}^{\text{end}})$$

such that

$$[Y,\sigma] \mapsto [H_c^*(Y;\mathbb{Q}),\sigma^*] := \sum_{i\geq 0}(-1)^i[H_c^i(Y;\mathbb{Q}),\sigma_i^*],$$

where σ_i^* denotes the automorphism of $H_c^i(Y,\mathbb{Q})$ induced by the action of σ. Here, the compactly supported cohomology is used in order to fit with the scissor relation (3) in the motivic Grothendieck group $K_0(\text{Var}_{\mathbb{C}}^{\hat{\mu}})$.

We can therefore define a homomorphism

$$\xi_{mot} : K_0(\text{Var}_{\mathbb{C}}^{\hat{\mu}}) \to (\mathbb{Q}(t)^*, \cdot)$$

by the composition $\xi_{mot} := \xi \circ \chi_b$, i.e.,

$$\xi_{mot}([Y,\sigma]) = Z_Y(t),$$

where

$$Z_Y(t) := \prod_{i\geq 0}\left[\det(\text{Id} - t\cdot\sigma_i^*, H_c^i(Y;\mathbb{Q}))\right]^{(-1)^{i+1}}$$

is the zeta function of the $\hat{\mu}$-action σ on Y.

Back to our geometric situation, the deck transformation T of the infinite cyclic cover $\widetilde{T}^*_{X,E,\Delta}$ induces automorphisms T_i^* on each group $H_c^i(\widetilde{T}^*_{X,E,\Delta})$. The corresponding zeta function of T is defined by:

$$Z_{\widetilde{T}^*_{X,E,\Delta}}(t) := \prod_{i\geq 0}\left[\det\left(\text{Id} - t\cdot T_i^*, H_c^i(\widetilde{T}^*_{X,E,\Delta})\right)\right]^{(-1)^{i+1}}.$$

Recall from Remark 4.1 that it suffices to take into consideration only the semisimple (hence of finite order) part of T_i^*.

The main result of this section describes the Betti realization of the motivic infinite cyclic cover $S_{X,E,\Delta}$.

Proposition 4.2. *The Betti realization of the motivic infinite cyclic cover of finite type is given by the cohomology with compact support of* $\widetilde{T}^*_{X,E,\Delta}$, *i.e.,*

$$\chi_b(S_{X,E,\Delta}) = \sum_{i\geq 0}(-1)^i[H^i_c(\widetilde{T}^*_{X,E,\Delta}), T^*_i] \in K_0(V^{\mathrm{aut}}_{\mathbb{Q}}). \tag{31}$$

Equivalently,

$$\xi_{mot}(S_{X,E,\Delta}) = Z_{\widetilde{T}^*_{X,E,\Delta}}(t). \tag{32}$$

In particular, (by taking degrees) the topological Euler characteristic of $\widetilde{T}^*_{X,E,\Delta}$ *is computed by*

$$\chi(\widetilde{T}^*_{X,E,\Delta}) = \chi(S_{X,E,\Delta}). \tag{33}$$

Proof. Consider the T-equivariant Mayer–Vietoris spectral sequence for the open cover $\{T^*_{E^\circ_i}\}_{i\in J}$ of $T^*_{X,E,\Delta}$, i.e.,

$$E_1^{p,q} = \bigoplus_{|I|=p+1} H^q_c(\widetilde{T}^*_{E^\circ_I}) \Rightarrow H^{p+q}_c(\widetilde{T}^*_{X,E,\Delta}),$$

in which we identify the intersections $\bigcap_{i\in I} T^*_{E^\circ_i} = T^*_{E^\circ_I}$ as in Proposition 2.4. By using the additivity of the universal Euler characteristic $W \to [W] \in K(V^{\mathrm{aut}}_{\mathbb{Q}})$, for $W \in V^{\mathrm{aut}}_{\mathbb{Q}}$, we have the identity:

$$\begin{aligned}\sum_{i\geq 0}(-1)^i[H^i_c(\widetilde{T}^*_{X,E,\Delta})] &= \sum_{i,j\geq 0}(-1)^{i+j}[E_1^{i,j}] \\ &= \sum_{i,j\geq 0}(-1)^{i+j}\left[\bigoplus_{|I|=j+1} H^i_c(\widetilde{T}^*_{E^\circ_I})\right] \\ &= \sum_{\emptyset\neq I\subseteq J}(-1)^{|I|-1}\left(\sum_{i\geq 0}(-1)^i[H^i_c(\widetilde{T}^*_{E^\circ_I})]\right).\end{aligned}$$

By using the definition of the motivic infinite cyclic cover $S_{X,E,\Delta}$, it thus suffices to check formula (31) for each stratum E°_I, i.e., we have to show that the following identity holds

$$\chi_b([\widetilde{E}^\circ_I](\mathbb{L}-1)^{|I|-1}) = \sum_{i\geq 0}(-1)^i[H^i_c(\widetilde{T}^*_{E^\circ_I})], \tag{34}$$

or equivalently (after applying ξ),

$$\begin{aligned}\xi_{mot}([\widetilde{E}_I^\circ](\mathbb{L}-1)^{|I|-1},\sigma_I) &= \prod_{i\geq 0}\left[\det(\mathrm{Id}-t\cdot T_i^*, H_c^i(\widetilde{T}^*_{E_I^\circ}))\right]^{(-1)^{i+1}} \\ &= Z_{\widetilde{T}^*_{E_I^\circ}}(t),\end{aligned} \tag{35}$$

with σ_I denoting the corresponding μ_{m_I}-action. On the other hand, by definition,

$$\xi_{mot}([\widetilde{E}_I^\circ](\mathbb{L}-1)^{|I|-1},\sigma_I) = Z_{\widetilde{E}_I^\circ\times(\mathbb{C}^*)^{|I|-1}}(t),$$

so it remains to prove the equality of zeta functions:

$$Z_{\widetilde{T}^*_{E_I^\circ}}(t) = Z_{\widetilde{E}_I^\circ\times(\mathbb{C}^*)^{|I|-1}}(t). \tag{36}$$

(Note that in (36), the product $\widetilde{E}_I^\circ \times (\mathbb{C}^*)^{|I|-1}$ can be replaced by any Zariski locally trivial fibration over $\widetilde{E}_I^\circ$ with fiber $(\mathbb{C}^*)^{|I|-1}$, as they give the same element in $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$.)

By Lemmas 3.1 and 3.2, the long exact sequence of homotopy groups associated to the $(\mathbb{C}^*)^{|I|}$-fibration $T^*_{E_I^\circ} \to E_I^\circ$ induces a locally trivial topological fibration

$$\widetilde{T}^*_{E_I^\circ} \to \widetilde{E}_I^\circ, \tag{37}$$

with connected fiber $\widetilde{(\mathbb{C}^*)^{|I|}} \simeq (\mathbb{C}^*)^{|I|-1}$, the infinite cyclic cover of $(\mathbb{C}^*)^{|I|}$ defined by the kernel of the epimorphism $\mathbb{Z}^{|I|} \twoheadrightarrow m_I\mathbb{Z}$ induced by the holonomy map Δ. As before, $\widetilde{T}^*_{E_I^\circ}$ is the infinite cyclic cover of $T^*_{E_I^\circ}$ defined by Δ, and $\widetilde{E}_I^\circ$ is the unbranched μ_{m_I}-cover of E_I° with holonomy Δ_I.

For a sufficiently fine cover of $\widetilde{E}_I^\circ$ by μ_{m_I}-invariant sets, the fibration (37) becomes trivial. Hence (36) follows from the multiplicativity of zeta functions, i.e., from the equality

$$Z_{U_1\cup U_2}(t) = \frac{Z_{U_1}(t)\cdot Z_{U_2}(t)}{Z_{U_0}(t)}$$

for T-invariant sets U_1, U_2 with $U_0 = U_1 \cap U_2$. This multiplicativity is easily deduced from the corresponding Mayer–Vietoris long exact sequence.

It is now easy to see that the common value of the terms in (36) is $(1-t^{m_I})^{-\chi(E_I^\circ)}$ if $|I| = 1$, and it is 1 otherwise. □

Remark 4.3. More generally, there is a *Hodge realization* homomorphism

$$\chi_h : K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}}) \to K_0(\mathrm{HS}^{\mathrm{mon}})$$

defined by the same formula as χ_b, with $K_0(\mathrm{HS}^{\mathrm{mon}})$ the Grothendieck group of monodromic Hodge structures (i.e., endowed with an automorphism of finite order), cf. [7].

In the case when the compactly supported cohomology of the infinite cyclic cover of T^*_E admits mixed Hodge structures (e.g., see [21, Section 6] for such a situation), the above proposition can be extended to show that the corresponding class in $K_0(\mathrm{HS}^{\mathrm{mon}})$ is the Hodge realization of the motivic infinite cyclic cover of T^*_E. On the other hand, for an arbitrary infinite cyclic cover $\widetilde{T}^*_{X,E,\Delta}$ of finite type, even when a construction of a mixed Hodge structure is absent, χ_h provides "its class".

5. Relation with motivic Milnor fiber

Denef and Loeser introduced the *local motivic Milnor fiber* $\mathcal{S}_{f,x}$ at a point x for a non-constant morphism $f : \mathbb{C}^{d+1} \to \mathbb{C}$ with $f(x) = 0$ (e.g., see [7, Def. 3.2.1, Def. 3.5.3] and the references therein) as a limit in the sense of [12, Section 2.8] (see [6, Lemma 4.1.1]):

$$\mathcal{S}_{f,x} := -\lim_{T \to +\infty} Z(T) \in K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{C}})[\mathbb{L}^{-1}] \tag{38}$$

of the *motivic zeta function*

$$Z(T) := \sum_{n \geq 1} [\mathcal{X}_{n,1}] \mathbb{L}^{-(d+1)n} T^n \in K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{C}})[\mathbb{L}^{-1}][[T]], \tag{39}$$

where $\mathcal{X}_{n,1}$ denotes the set of $(n+1)$-jets φ of $\mathbb{C}^{d+1}$ centered at x such that $f \circ \varphi = t^n + \dots$. Note that there is a good action of the group μ_n (hence of $\hat{\mu}$) on $\mathcal{X}_{n,1}$ by $\lambda \times \varphi \mapsto \varphi(\lambda \cdot t)$.

The following result relates the concepts of motivic Milnor fiber and the motivic infinite cyclic cover, respectively.

Theorem 5.1. *Let $f : \mathbb{C}^{d+1} \to \mathbb{C}$ be a non-constant morphism with $f(x) = 0$, and let $p : X \to \mathbb{C}^{d+1}$ be a log-resolution of the singularities of pair $(\mathbb{C}^{d+1}, f^{-1}(0))$. Choose p in such a way that $(p^{-1}(x))_{\mathrm{red}}$ is a union of components of $(p^{-1}(f^{-1}(0)))_{\mathrm{red}}$. Let $E = \sum_{j \in J} E_j$ be the irreducible component decomposition of $p^{-1}(f^{-1}(0))_{\mathrm{red}}$, and let $A = \{i \in J \mid E_i \subset p^{-1}(x)\}$. Then the following hold:*

(1) *For $\epsilon > 0$ small enough, and $B(x, \epsilon)$ a ball of radius ϵ centered at $x \in \mathbb{C}^{d+1}$, the map p provides a biholomorphic identification between $B(x, \epsilon) \setminus \{f = 0\}$ and $T^*_{E^A}$, the punctured regular neighborhood of the divisor $E^A := \sum_{i \in A} E_i$. In particular, the map $\gamma \to \int_\gamma \frac{df}{f}$ can be viewed as a holonomy homomorphism: $\Delta : \pi_1(T^*_{E^A}) \to \mathbb{Z}$ of the punctured neighborhood of E^A. This holonomy map takes the boundary δ_i of any small disk transversal to the irreducible component E_i of E^A to the multiplicity m_i of E_i in the divisor of $f \circ p$, i.e., $\Delta(\delta_i) = m_i$ for all $i \in A$.*
(2) *One has the following identity in $K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{C}})[\mathbb{L}^{-1}]$:*

$$\mathcal{S}_{f,x} = S^A_{X,E,\Delta}.$$

Proof. We shall use the following formula (e.g., [7, Def.3.5.3], which in turn is motivated by the calculation in [6, Theorem 2.2.1] and [5, Theorem 2.4]) for the motivic Milnor fiber $\mathcal{S}_{f,x}$ in terms of a log-resolution $p : X \to \mathbb{C}^{d+1}$ of $f^{-1}(0)$:

$$\mathcal{S}_{f,x} = \sum_{\substack{\emptyset \neq I \subseteq J \\ I \cap A \neq \emptyset}} (-1)^{|I|-1} c_I (\mathbb{L} - 1)^{|I|-1}, \tag{40}$$

where c_I is the class of the unramified Galois cover $\widetilde{\widetilde{E}}_I^\circ \in \mathrm{Var}_{\mathbb{C}}^{\hat{\mu}}$ of E_I°, with Galois group μ_{m_I}, defined as follows. Let m_i be the multiplicity of E_i in the divisor of $f \circ p$ and $m_I = \gcd(m_i | i \in I)$. Given an affine Zariski open subset U of X such that $f \circ p = uv^{m_I}$ on U, with $u \in \Gamma(U, \mathcal{O}_U)$ a unit and v a morphism from U to $\mathbb{C}$, the restriction $\widetilde{\widetilde{E}}_I^\circ|_U$ of $\widetilde{\widetilde{E}}_I^\circ$ over $E_I^\circ|_U := E_I^\circ \cap U$ is defined by

$$\widetilde{\widetilde{E}}_I^\circ|_U = \{(z, y) \in \mathbb{C} \times E_I^\circ|_U \mid z^{m_I} = cu^{-1}\}. \tag{41}$$

There is a natural μ_{m_I}-action defined by multiplying the z-coordinate with the elements of μ_{m_I}, whose corresponding quotient yields the covering map: $\widetilde{\widetilde{E}}_I^\circ|_U \to E_I^\circ|_U$. We denote this action σ_I'. For proving our theorem, it suffices to show that, as elements of $\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}}$, the cover $(\widetilde{\widetilde{E}}_I^\circ|_U, \sigma_I')$ coincides with the cover $(\widetilde{E}_I^\circ|_U, \sigma_I)$ from Definitions 3.3 and 3.5, where we let as above $\widetilde{E}_I^\circ|_U$ denote the restriction of $\widetilde{E}_I^\circ$ over U.

Let M_f denote the Milnor fiber $\{f = c\} \cap B(x, \epsilon) \subset B(x, \epsilon) \setminus \{f = 0\} \cong T^*_{E^A}$. For a sufficiently small subset $U \subset E_I^\circ$ we can choose a trivialization of $T_{E^A}|_U$ which yields a trivialization as a $(\mathbb{C}^*)^{|I|}$-bundle of the subset $T^*_{U,E_I^\circ} := T^*_{E_I^\circ}|_U$ of $T^*_{E^A}$. Let

$$M_{U,E_I^\circ} = M_f \cap T^*_{U,E_I^\circ} \subset T^*_{U,E_I^\circ} = E_I^\circ|_U \times (\mathbb{C}^*)^r,$$

with $r = |I|$. In the latter identification, M_{U,E_I° is the hypersurface given by $z_1^{m_1} \cdots z_r^{m_r} = cu^{-1}$ (where z_i are the coordinates in the torus). It follows that fibers of M_{U,E_I° over $E_I^\circ|_U$ are disjoint unions of m_I translated subgroups $z_1^{\frac{m_1}{m_I}} \cdots z_r^{\frac{m_r}{m_I}} = \lambda\omega_{m_I}$ where $\lambda^{m_I} = cu^{-1}$ and $\omega_{m_I} \in \mu_{m_I}$. Each such translated subgroup is biholomorphic to a torus $(\mathbb{C}^*)^{r-1}$. In fact, the Stein factorization presents M_{U,E_I° as a $(\mathbb{C}^*)^{r-1}$-torus fibration over $\widetilde{\widetilde{E}}_I^\circ|_U$, with the map $M_{U,E_I^\circ} \to \widetilde{\widetilde{E}}_I^\circ|_U$ induced by $(z_1, ... z_r) \mapsto z = z_1^{\frac{m_1}{m_I}} \cdots z_r^{\frac{m_r}{m_I}}$.

Next consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_1((\mathbb{C}^*)^{r-1}) & \to & \pi_1((\mathbb{C}^*)^r) & & \\
\downarrow & & \downarrow & \searrow & \\
\pi_1(M_{U,E_I^\circ}) & \to & \pi_1(T^*_{U,E_I^\circ}) & \overset{\Delta}{\to} & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(E_I^{\circ\approx}|_U) & \to & \pi_1(E_I^\circ|_U) & \overset{\Delta_{m_I}}{\to} & \mathbb{Z}_{m_I}
\end{array}$$

induced by the fibrations described above. Here Δ is the holonomy described in (1) and Δ_{m_I} is the map induced by Δ as described in Lemma 3.1.

To conclude the proof of the theorem it is enough to show that image of $\pi_1(\approx{E}_I^\circ|_U)$ belongs to the kernel of the map Δ_{m_I}, since $\ker(\Delta_{m_I}) = \pi_1(\widetilde{E}_I^\circ|_U)$ by our construction and both groups $\pi_1(\widetilde{E}_I^\circ|_U)$ and $\pi_1(\approx{E}_I^\circ|_U)$ have index m_I in $\pi_1(E_I^\circ|_U)$. Notice that the restriction of Δ on $M_f \subset T^*_{E^A}$ yields $\Delta : \pi_1(M_f) \to \mathbb{Z}$, which is trivial since for any $\gamma \subset M_f$ one has $\int_\gamma \frac{df}{f} = 0$ (as $f(\gamma)$ is constant). Hence the composition of maps in the middle row of the above diagram is trivial. By commutativity, the image of the composition

$$\pi_1(M_{U,E_I^\circ}) \to \pi_1(\approx{E}_I^\circ|_U) \to \pi_1(E_I^\circ|_U) \to \mathbb{Z}_{m_I}$$

is also trivial. Moreover, the homomorphism $\pi_1(M_{U,E_I^\circ}) \to \pi_1(\approx{E}_I^\circ|_U)$ is surjective since it is induced by the map $M_{U,E_I^\circ} \to \approx{E}_I^\circ|_U$ which is a fibration with connected fibers. Therefore the composition $\pi_1(\approx{E}_I^\circ|_U) \to \pi_1(E_I^\circ|_U) \to \mathbb{Z}_{m_I}$ is trivial and the claim follows. □

Remark 5.2. Note that Theorems 5.1 and 3.7 give a direct proof of the fact that the right-hand side of formula (40) expressing the Denef–Loeser motivic Milnor fiber in terms of a log-resolution is actually independent of the choice of log-resolution. This was apriori known only because of the relation (38) with the motivic zeta function (which is intrinsically defined by Denef–Loeser in terms of arc spaces as in (39)), see also the discussion in [7, Section 3.5]. It should also be noted that our proof of independence of (40) of the choice of log resolution does not make sense of the third relation (5) in the motivic Grothendieck group $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$. As a consequence, our results also imply the well-definedness of the Denef–Loeser motivic nearby and vanishing cycles without the use of the third relation (5) in the motivic Grothendieck group (which was needed for the approach via arc spaces).

Let us consider now a non-constant morphism $f : \mathbb{C}^{d+1} \to \mathbb{C}$ with $f(0) = 0$. As described at the end of Section 2, by Milnor's fibration theorem [24], there is a locally trivial fibration $\pi : B_{\epsilon,\delta} \to D^*_\delta$ associated to f and the origin $0 \in \mathbb{C}^n$. Let us call T_f the corresponding monodromy map. Since the infinite cyclic cover of $B_{\epsilon,\delta}$ and the Milnor fiber M_f at the origin are homotopically equivalent, we have the following corollary as a consequence of Theorems 4.2 and 5.1. This is a weak version of Theorem 4.2.1 in [6], see also [5].

Corollary 5.3. *The Betti realization of the motivic Milnor fiber of f at the origin coincides with the Betti invariant of the monodromy T_f, i.e.,*

$$\chi_b(\mathcal{S}_{f,0}) = \sum_i (-1)^i [H^i_c(M_f), T^*_{M_f}].$$

6. Motivic Milnor fibers at infinity and motivic Milnor fibers associated with rational functions

In this last section, we outline further geometric situations in which our main construction allows to obtain motivic invariants for which we also obtain Betti realizations.

Let $f, g \in \mathbb{C}[x_1, \cdots, x_n]$ be two polynomials, with $\deg(f) - \deg(g) = k \geq 0$. Consider the pencil of hypersurfaces in $\mathbb{P}^n = \text{Proj } \mathbb{C}[x_0, x_1, \cdots, x_n]$ of degree $\deg(f)$ given by

$$\lambda \bar{f} + \mu \bar{g} x_0^k = 0,$$

where $\bar{f}$, $\bar{g}$ denote the homogenizations of f and g, respectively, and $[\lambda : \mu] \in \mathbb{P}^1$. The rational map $\pi_{f,g} : \mathbb{P}^n \to \mathbb{P}^1$ corresponding to this pencil is given by $[x_0 : \cdots : x_n] \mapsto [\bar{f} : \bar{g} x_0^k]$. Let $\phi : \widetilde{\mathbb{P}^n}_{f,g} \to \mathbb{P}^n$ be a resolution of the indeterminacy points of the rational map $\pi_{f,g}$ (i.e., the set of solutions of $\bar{f} = \bar{g} x_0^k = 0$), cf. [11, 7.1.2]; in a small ball about an indeterminacy point the restriction of the map $\pi_{f,g}$ to the complement of $\{g = 0\}$ is given by f/g, where the target of f/g is identified with $\mathbb{P}^1 \setminus \{[1 : 0]\}$. Let us denote by $\widetilde{\pi}_{f,g}$ the composition $\widetilde{\mathbb{P}^n}_{f,g} \xrightarrow{\phi} \mathbb{P}^n \xrightarrow{\pi_{f,g}} \mathbb{P}^1$. Note that the proper transforms under ϕ of divisors $\pi_{f,g}^{-1}([\lambda : \mu])$ and $\pi_{f,g}^{-1}([\lambda' : \mu'])$ have empty intersection provided $[\lambda : \mu] \neq [\lambda' : \mu']$. After possibly additional blow-ups, we can assume (using the same notations) that the fibers $E_0 = \widetilde{\pi}_{f,g}^{-1}([0 : 1])$ and $E_\infty = \widetilde{\pi}_{f,g}^{-1}([1 : 0])$ (i.e., the total transform of the divisors $\bar{f} = 0$ and $\bar{g} x_0^k = 0$, respectively) are both normal crossing divisor on $\widetilde{\mathbb{P}^n}_{f,g}$. We shall assume from now on that $\widetilde{\mathbb{P}^n}_{f,g}$ already satisfies this condition.

The following is a standard consequence of transversality theory in the context of stratified spaces.

Proposition 6.1. *Let $F \subset \widetilde{\mathbb{P}^n}_{f,g}$ be the union of components of the total transform of the pencil such that for generic $t \in \mathbb{P}^1$ the proper preimage of t has non-empty intersection with F. Then:*

1. *The variety F is always non-empty, and its irreducible components map surjectively onto $\mathbb{P}^1$. There is a finite subset $D \subset \mathbb{P}^1$ such that $\widetilde{\pi}_{f,g}$ is a locally trivial topological fibration over $\mathbb{P}^1 \setminus D$ and, for any $t \in \mathbb{P}^1 \setminus D$, the fiber $\widetilde{\pi}_{f,g}^{-1}(t)$ is transversal to F.*
2. *The restriction of $\widetilde{\pi}_{f,g}$ to $\widetilde{\pi}_{f,g}^{-1}(\mathbb{P}^1 \setminus D) \setminus (F \cap \widetilde{\pi}_{f,g}^{-1}(\mathbb{P}^1 \setminus D))$ is a locally trivial topological fibration with fiber homeomorphic to $\widetilde{\pi}_{f,g}^{-1}(t) \setminus (F \cap \widetilde{\pi}_{f,g}^{-1}(t))$.*
3. *Let $S \subset \mathbb{P}^1$ be a sufficiently small disk in $\mathbb{P}^1$ centered at $[0 : 1]$ (resp. at $[1 : 0]$) such that $S \cap D = \emptyset$, and let S^* be the disc S punctured at its center. Then $\widetilde{\pi}_{f,g}^{-1}(S^*) \setminus (F \cap \widetilde{\pi}_{f,g}^{-1}(S^*))$ is homeomorphic to a small punctured regular neighborhood of $E_0 \setminus (E_0 \cap F)$ (resp. $E_\infty \setminus (E_\infty \cap F)$) in $\widetilde{\mathbb{P}^n}_{f,g} \setminus F$.*
4. *Let $c \in \mathbb{C}^n \subset \mathbb{P}^n$ be such that $f(c) = g(c) = 0$, i.e., c is an indeterminacy point of the rational map $\pi_{f,g}$ outside the hyperplane at infinity. For sufficiently small ϵ, let B_ϵ be a ball of radius ϵ about c (so that the boundary of $B_{\epsilon'}$ is transversal to both $\{f = 0\}$, $\{g = 0\}$ and their intersection, for all $\epsilon' < \epsilon$). Finally, for $\delta << \epsilon$, let*

$S^*_\delta \subset S^*$ be a punctured disk, where S is like in (3). Then, the restriction of the map $\widetilde{\pi}_{f,g}$ from (2) to $\phi^{-1}(B_\epsilon) \cap \widetilde{\pi}^{-1}_{f,g}(S^*_\delta)$ is a locally trivial topological fibration over S^*_δ.

Using this set up we can now make the following definition.

Definition 6.2. In the notations of Proposition 6.1,

1. the Milnor fiber $M_{f,g,0}$ (resp. $M_{f,g,\infty}$) for the value 0 (resp. for the value ∞) of a rational function $\frac{f}{g}$ is the manifold $\widetilde{\pi}^{-1}_{f,g}(t) \setminus (F \cap \widetilde{\pi}^{-1}_{f,g}(t))$ for any $t \in \mathbb{P}^1$ closed enough to $[0:1]$ (resp. to $[1:0]$). The monodromy of this Milnor fiber is the monodromy map of the locally trivial fibration from Proposition 6.1(2). We denote the monodromy of this fibration by $T_{f,g,0}$ (resp. by $T_{f,g,\infty}$).
2. the Milnor fiber $M_{f,g,c,0}$ (resp. $M_{f,g,c,\infty}$) of a *germ* of rational function at an indeterminacy point c for the value 0 (resp. value ∞) is a generic fiber of the fibration from Proposition 6.1(4). We denote the monodromy of this fibration by $T_{f,g,c,0}$ (resp. $T_{f,g,c,\infty}$).

We shall refer to the composition

$$\nabla : \pi_1 \left(\widetilde{\pi}^{-1}_{f,g}(S^*) \setminus (F \cap \widetilde{\pi}^{-1}_{f,g}(S^*)) \right) \to \pi_1(S^*) = \mathbb{Z}$$

as the holonomy map of the punctured neighborhood of $E_0 \setminus F$ (resp. $E_\infty \setminus F$) as in Proposition 6.1(3).

Remark 6.3.

1. Generalizations of the notion of Milnor fiber in the context of rational functions were initiated by Gusein-Zade, Luengo and Melle-Hernandez [13,14], but see also [3,31].
2. Recall that given $f \in \mathbb{C}[x_1, \cdots, x_n]$, the *Milnor fiber of f at infinity* is defined as $M_f = f^{-1}(t)$ where $|t| >> 0$. Its topological type is independent of t, provided $|t|$ is sufficiently large. Moreover, its cohomology $H^{n-1}(M_f, \mathbb{Z})$ is endowed with the monodromy operator induced by the trivialization of the bundle $\psi^*(M_{|t|=a})$, where $M_{|t|=a}$ is the preimage under f of the circle $S_a = \{t \in \mathbb{C} | |t| = a, a \in \mathbb{R}\}$ and $\psi : [0,1] \to S_a$ is given by $s \mapsto ae^{2\pi i s}$ (cf. [19,20,30]). This notion coincides with $M_{f,1,\infty}$ in Definition 6.2(1).

In the above notations, we can now introduce motives associated to such topological objects (compare with [27]).

Definition 6.4. The *motivic Milnor fiber for the value zero (resp. for the value infinity) of a rational function* $\frac{f}{g}$ is the class $S^{A}_{\widetilde{\mathbb{P}}^n_{f,g}, E_0, \nabla} \in K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{C}})$ (resp. the class $S^{A}_{\widetilde{\mathbb{P}}^n_{f,g}, E_\infty, \nabla} \in$

$K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$), in the sense of Definition 3.5, with A indexing the collection of components of E_0 (resp. E_∞) not contained in F.

Definition 6.5. The *motivic Milnor fiber of a germ of rational function* $\frac{f}{g}$ at an indeterminacy point c, with $f(c) = g(c) = 0$, for the value zero (resp. for the value infinity) is the class $S^{A(c)}_{\widetilde{\mathbb{P}}^n_{f,g},E_0,\nabla} \in K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$ (resp. the class $S^{A(c)}_{\widetilde{\mathbb{P}}^n_{f,g},E_\infty,\nabla} \in K_0(\mathrm{Var}_{\mathbf{C}}^{\hat{\mu}})$), in the sense of Definition 3.5, with $A(c)$ indexing the collection of components of E_0 (resp. E_∞) not contained in F and that map to the value c under the resolution ϕ of the rational map $\pi_{f,g}$.

Remark 6.6. If $g = 1$ one obtains a notion of *motivic Milnor fiber of f at infinity*, compare for example with work by Matsui–Takeuchi [22] and Raibaut [26]. Another definition of motivic Milnor fibers for rational functions has been given by Raibaut in [27].

Finally, as in the case of Milnor fibers of germs of polynomials, motivic Milnor fibers of rational functions have Betti realizations and there are generalizations of Corollary 5.3 in this setting.

Corollary 6.7. *The Betti realization of the motivic Milnor fiber of a rational function f/g for the value zero (resp. for the value infinity) coincides with the Betti invariant of the monodromy $T_{f,g,0}$ (resp. of the monodromy $T_{f,g,\infty}$), i.e.*

$$\chi_b(S^{A}_{\widetilde{\mathbb{P}}^n_{f,g},E_\bullet,\nabla}) = \sum_i (-1)^i [H^i_c(M_{f,g,\bullet}), T^*_{M_{f,g,\bullet}}],$$

where $\bullet$ stands for 0 (resp. ∞).

Corollary 6.8. *The Betti realization of the motivic Milnor fiber for the value zero (resp. for the value) infinity of a germ of a rational function f/g at an indeterminacy point c coincides with the Betti invariant of the monodromy $T_{f,g,c,0}$ (resp. of the monodromy $T_{f,g,c,\infty}$), i.e.*

$$\chi_b(S^{A(c)}_{\widetilde{\mathbb{P}}^n_{f,g},E_\bullet,\nabla}) = \sum_i (-1)^i [H^i_c(M_{f,g,c,\bullet}), T^*_{M_{f,g,c,\bullet}}],$$

where $\bullet$ stands for 0 (resp. ∞).

Remark 6.9. $M_{f,g,0}$ and $M_{f,g,\infty}$ are members of a family of complex (in fact, quasi-projective) manifolds. This can be used to associate a limit mixed Hodge structure, cf. [29], whose motive is the Hodge realization of the above motivic Milnor fibers.

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