CAT(0) GEOMETRY OF COMPLEX CURVE COMPLEMENTS AND FAMILIES

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Abstract. Motivated by the question of whether braid groups are CAT(0), we investigate the CAT(0) behavior of fundamental groups of plane curve complements and certain universal families. If C is the branch locus of a generic projection of a smooth, complete intersection surface to $\mathbb{P}^2$, we show that $\pi_1(\mathbb{P}^2\backslash C)$ is CAT(0). In the other direction, we prove that the fundamental group of the universal family associated with the singularities of type E_6, E_7, and E_8 is not CAT(0).

1. Introduction

The purpose of the present article is to investigate the properties of complex plane curve complements and their fundamental groups via the lens of geometric group theory. On the one hand, plane curve singularities bear an intimate connection with the topology of 3-dimensional iterated torus link complements (see, for example, [EN16]). On the other hand, work going back to Zariski [Zar36] and Moishezon [Moi81] highlights the close relationship between fundamental groups of many plane curves complements and braid groups. More generally, it turns out that Artin groups, as natural generalizations of braid groups, are also related to the fundamental groups of some specific plane curves (cf.[ABCAMM18, Theorem 2.6]). Since both braid groups, more generally Artin groups and 3-manifold groups (with or without boundaries) are well-studied objects in geometric group theory for the richness of their intrinsic geometry, fundamental groups of plane curve complements form a natural setting to explore commonalities between these two classes.

Fundamental groups of plane curve complement fit into the larger class of *quasi-projective groups*, *i.e.* fundamental groups of smooth, quasi-projective varieties. Both classes are broad, and restrictions on such groups can often be quite subtle. Nevertheless, some of the topological constraints described above are reflected even in the larger quasi-projective setting. For example, Friedl–Suciu [FS14] have shown that if a quasi-projective group G is isomorphic to the fundamental group of a compact 3-manifold N with empty or toroidal boundary, then N is a graph manifold. In this sense, G also resembles the fundamental group of an iterated torus link complement, as in the case of a plane curve singularity.

Geometrically, results of Leeb [Lee95] imply that graph manifolds with non-empty boundary (and hence iterated torus link complements) admit Riemannian metrics of nonpositive curvature. In particular, their fundamental groups are CAT(0) in the sense that they admit proper, cocompact actions on CAT(0) spaces. CAT(0) geometry, introduced by Gromov [Gro87], is a natural extension of the concept of a nonpositive curvature to general metric spaces and conjecturally, braid groups are CAT(0) as well. Motivated by the intrinsic connections between 3-dimensional graph manifold groups, braid groups, and CAT(0) geometry, in this paper, we focus on the relation between fundamental groups of plane curve complements and CAT(0) geometry. Our principal goal is to provide both positive and

negative results in this vein, as well as raise some questions for further investigation. In the positive direction, we show

Theorem A. *Let $C \subset \mathbb{P}^2$ be a plane curve.*

(1) Fix $O \in \mathbb{P}^2 \backslash C$ and let L be the union of singular fibers from the projection of the blowup $\mathrm{Bl}_O(\mathbb{P}^2)\backslash C \to \mathbb{P}^1$. If $\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L)$ fibers with finite monodromy, then $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L))$ is CAT(0).

(2) If C is the branch locus of a generic finite map $X \to \mathbb{P}^2$, arising from a smooth complete intersection surface $X \subset \mathbb{P}^N$, then $\pi_1(\mathbb{P}^2 \backslash C)$ is CAT(0).

For background on blowups and construction of the fibration described in part (1), see Sections 2.1 and A.1 in the Appendix. The first part of this theorem will be known to the experts, and in this case $\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L)$ is the total space of a punctured $\mathbb{P}^1$-bundle over a punctured $\mathbb{P}^1$ admitting a complete Riemannian metric of nonpositive sectional curvature. The second part of this theorem relies on a result of Robb [Rob97] concerning fundamental groups of the complements to branching curves of generic complete intersections.

In the negative direction, our obstruction for a group to be CAT(0) comes from the previous work of the third author [Zhu23], generalizing methods of [LIP21]. We consider examples of quasi-projective varieties whose fundamental groups are extensions of free groups of finite rank. First, we exhibit a free-by-free group with infinite monodromy, which is not CAT(0), derived from the union of a 3-cuspidal quartic curve and a collection of lines (Theorem 4.2). This braid monodromy was first studied by Catanese–Wajnryb [CW04], and in this case, we compute the monodromy explicitly.

The second set of non-CAT(0) examples arise as monodromy groups associated with simple singularities of curves. Such singularities are enumerated by root systems R of type A_n $(n \geqslant 1)$, D_n $(n \geqslant 4)$ or E_n $(6 \leqslant n \leqslant 8)$. If U_R is the corresponding semi-universal deformation space, then $\pi_1(U_R)$ is isomorphic to the Artin group A_R. The deformation space U_R is the base space of a fiber bundle V_R whose fibers are smooth affine curves. Consequently, $\pi_1(V_R)$ is an extension of A_R by a free group of finite rank.

Theorem B. *Let R be one of the root systems E_6, E_7, or E_8 and let V_R be the total space of the affine family over the semi-universal deformation of the simple singularity of type R. Then $\pi_1(V_R)$ is not CAT(0).*

A crucial ingredient here is a result of Wajnryb [Waj99], which states that for these three root systems, the geometric monodromy is not injective. In contrast, Perron–Vannier [PV92] showed that when $R = A_n$ or D_n, the geometric monodromy is injective. At this time, the authors do not know whether $\pi_1(V_R)$ is CAT(0) in these cases.

Acknowledgments: The third author would like to thank Matthew Durham, Jose Ignacio Cogolludo-Agustin, and Chenxi Wu for useful discussions.

2. Preliminaries on plane curve complements and CAT(0) groups

In this section, we review some necessary material on computing fundamental groups of plane curve complements. We refer the reader to the Appendix for a more comprehensive overview of the topology of plane curves. We also recall basic definitions and properties of CAT(0) spaces and groups, introduce the LIP property, and describe the main obstruction we will use to show certain group extensions are not CAT(0).

2.1. **The Zariski–van Kampen Theorem.**
(See Appendices A.1 and A.2 and [Lib21] for additional details). The blow-up $\mathrm{Bl}_O\mathbb{P}^2$ of $\mathbb{P}^2$ at a point O admits a fibration over $\mathbb{P}^1$ with $\mathbb{P}^1$-fibers. Given a plane curve C and $O \notin C$, we obtain a (singular) fibration from the complement of the plane curve $\mathrm{Bl}_O(\mathbb{P}^2)\backslash C \to \mathbb{P}^1$ with finitely many singular fibers, and whose generic fibers are complements $\mathbb{P}^1\backslash\{x_1, ..., x_d\}$ to a finite set of points of cardinality d, where $d = \deg(C)$. Let $L = \cup_{i=1}^n L_i$ be the union of lines corresponding to the singular fibers. After removing L, we get a (locally trivial) punctured sphere bundle over a punctured sphere:

$$\mathbb{P}^1\backslash\{x_1, ..., x_d\} \to \mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L) \to \mathbb{P}^1\backslash\{y_1, ..., y_n\}, \tag{1}$$

where n is the number of singular fibers.

If $n \geqslant 1$, the long exact sequence of homotopy groups for (1) gives rise to a short exact sequence on π_1:

$$1 \to \pi_1(\mathbb{P}^1\backslash\{x_1, ..., x_d\}) \to \pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L)) \to \pi_1(\mathbb{P}^1\backslash\{y_1, ..., y_n\}) \to 1.$$

We have $\pi_1(\mathbb{P}^1\backslash\{x_1, ..., x_d\}) \cong F_{d-1}$ and $\pi_1(\mathbb{P}^1\backslash\{y_1, ..., y_n\}) \cong F_{n-1}$, where F_k denotes the free group on k generators. Substituting in the sequence above yields

$$1 \to F_{d-1} \to \pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L)) \to F_{n-1} \to 1 \tag{2}$$

This short exact sequence is actually split since F_{n-1} is free. Thus the fundamental group of the complement of $C \cup L$ is a semi-direct product:

$$\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L)) \cong F_{d-1} \rtimes_\rho F_{n-1},$$

where ρ is given by a lift of the induced π_1-representation $\rho\colon F_{n-1} \to \mathrm{Out}(F_{d-1})$.

If we rewrite $\pi_1(\mathbb{P}^1\backslash\{x_1, \ldots, x_d\})$ as $\langle g_1, \ldots, g_d | \prod_{i=1}^d g_i = 1\rangle$, and $\pi_1(\mathbb{P}^1\backslash\{y_1, \ldots, y_n\})$ as $\langle t_1, .., t_n | \prod_{i=1}^n t_i = 1\rangle$ then we get a presentation

$$\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C\cup L)) \cong \left\langle g_i, t_j,\ 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant n \;\middle|\; \prod_{i=1}^d g_i = 1, \prod_{j=1}^n t_j = 1, g_i^{t_j} = \rho(t_j)(g_i) \right\rangle.$$

Remark 2.1. *Here, and throughout, for any group G and elements $g, h \in G$, we will use the notation $h^g := ghg^{-1}$; and for the inverse of g, we will use the notation $\overline{g}$.*

Filling in the singular fibers is equivalent to killing the generators t_j on the level of π_1. On the other hand, since $O \notin C$ blowing up at O does not affect the fundamental group of the complement, *i.e.* $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash C) \cong \pi_1(\mathbb{P}^2\backslash C)$. It follows that $\pi_1(\mathbb{P}^2\backslash C)$ has presentation:

$$\pi_1(\mathbb{P}^2\backslash C) \cong \left\langle g_i,\ 1 \leqslant i \leqslant d \;\middle|\; \prod_{i=1}^d g_i = 1, g_i = \rho(t_j)(g_i),\ 1 \leqslant j \leqslant n \right\rangle.$$

Since $\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L)$ is the total space of a fiber bundle, we also have a monodromy homomorphism $\phi : \pi_1(\mathbb{P}^1\backslash\{y_1, ..., y_n\}) \to \mathrm{Mod}(\mathbb{P}^1\backslash\{x_1, ..., x_d\})$, where for any finite type surface S, $\mathrm{Mod}(S)$ denotes the *mapping class group of S*. The representation ρ defined above factors through ϕ, thus restricting the possible free-by-free groups arising as $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L))$. In Section 4.1, we will present an example of the computation of the monodromy homomorphism and fundamental group of the complement along the lines of the above description.

2.2. **CAT(0) groups and the LIP property.**
Let X be a proper geodesic metric space. Given any three points p, q, r in X, we can find a geodesic triangle $\Delta = \Delta(p, q, r)$ with p, q, r as its vertices. By the triangle inequality, there exists a unique triangle $\Delta' = \Delta'(p', q', r') \subset \mathbb{R}^2$ with the same side lengths as Δ, and a map $f\colon \Delta \to \Delta'$ mapping p, q, r to p', q', r', respectively, and restricting to an isometry on each edge. We call Δ' a *comparison triangle* and f a *comparison map*.

Definition 2.2. A proper, geodesic metric space X is called CAT(0) if the following holds. Given a geodesic triangle $\Delta = \Delta(p, q, r) \subset X$ with comparison triangle $\Delta' = \Delta'(p', q', r') \subset \mathbb{R}^2$ and comparison map $f\colon \Delta \to \Delta'$, for any $t_1 \in [p, q]$ and $t_2 \in [p, r]$ we have

$$d_X(t_1, t_2) \leqslant d_{\mathbb{R}^2}(f(t_1), f(t_2)).$$

The notion of a CAT(0) space generalizes the concept of nonpositive curvature to more general metric spaces. Many results, such as the Cartan–Hadamard theorem, extend naturally to CAT(0) spaces. In particular, CAT(0) spaces are geodesically convex and therefore contractible. As in the case of curvature, being CAT(0) is a local notion and comes with an analogous local-to-global principle. Thus, if a space is simply connected and locally CAT(0), it is globally CAT(0). It is not difficult to show that CAT(0) spaces, the product of two CAT(0) spaces with the ℓ^2-metric is again CAT(0) [BH99].

Definition 2.3. A group G is CAT(0) if it acts properly and cocompactly on a CAT(0) space by isometries.

Examples of CAT(0) spaces include simplicial trees and nonpositively curved Riemannian manifolds. Accordingly, examples of CAT(0) groups include free groups, free abelian groups, and hyperbolic surface groups. Conjecturally, all Artin groups are CAT(0) as well, but this has only been verified in specific cases and is open even for finite type Artin groups such as braid groups B_n, $n > 7$.

Llosa-Isenrich–Py [LIP21] studied when a surface-by-surface group is CAT(0) if both base and fiber genus are at least 2. They derived a necessary condition on the algebraic monodromy: it is either injective or has finite image. In [Zhu23], the third author extended their results by introducing the following definition and general obstruction for an extension to be CAT(0).

Definition 2.4 (Property LIP)**.** A group Q has *Property LIP* if, for every infinite normal subgroup $N \trianglelefteqslant Q$, there is an infinite finitely generated subgroup $N_0 < N$ so that the centralizer $C_Q(N_0)$ is finite.

Theorem 2.5 (Theorem 1.5, [Zhu23])**.** *Suppose G fits into a short exact sequence*

$$1 \to R \to G \to \Gamma \to 1$$

where R is finitely generated and has trivial center, and Q has Property LIP. Then if G is CAT(0)*, the algebraic monodromy $\rho\colon Q \to \mathrm{Out}(R)$ either has finite image or finite kernel.*

Remark 2.6. *The initial version of this theorem assumed that R is a surface group, but later, the author realized that the proof in* [Zhu23] *only requires R to be finitely generated (see* [Zhu23]*-p.4649, the eighth line from the bottom), have a trivial center (see* [Zhu23, Remark 3.2] *and the proof of* [LIP21, Lemma 30] *to make the theorem work.*

Although the definition of Property LIP is easy to state, a simple criterion which guarantees it holds is provided by the following.

Proposition 2.7 (Theorem 1.8, [Zhu23]). *If Q is acylindrically hyperbolic, then Q has Property LIP.*

We recall that a group is called acylindrically hyperbolic if it admits a nonelementary acylindrical action on a hyperbolic geodesic metric space. Since the precise definition of acylindrical action will not figure prominently in the sequel, we will not review it here, and instead refer the reader to [Osi16].

3. Positive results

In this section we explore several infinite families of curves C for which $\pi_1(\mathbb{P}^2\backslash(C \cup L))$ or $\pi_1(\mathbb{P}^2\backslash C)$ is CAT(0). The first class consists of those with finite monodromy. For these, we show that $\pi_1(\mathbb{P}^2\backslash(C \cup L))$ is CAT(0) and moreover that the total space $\mathrm{Bl}_O(\mathbb{P}^2\backslash(C \cup L)$ in fact admits a complete, finite-volume nonpositively curved Riemannian metric. The second class of examples is derived from Robb's computation of the fundamental group of a branching curve for a generic projection of a complete intersection surface. We will deduce Theorem A as a combination of Proposition 3.2 and Theorem 3.6.

3.1. **Finite monodromy.**

As a consequence of Theorem 2.5, the existence of a CAT(0) extension group results in a dichotomy for the monodromy, namely that it either be finite or have finite kernel. In this section, we investigate the fundamental groups of plane curve complements with finite monodromy.

Example 3.1. For $n \geqslant 2$, consider the family of plane curves given by the equation

$$x^{n-1}z - y^n = 0.$$

This curve has a single multiple cusp of order n at $[0:0:1]$. The projection onto x has one singular fiber at $x = 0$, hence the base of fibration is a twice-punctured sphere while the fiber is an n-punctured sphere. This yields the extension

$$1 \to F_{n-1} \to G \to \mathbb{Z} \to 1$$

with monodromy isomorphic $\mathbb{Z}/n\mathbb{Z}$, and acts by cyclically permuting the punctures of the fiber.

For plane curves with finite monodromy we prove:

Proposition 3.2. *Let $C \subseteq \mathbb{P}^2$ be a plane curve and suppose $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash C)$ fits into a short exact sequence*

$$1 \to F_n \to \pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash C) \to F_m \to 1$$

with finite monodromy. Then $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash C)$ is CAT(0). Moreover, $\mathrm{Bl}_O(\mathbb{P}^2\backslash C)$ admits a finite-volume Riemannian metric of nonpositive curvature.

This will be an easy consequence of the next two lemmas.

Lemma 3.3. *Suppose G fits into a short exact sequence*

$$1 \to F_n \to G \to F_m \to 1$$

with finite monodromy $H \leqslant \mathrm{Out}(F_n)$. Then G is the fundamental group of a compact, NPC 2-complex X whose universal cover is a product of trees.

Proof. By the (Nielsen) realization theorem for $\mathrm{Out}(F_n)$ due independently to Culler [Cul84], Khramtsov [Khr85] and Zimmermann [Zim81], there exists a rank n graph Γ realizing H by isometries (here the metric assigns a fixed positive length λ to each edge of Γ). Let R_m be a wedge of m circles, and let T_m be the universal cover of R_m. Consider the orthogonal product $X' = \Gamma \times T_m$, which is locally $\mathrm{CAT}(0)$ since each of the factors are. Then F_m acts on X' by isometries via H on the Γ-factor and by deck transformations on T_m. The action is free, proper and cocompact, since it is on the second factor and Γ is compact. Thus, the quotient X is locally $\mathrm{CAT}(0)$ and has a fundamental group of G by construction. The universal cover of X is $\widetilde{\Gamma} \times T_m$, which is a product of trees. □

Lemma 3.4. *Let X be the total space of locally trivial $S_{0,k}$-bundle over $S_{0,l}$ with $l \geqslant 1$. Then X is determined up to equivalence by the monodromy homomorphism $\rho\colon \pi_1(S_{0,l}) \to \mathrm{Mod}(S_{0,k})$.*

Proof. Since $S_{0,l}$ is homotopy equivalent to a wedge of $l-1$ circles, any such bundle is determined up to equivalence, via the clutching construction, by a homomorphism $\pi_1(S_{0,l}) \to \pi_0 \mathrm{Diff}(S^2, \{p_1, \dots p_k\}) := \mathrm{Mod}(S_{0,k})$. □

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. That $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash C)$ is $\mathrm{CAT}(0)$ follows directly from Lemma 3.3. For the second part, we know that $\mathrm{Bl}_O(\mathbb{P}^2\backslash C)$ is the total space of a locally trivial $S_{0,n+1}$-bundle over $S_{0,m+1}$, where $S_{0,k}$ denotes the k-punctured 2-sphere. The monodromy group H is a finite subgroup of the mapping class group of $S_{0,n+1}$. By the solution to the Nielsen realization problem for punctured surfaces [Zie81, Theorem 11.14], we can find a finite-volume hyperbolic surface V homeomorphic to $S_{0,n+1}$ which realizes the action of H by isometries.

Now let W be any finite-volume hyperbolic surface homeomorphic to $S_{0,m+1}$. The universal cover of both $S_{0,n+1}$ and $S_{0,m+1}$ can be identified with the hyperbolic plane $\mathbb{H}^2$. Mimicking the construction in the proof of Lemma 3.3, we consider the product $V \times \mathbb{H}^2$, and let $\pi_1(W) \cong F_m$ act by isometries on V and by deck transformations in the second component. Again the action is free and proper. The quotient X is therefore a Riemannian manifold whose universal cover can be identified with $\mathbb{H}^2 \times \mathbb{H}^2$, hence is nonpositively curved.

By construction, X is the total space of an isotrivial $S_{0,n+1}$-bundle over $S_{0,m+1}$. In particular, X is finitely covered by a product $V \times W'$ where W' is the finite cover of W corresponding to the kernel of the monodromy homomorphism. Since V and W have a finite volume, this implies $V \times W'$ has finite volume, and thus that X does as well. The fact that X is homeomorphic to $\mathrm{Bl}_O(\mathbb{P}^2\backslash C)$ follows from Lemma 3.4. □

3.2. **Branch loci of complete intersections.**

For $n \geqslant 3$, let $X \subset \mathbb{P}^n$ be a smooth surface and consider a projection $\pi\colon X \to \mathbb{P}^2$. This will exhibit X as a branched cover over $\mathbb{P}^2$, branched along a singular curve $C \subset \mathbb{P}^2$. For a generic projection, C will have only cusps and nodes.

The branching curve of a generic projection of a smooth cubic surface in $\mathbb{P}^3$ is a sextic with six cusps on a conic. Zariski [Zar29] showed that in this case the fundamental group of the complement is $\pi_1(\mathbb{P}^2\backslash C) \cong \mathbb{Z}/2 * \mathbb{Z}/3 \cong \mathrm{PSL}_2(\mathbb{Z})$. Zariski famously used this calculation to show that the moduli space of irreducible sextics with six cusps in $\mathbb{P}^2$ is disconnected (for sextic curves with six cusps not on a conic, the fundamental group is $\mathbb{Z}/6$).

Moishezon [Moi81] generalized Zariski's result by calculating the analogous fundamental group for generic projections of a smooth hypersurface in $\mathbb{P}^3$ of arbitrary degree d. For these, the complement of the branch curve C has fundamental group $\pi_1(\mathbb{P}^2\backslash C) \cong B_d/Z(B_d)$, where B_d is the braid group of the disk on d strands, and $Z(B_d)$ is its center. This quotient can be identified with a finite index subgroup of the mapping class group of $S_{0,d+1}$.

Recall that the braid group B_d maps onto the symmetric group S_d with kernel the pure braid group P_d. The abelianization of B_d is $\mathbb{Z}$, and is obtained by sending each of the standard positive Artin generators to $1 \in \mathbb{Z}$. The center $Z(B_d) \cong \mathbb{Z}$ is generated by the square of the "long braid," and since it is a positive braid, it maps to $d(d-1) \in \mathbb{Z}$. Moreover, it is itself a pure braid, and after passing to P_d we have a splitting $P_d \cong P_{0,d} \times Z(B_d)$ where $P_{0,d}$ is the kernel of the restriction of the abelianization map of B_d to P_d. Thus the image of P_d in the quotient $B_d/Z(B_d)$ is simply $P_{0,d}$.

Robb extended Moishezon's result to the case where $X \subset \mathbb{P}^n$ is a smooth, complete intersection of codimension greater than 1 [Rob97]. Let $x_1, \ldots, x_{d-1}$ be the standard generators of B_d. For $d \geqslant 4$, define

$$\widetilde{B}_d := B_d / \left\langle\!\left\langle [x_2, (x_3x_1)^{-1}x_2(x_3x_1)] \right\rangle\!\right\rangle \tag{3}$$

where $\langle\!\langle \cdot \rangle\!\rangle$ indicates normal closure. Robb shows that the center of $\widetilde{B}_d$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$. Let η and μ be the generators of $\mathbb{Z}$ and $\mathbb{Z}/2$, respectively. By the adjunction formula and Riemann-Hurwitz, the degree $\deg(S)$ will be even and a multiple of $d = \deg(X)$. Define an integer

$$m = \begin{cases} \frac{\deg(S)}{d} & d \text{ even} \\ \frac{\deg(S)}{2d} & d \text{ odd} \end{cases}$$

Robb's extension of Zariski and Moishezon's result is the following.

Theorem 3.5 ([Rob97])**.** *Let $X \subset \mathbb{P}^n$ be a smooth, non-degenerate complete intersection of codimension greater than 1. Suppose $C \subset \mathbb{P}^2$ is the branch locus of a generic projection $X \to \mathbb{P}^2$, and let C_a be a generic affine portion of C. Then for some $e \in \{0, 1\}$,*

$$\pi_1(\mathbb{C}^2\backslash C_a) \cong \widetilde{B}_d \text{ and } \pi_1(\mathbb{P}^2\backslash C) \cong \widetilde{B}_d/\langle \eta^m \mu^e \rangle.$$

One knows that $\mathrm{PSL}_2(\mathbb{Z})$ is CAT(0); indeed it is virtually free. Conjecturally, the braid group B_d for all $d \geqslant 1$ is CAT(0), and this has been verified for $d \leqslant 7$ [BM10, HKS16, Jeo23]. A result of Bowers–Ruane [BR96] implies that if $G \times \mathbb{Z}^n$ is CAT(0), then G is CAT(0). Hence, by the remarks above concerning P_d and $Z(B_d)$, we see that if B_d is CAT(0) then $B_d/Z(B_d)$ is as well. We now show that both of the groups $\widetilde{B}_d$ and $\widetilde{B}_d/\langle \eta^m \mu^e \rangle$ appearing in Theorem 3.5 is CAT(0). In fact, we will prove:

Theorem 3.6. *For any $d \geqslant 4$, $\widetilde{B}_d$ is virtually abelian. In particular, both groups $\pi_1(\mathbb{C}^2\backslash C_a)$ and $\pi_1(\mathbb{P}^2\backslash C)$ appearing in Theorem 3.5 are CAT(0).*

The second statement will be implied by the first, as a consequence of the following easy lemma.

Lemma 3.7. *Suppose G is finitely generated and virtually abelian. Then G acts properly and cocompactly on a CAT(0) space.*

Proof. Let $H \trianglelefteq G$ be a finite index normal subgroup isomorphic to $\mathbb{Z}^n$. Then G fits into a short exact sequence

$$1 \to H \cong \mathbb{Z}^n \to G \xrightarrow{\varphi} F \to 1, \tag{4}$$

where F is a finite group. From the extension in (4), we obtain an algebraic monodromy homomorphism $\rho\colon F \to \mathrm{GL}_n(\mathbb{Z})$, making $\mathbb{Z}^n$ into an $\mathbb{Z}[F]$-module. Let F' be the image of ρ. The space of marked, flat n-dimensional tori is the symmetric space $X_n := \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. Since F' is a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ and X_n is a CAT(0) space, we can find a marked, flat n-torus T on which F' acts by isometries. Thus we obtain an explicit lattice $\Lambda \leqslant \mathbb{R}^n$ and a homomorphism $\hat{\rho}\colon G \to N(\Lambda)$, where the latter is the normalizer of Λ in $\mathrm{Isom}(\mathbb{R}^n)$. The image G' of $\hat{\rho}$ acts effectively on $\mathbb{R}^n$ by isometries.

On the other hand, the kernel of $\hat{\rho}$ is a finite normal subgroup $K \trianglelefteq G$, hence $K \cap H = \{1\}$, and K maps injectively to F. Choose a sufficiently high-dimensional closed Euclidean disk D^k on which F acts effectively by isometries and consider the product $X = \mathbb{R}^n \times D^k$. Then G acts via $\hat{\rho}$ on $\mathbb{R}^n$ and by factoring through φ on D^k. This action is effective by construction, and proper since each point stabilizer is conjugate to subgroup of F. □

Proof of Theorem 3.6. Since $\widetilde{B}_d$ is the quotient of B_d by a commutator, the abelianization of B_d factors through $\widetilde{B}_d$. Hence, their abelianizations are equal and isomorphic to $\mathbb{Z}$. Let $\widetilde{P}_d \leqslant \widetilde{B}_d$ be the finite-index subgroup which is the image of P_d under the quotient. We know that P_d splits as a product $P_{0,d} \times Z(B_d) \cong P_{0,d} \times \mathbb{Z}$, where $P_{0,d}$ lies in the kernel of the abelianization map. It follows that $\widetilde{P}_d$ also splits as a product $\widetilde{P}_{0,d} \times \mathbb{Z}$ where $\mathbb{Z}$ lies in the center $Z(\widetilde{P}_d)$. Since $\widetilde{P}_d$ has finite index in $\widetilde{B}_d$, it is enough to show that $\widetilde{P}_{0,d}$ is virtually abelian by Lemma 3.7.

Robb [Rob97, Proposition 2.2] proves that $\widetilde{P}_{0,d}$ is generated by elements $v_1, \ldots, v_{d-1}, \mu$ subject to relations

- $\mu^2 = 1$.
- For all $i \neq j$, $[v_i, v_j] = \begin{cases} 1 & |i-j| \geqslant 2 \\ \mu & |i-j| = 1 \end{cases}$
- $[\mu, v_i] = 1$ for all i.

This implies $\widetilde{P}_{0,d}$ is a central extension

$$1 \to \mathbb{Z}/2 \to \widetilde{P}_{0,d} \to \mathbb{Z}^{d-1} \to 1$$

Since μ has order 2, the subgroup generated by the squares of the v_i and μ has finite index and is isomorphic to $\mathbb{Z}^{d-1} \oplus \mathbb{Z}/2$. Thus $\widetilde{P}_{0,d}$, and hence $\widetilde{B}_d$, is virtually free abelian. □

4. Negative Results

We now turn to examples of quasi-projective groups that are not CAT(0). First, we present a detailed example showing that the complement of the 3-cuspidal quartic together with the singular fibers under a (non-generic) projection does not have CAT(0) fundamental group. In general, complements of *reducible* plane curves will not be CAT(0), and this is an example of this phenomenon. Second, we consider the family of curves arising from the universal deformation of an isolated plane curve singularity of ADE type. In the case of E_6, E_7, and E_8, we use a result of Wajnryb to show that the fundamental group of the universal family is not CAT(0), proving Theorem B.

4.1. The real 3-cuspidal quartic.

If $d \leqslant 3$, $\mathrm{Mod}(\mathbb{P}^1 \backslash \{x_1, ..., x_d\})$ is finite, hence the image of the monodromy homomorphism is necessarily finite. In this section, we will closely examine an explicit example of a 3-cuspidal quartic (*i.e.* $d = 4$) curve C whose braid monodromy is infinite. Although $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2) \backslash C) = \pi_1(\mathbb{P}^2 \backslash C)$ will be finite and thus CAT(0), we will nevertheless show that $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2) \backslash (C \cup$

L)) is not CAT(0), where L is the union of singular lines arising from a particular non-generic projection. We closely follow the exposition of the real 3-cuspidal quartic curve from Catanese and Wajnryb, [CW04, Section 4].

A *3-cuspidal quartic* is an irreducible plane curve of degree 4 having 3 ordinary cusps as singularities (see Example A.1 in the Appendix for a more complete description of its construction). Any two such curves are equivalent up to equisingular deformation, and Zariski showed that the fundamental group of the complement of the 3-cuspidal quartic is $B_3(S^2)$, the braid group of the sphere on 3 strands. If $C \subset \mathbb{P}^2$ is an irreducible quartic curve that is not a 3-cuspidal quartic, then the fundamental group $\pi_1(\mathbb{P}^2 \backslash C)$ is abelian (cf. [Dim12, (4.3)]).

Example 4.1 ([CW04], Section 4)**.** The following is a real quartic with three cusps:

$$C = \{F([x:y:z] = (x^2+y^2)^2 + x^3z + 9xy^2z + \frac{27}{4}y^2z^2 = 0\}.$$

Since $O := [0:1:0] \notin C$, we can define a projection $\phi_C : C \to \mathbb{P}^1$ after blowing up at O, which projects $[x:y:z] \mapsto [x:z]$ away from O. For any $t \in \mathbb{P}^1$, ϕ_C is ramified at t if and only if there exists $p \in C_t$ such that $\frac{\partial F}{\partial y}(p) = 0$. There are two such ramification points: a double tangency at $y_4 = [1:0]$ and a single tangency $y_2 = [-1:1]$. There are 2 more singular fibers from $y_1 = [\frac{-9}{8}:1]$ (double cusp), $y_4 = [0:1]$ (cusp).

Let $L = \cup_{i=1}^4 L_i$ be the union of the four singular fibers. After deleting L, let the generators of $\pi_1(\mathbb{P}^1 \backslash \{y_1, \cdots, y_4\})$ corresponding to y_1, y_2 and y_3 be τ_1, τ_2 and τ_3, respectively. In the chart $z \neq 0$, we choose a basepoint on the positive real axis and orient these generators so that they follow a path in the upper half plane, go counterclockwise around exactly one of the punctures (all of which are real), and then go back to the basepoint in the upper half-plane. Since C has degree 4, we have a fiber bundle $\mathbb{P}^1 \backslash \{x_1, ..., x_4\} \to \mathrm{Bl}_O(\mathbb{P}^2) \backslash (C \cup L)) \xrightarrow{\phi} \mathbb{P}^1 \backslash \{y_1, ..., y_4\}$ as in the setting described above.

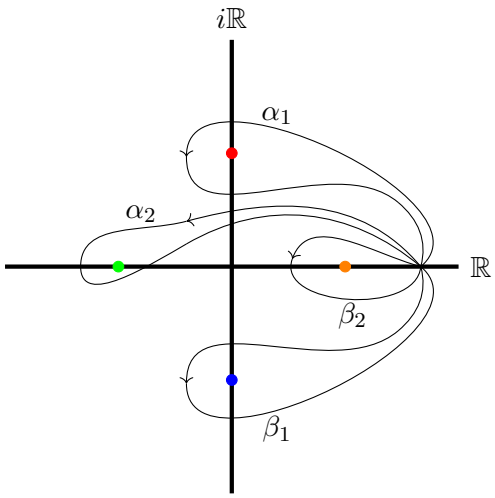


FIGURE 1. A choice of generating set for the fundamental group of the fiber at a point along the positive real axis in the base. The punctures x_1, x_2, x_3 and x_4 correspond to α_1, α_2, β_1 and β_2, respectively.

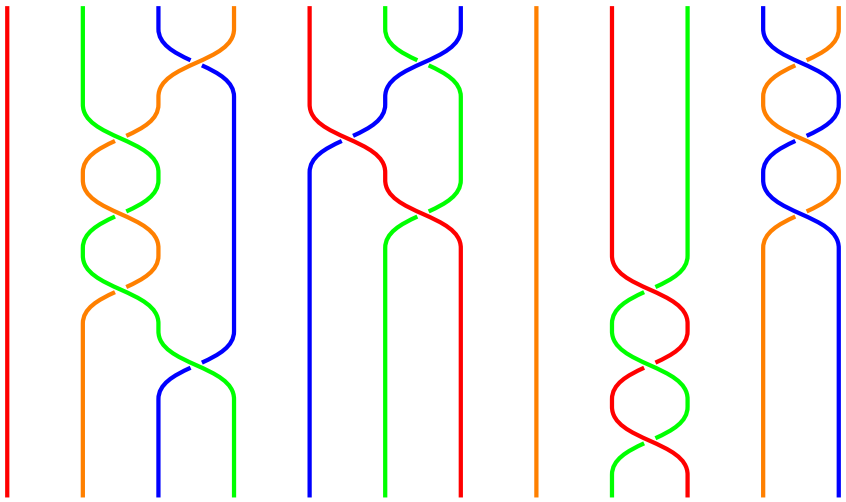


FIGURE 2. From left to right, the three 4-stranded braids correspond to τ_1, τ_2, and τ_3. The strands of each braid correspond to x_1, x_2, x_3, x_4 in left to right order, and match the colors in Figure 1.

Along the positive real axis, the fiber contains two real punctures and two purely imaginary ones. We order the punctures and choose representatives for simple loops around the pure imaginary punctures α_1 for x_1, α_2 for x_2, and the real punctures β_1 for x_3, β_2 for x_4 as in Figure 1. The τ_i act as the following braids in terms of the standard Artin generators $\{\sigma_i\}_{i=1}^3$:

$$\begin{aligned}\tau_1 &= \sigma_3\sigma_2^3\sigma_3^{-1}\\ \tau_2 &= \sigma_2\sigma_1\sigma_2^{-1}\\ \tau_3 &= \sigma_1^3\sigma_3^3.\end{aligned}$$

These braids are pictured in Figure 2. In terms of the chosen basis $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ for $\pi_1(\mathbb{P}^1\backslash\{x_1, ..., x_4\})$ the algebraic monodromy is given by:

$$\begin{aligned}
\tau_1\colon \alpha_1 &\longmapsto \alpha_1 & \tau_2\colon \alpha_1 &\longmapsto \alpha_1\beta_2\beta_1\bar{\beta}_2\overline{\alpha_1} & \tau_3\colon \alpha_1 &\longmapsto \alpha_1\alpha_2\alpha_1\alpha_2\overline{\alpha_2}\overline{\alpha_1}\\
\alpha_2 &\longmapsto \alpha_2\beta_2\alpha_2\beta_2\overline{\alpha_2}\bar{\beta}_2\overline{\alpha_2} & \alpha_2 &\longmapsto \alpha_1\beta_2\bar{\beta}_1\bar{\beta}_2\alpha_2\beta_2\beta_1\bar{\beta}_2\overline{\alpha_1} & \alpha_2 &\longmapsto \alpha_1\alpha_2\alpha_1\overline{\alpha_1}\overline{\alpha_2}\overline{\alpha_1}\\
\beta_1 &\longmapsto \beta_1 & \beta_1 &\longmapsto \bar{\beta}_2\alpha_1\beta_2 & \beta_1 &\longmapsto \beta_2\beta_1\beta_2\bar{\beta}_1\bar{\beta}_2\\
\beta_2 &\longmapsto \alpha_2\beta_2\alpha_2\bar{\beta}_2\overline{\alpha_2} & \beta_2 &\longmapsto \beta_2 & \beta_2 &\longmapsto \beta_2\beta_1\beta_2\beta_1\bar{\beta}_2\bar{\beta}_1\bar{\beta}_2
\end{aligned}$$

where, for ease of notation, we have used $\overline{\alpha}$ to denote α^{-1}. After an easy simplification, the fundamental group of the complement of the 3-cuspidal quartic curve, $\pi_1(\mathbb{P}^2\backslash C)$, is

$$\langle \alpha_1, \alpha_2 | \alpha_2^{\alpha_1} = \alpha_1^{\alpha_2}, \alpha_2\alpha_1^2\alpha_2 = 1\rangle \cong B_3(S^2),$$

which is finite, and, in particular, CAT(0).

Combined with the results of Section 2.1, we obtain a presentation $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C\cup L)) = \langle S|R\rangle$, where

$$S = \{\alpha_1, \alpha_2, \beta_1, \beta_2, \tau_1, \tau_2, \tau_3\},$$
$$R = \left\{ \begin{array}{l} \alpha_1^{\tau_1} = \alpha_1, \alpha_1^{\tau_2} = \beta_1^{\alpha_1\beta_2}, \alpha_1^{\tau_3} = \alpha_2^{\alpha_1\alpha_2\alpha_1}; \\ \alpha_2^{\tau_1} = \beta_2^{\alpha_2\beta_2\alpha_2}, \alpha_2^{\tau_2} = \alpha_2^{\alpha_1\beta_2\beta_1\overline{\beta_2}}, \alpha_2^{\tau_3} = \alpha_1^{\alpha_1\alpha_2}; \\ \beta_1^{\tau_1} = \beta_1, \beta_1^{\tau_2} = \alpha_1^{\overline{\beta_2}}, \beta_1^{\tau_3} = \beta_2^{\beta_1\beta_2}; \\ \beta_2^{\tau_1} = \alpha_2^{\alpha_2\beta_2}, \beta_2^{\tau_2} = \beta_2, \beta_2^{\tau_3} = \beta_1^{\beta_2\beta_1\beta_2}; \\ \alpha_1\alpha_2\beta_1\beta_2 = 1. \end{array} \right\}$$

Solving for $\beta_1 = \bar{\beta}_2\overline{\alpha_2}\overline{\alpha_1}$, we can simplify the above presentation to the following:

$$S = \{\alpha_1, \alpha_2, \beta_2, \tau_1, \tau_2, \tau_3\},$$
$$R = \left\{ \begin{array}{l} \alpha_1^{\tau_1} = \alpha_1, \alpha_1^{\tau_2} = (\bar{\beta}_2\bar{\alpha}_2\bar{\alpha}_1)^{\alpha_1\beta_2}, \alpha_1^{\tau_3} = \alpha_2^{\alpha_1\alpha_2\alpha_1}; \\ \alpha_2^{\tau_1} = \beta_2^{\alpha_2\beta_2\alpha_2}, \alpha_2^{\tau_2} = \alpha_2^{\alpha_1\bar{\alpha}_2\bar{\alpha}_1\overline{\beta_2}}, \alpha_2^{\tau_3} = \alpha_1^{\alpha_1\alpha_2}; \\ \beta_2^{\tau_1} = \alpha_2^{\alpha_2\beta_2}, \beta_2^{\tau_2} = \beta_2, \beta_2^{\tau_3} = (\bar{\beta}_2\bar{\alpha}_2\bar{\alpha}_1)^{\bar{\alpha}_2\bar{\alpha}_1\beta_2} \end{array} \right\}$$

Now we consider the geometric monodromy of the associated punctured sphere bundle

$$\mathbb{P}^1\backslash\{x_1, ..., x_4\} \to \mathrm{Bl}_O(\mathbb{P}^2)\backslash(C\cup L) \xrightarrow{\phi} \mathbb{P}^1\backslash\{y_1, ..., y_4\}.$$

By [FM11, Proposition 2.7], there is an isomorphism

$$\mathrm{Mod}(S_{0,4}) = (\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}) \rtimes \mathrm{PSL}_2(\mathbb{Z}),$$

while

$$B_4(S^2) \cong (\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z})$$

Let $q\colon T^2 \to T^2/\langle\iota\rangle = S_{0,4}$ be the quotient map, where ι is the hyperelliptic involution. Identify the torus T^2 with a quotient of the unit square (rotated by $\pi/4$) as in Figure 3. With respect to the usual group structure on T^2, the involution ι can be identified with $-I$ and the ramification locus on T^2 consists of the four points $\tilde{A}_1 = (1/2, 1/2)^T, \tilde{A}_2 = (0, 1/2), \tilde{B}_1 = (0, 0)^T, \tilde{B}_2 = (1/2, 0)^T$, which cover the branch points x_1, x_2, x_3, x_4 as in Figure 1.

These points generate the 2-torsion subgroup $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \leqslant T^2$, and are preserved by the action of $\mathrm{SL}_2(\mathbb{Z})$. Translations of the torus by these four elements comprise the normal Klein four subgroup in the semi-direct product decompositions of $\mathrm{Mod}(S_{0,4})$ and $B_4(S^2)$ above. Thus, the product of two elements $(v_1, M_1), (v_2, M_2) \in B_4(S^2) \cong (\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}) \rtimes \mathrm{SL}(2, \mathbb{Z})$ is given by

$$(v_1, M_1)\cdot(v_2, M_2) = (v_1 + M_1\cdot v_2, M_1M_2)$$

where $A \in \mathrm{SL}_2(\mathbb{Z})$ acts on $v \in \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ via reduction mod 2.

The generators of $\mathrm{SL}_2(\mathbb{Z})$ are the two Dehn twists

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

both of which fix $\tilde{B}_1$. We can relate these to the standard Artin generators of $B_4(S^2)$ as follows. Each positive half-twist generator in $S_{0,4}$ lifts to a map of T^2 which is isotopic one of these two Dehn twists, but fixing different pairs of the ramification points. To fix notation, let

$$x := \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, y := \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

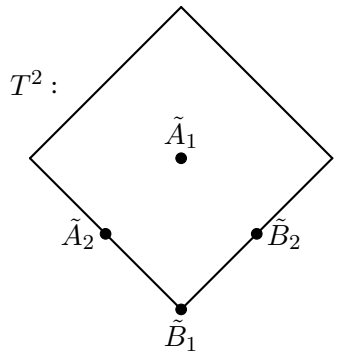


FIGURE 3. Representation of T^2 with the 4 ramification points of ι at $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$, and $\tilde{B}_2$.

be the two generators of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, thought of translations on T^2. The three generators σ_1, σ_2, σ_3 then correspond to the following pairs group elements

$$\begin{aligned}
\sigma_1 &= (0, S) \\
\sigma_2 &= (x, I) \cdot (0, T) \cdot (x, I) = (x, T) \cdot (x, I) = (x + y + x, T) = (y, T) \\
\sigma_3 &= (y, I) \cdot (0, S) \cdot (y, I) = (y, S) \cdot (y, I) = (y + x + y, S) = (x, S)
\end{aligned}$$

Now we can compute the image of the three generators of the fundamental group of the base space, τ_1, τ_2, τ_3, in the braid group $B_4(S^2)$, in terms of matrices:

$$\begin{aligned}
\tau_1 = \sigma_3 \sigma_2^3 \sigma_3^{-1} &= (x, S) \cdot (y, T^3) \cdot (x, S^{-1}) \\
&= (x + x + y, ST^3) \cdot (x, S^{-1}) \\
&= (y + y, ST^3S^{-1}) = (0, ST^3S^{-1}) \\
\tau_2 = \sigma_2 \sigma_1 \sigma_2^{-1} &= (y, T) \cdot (0, S) \cdot (y, T^{-1}) \\
&= (y, TS) \cdot (y, T^{-1}) = (x + y, TST^{-1}) \\
\tau_3 = \sigma_1^3 \sigma_3^3 &= (0, S^3) \cdot (x, S^3) = (x, S^6)
\end{aligned}$$

Recall that $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \cong \langle A \rangle *_{\langle -I \rangle} \langle B \rangle$, where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Thus, $A^2 = B^3 = -I$ and

$$\begin{aligned}
AB &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = S \\
AB^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = T
\end{aligned}$$

Substituting these above, we obtain

$$\begin{aligned}\tau_1 &= \big(0, (AB)(AB^2)^3(AB)^{-1}\big) = \big(0, ABAB^2AB^2ABA\big)\\ \tau_2 &= \big(x+y, (AB)^2(AB)^3(AB^2)^{-1}\big) = \big(x+y, AB^2AB^2A\big)\\ \tau_3 &= \big(x, (AB)^6\big)\end{aligned}$$

Observe that

$$\tau_2^{-3} = \big(x+y, AB(AB^2AB^2A)BA\big) = (x+y, I)\cdot\tau_1,$$

and

$$\tau_2\tau_3 = \big(x+y, AB^2AB^2A\big)\cdot\big(x, (AB)^6\big) = \big(x, (BA)^3B\big).$$

It follows that $\langle\tau_1,\tau_2,\tau_3\rangle = \langle g_1, g_2, g_3\rangle$, where

$$\begin{aligned}g_1 &= (x+y, I) = \tau_1\tau_2^3,\\ g_2 &= \Big(0, (AB^2)^2A\Big) = g_1\tau_2 = \tau_1\tau_2^4\\ g_3 &= \big(x, (BA)^3B\big) = \tau_2\tau_3.\end{aligned}$$

The images of $(AB^2)^2A, (BA)^3B \in \mathrm{SL}_2(\mathbb{Z})$ under the projection to $\mathrm{PSL}_2(\mathbb{Z})$ generate a free group of rank 2. Indeed, each has infinite order, and since they start and end in different letters, their inverses do too. Thus no cancellation occurs after taking products. It follows that $\langle g_2, g_3\rangle \cong F_2$. Now observe that g_1 commutes with both g_2 and g_3; hence, the image of the monodromy is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times F_2$. The extension of the fundamental group

$$1 \to F_3 \to \pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C\cup_{i=1}^3 L_i)) \to F_3 \to 1,$$

with infinite monodromy $\mathbb{Z}/2\mathbb{Z}\times F_2$ and infinite kernel of the monodromy. Clearly, F_3 has the LIP Property and trivial center, so by Theorem 2.5, we have the following theorem:

Theorem 4.2. *The fundamental group of the complement of the union of 3-cuspidal quartic and the singular fibers, $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C\cup L))$, is not CAT(0).*

When we fill in all the singular fibers, $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash C)$ is finite and hence CAT(0), while here $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C\cup L)$ is not. A natural question is whether adding back a subset of the singular fibers makes a difference.

Let L_0 be the union of the two fibers containing cusps. Since the fibers coming from tangencies depend on the projection, a reasonable modification is to add back the fibers coming from tangencies, i.e., setting $\tau_2 = 1$ and $\tau_1\tau_2\tau_3 = 1$. The presentation of the fundamental group becomes $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash L_0) = \langle S|R\rangle$, where

$$S = \{\alpha_1, \alpha_2, \beta_2, \tau_1\}$$

$$R = \left\{\begin{array}{l}\alpha_1^{\tau_1} = \alpha_1, \alpha_1 = (\bar\beta_2\bar\alpha_2\bar\alpha_1)^{\alpha_1\beta_2}, \alpha_1^{\bar\tau_1} = \alpha_2^{\alpha_1\alpha_2\alpha_1};\\ \alpha_2^{\tau_1} = \beta_2^{\alpha_2\beta_2\alpha_2}, \alpha_2 = \alpha_2^{\alpha_1\bar\alpha_2\bar\alpha_1\bar\beta_2}, \alpha_2^{\bar\tau_1} = \alpha_1^{\alpha_1\alpha_2};\\ \beta_2^{\tau_1} = \alpha_2^{\alpha_2\beta_2}, \beta_2^{\bar\tau_1} = (\bar\beta_2\bar\alpha_2\bar\alpha_1)^{\bar\alpha_2\bar\alpha_1\beta_2}.\end{array}\right\}$$

Using that $\alpha_1^{\tau_1} = \alpha_1$ and $\alpha_1^{\bar\tau_1} = \alpha_2^{\alpha_1\alpha_2\alpha_1}$, it follows that $\alpha_1\alpha_2\alpha_1 = \alpha_2\alpha_1\alpha_2$, and by $\alpha_1 = (\bar\beta_2\bar\alpha_2\bar\alpha_1)^{\alpha_1\beta_2}$, we obtain

$$\alpha_2 = \overline{\alpha_1}\bar\beta_2\overline{\alpha_1}.$$

In particular, we can solve for β_2 in terms of α_1, α_2, so $\langle\alpha_1,\alpha_2,\beta_2\rangle = \langle\alpha_1,\alpha_2\rangle$. By $\alpha_2^{\bar\tau_1} = \alpha_1^{\alpha_1\alpha_2} = \alpha_1\alpha_2\alpha_1\bar\alpha_2\bar\alpha_1$ and $\alpha_1\alpha_2\alpha_1 = \alpha_2\alpha_1\alpha_2$, it follows that $\alpha_2^{\bar\tau_1} = \alpha_2$.

Therefore τ_1 commutes with α_1 and α_2. The existence of the projection onto $\langle\tau_1\rangle \cong \mathbb{Z}$ then implies

$$\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L_0)) \cong \langle\tau_1\rangle \times \langle\alpha_1, \alpha_2\rangle \cong \mathbb{Z} \times \pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash C)$$

where $\langle\alpha_1, \alpha_2\rangle \cong \pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash C)$ since killing τ_1 is equivalent to filling all the singular fibers. In particular, since $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash C)$ is finite, $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L_0))$ is CAT(0) by Lemma 3.7. Thus, we have proved

Proposition 4.3. *Let C be the 3-cuspidal quartic, and let L_0 be the union of the two singular fibers containing cusps. Then $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L_0))$ is CAT(0).*

In contrast, given any plane curve complement with infinite monodromy, deleting nonsingular fibers in addition to the singular fibers will make the resulting complement, not CAT(0). Indeed, deleting a nonsingular fiber creates an additional generator in the fundamental group of the base that acts trivially on the fiber. In particular, the new monodromy will have the same monodromy but an infinite kernel. For some examples of curves with infinite monodromy, we reference several results obtained by Moishezon:

Theorem 4.4 (Theorems 1, 2, and 3, [Moi81])**.**

(i) Let C be an irreducible plane curve of degree d that is either smooth or rational with nodes as the only singularities. Then, for a generic projection, the image of braid monodromy is the full Artin braid group B_d.

(ii) Let C be a union of d lines in general position. Then, for a generic projection, the image of braid monodromy is the pure braid group P_d.

As a quick application of Theorem 4.4, we obtain the following corollary, giving infinitely many examples of non-CAT(0) plane curve complements. Similar considerations apply to case (ii) above.

Corollary 4.5. *Let C be an irreducible plane curve of degree $d \geqslant 4$ that is smooth or rational with nodes as the only singularities, and let $L = \cup_{i=1}^{n} L_i$ be the union of singular fibers for a generic projection. Then $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L))$ is not CAT(0).*

Proof. For $d \geqslant 4$, $\mathrm{Mod}(S_{0,d})$ is infinite, and by Theorem 4.4(i), in this case the braid monodromy is all of B_d. Since both compositions $B_d \to B_d(S^2)$ and $B_d(S^2) \to \mathrm{Mod}(S_{0,d})$ are surjective, it follows that the geometric monodromy $\rho\colon \pi_1(S_{0,n}) \to \mathrm{Mod}(S_{0,d})$ is surjective. However, as $\mathrm{Mod}(S_{0,d})$ is not free, ρ cannot be injective. We conclude that $\pi_1(\mathrm{Bl}_O(\mathbb{P}^2)\backslash(C \cup L))$ is not CAT(0) by Theorem 2.5. □

4.2. **Rational cuspidal curves with higher order singularities.**
Flenner and Zaidenberg [FZ96] classified rational cuspidal curves $C \subset \mathbb{P}^2$ with at least 3 singularities, one of which has multiplicity $\deg(C) - 2$. A study of fundamental groups of these curves was initiated by Artal Bartolo [AB97]. Unlike the case of simple cusps, we will show that in this case, the fundamental group of the complement does not need to be CAT(0).

Let C be a rational cuspidal curve of degree d. In the special case where there are exactly three cusps with one multiplicity of $d-2$, there are, in fact, only 3 singular points, which can be enumerated as follows. For each triple $(d, a, b) \in \mathbb{Z}^3$ with $d \geqslant 3$ $a \geqslant b > 0$ and $a + b = d - 2$, there is exactly one curve $C_{d,a,b}$ have three cusps, each of whose germs is a single Puiseux pair of type $(d-1, d-2)$, $(2a+1, 2)$ and $(2b+1, 2)$, respectively. The case $C_{4,1,1}$ is the tricuspidal quartic discussed in section 4.1.

Recall that for integers $p, q, r > 0$ the triangle group $T_{p,q,r}$ has presentation:

$$T_{p,q,r} = \langle x, y \mid x^p = y^q = (yx)^r = 1 \rangle \tag{5}$$

If $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} > \pi$ (resp. $= \pi$, resp. $< \pi$) then there exists a tessellation of the sphere $\mathbb{S}^2$ (resp. Euclidean plane $\mathbb{E}^2$, resp. hyperbolic plane $\mathbb{H}^2$) by triangles with those interior angles, and $T_{p,q,r}$ is the group of orientation-preserving symmetries of that tessellation.

Theorem 4.6 ([AB97])**.** *Let $C = C_{d,a,b}$ as above and let $2n + 1 = \gcd(2a + 1, 2b + 1)$ for some $n \geqslant 0$.*

(i) If $n = 0$ then $\pi_1(\mathbb{P}^2 \backslash C)$ is finite cyclic.
(ii) If $n > 0$ and (d, a, b) is $(4, 1, 1)$ or $(7, 4, 1)$, then $\pi_1(\mathbb{P}^2 \backslash C)$ is a finite group.
(iii) Otherwise, $n > 0$ and $\pi_1(\mathbb{P}^2 \backslash C)$ is a central extension

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{P}^2 \backslash C) \to T_{2,2n+1,d-2} \to 1. \tag{6}$$

It is not difficult to see that the triangle group appearing in case (iii) must be hyperbolic. Central extensions of this form are 3-dimensional cocompact lattices with one of two types of Thurston geometries, namely the product geometry $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, the geometry associated to the universal cover of the Lie group $\mathrm{SL}_2(\mathbb{R})$.

Since $\mathbb{H}^2 \times \mathbb{R}$ is CAT(0), any cocompact lattice with this geometry will be CAT(0). On the other hand, $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is not CAT(0), and moreover any cocompact lattice of this type cannot act properly cocompactly by isometries on a CAT(0) space. This is because for a group acting properly and cocompactly on a CAT(0) space, an infinite central subgroup must virtually split off as a direct factor as a consequence of the flat torus theorem [BH99, Theorem II.7.1]. Any cocompact lattice with $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is a central extension by $\mathbb{Z}$ which is not virtually split. Since central extensions of group G by $\mathbb{Z}$ are classified by elements of $H^2(G; \mathbb{Z})$, the two geometries are distinguished by whether the extension cocycle associated to the short exact sequence in (6) has finite order or not:

Lemma 4.7. *Let $T = T_{p,q,r}$ be a hyperbolic triangle group with $p, q, r < \infty$. Suppose $\alpha \in H^2(T; \mathbb{Z})$ and let G_α be the extension of T corresponding to α. Then G_α is virtually split if and only α has finite order.*

Proof. If α is virtually split, we can find a finite index subgroup $\iota\colon T' \hookrightarrow T$ such that $\iota^*(\alpha) = 0$. The existence of the transfer $\tau\colon H^2(T'; \mathbb{Z}) \to H^2(T; \mathbb{Z})$ implies that

$$[T\colon T'] \cdot \alpha = \tau\iota^*(\alpha) = \tau(0) = 0,$$

so α has finite order. Conversely, assume α has finite-order, and let $T' \leqslant T$ a finite-index torsion-free subgroup, which exists by Selberg's lemma. Observe that $T' \cong \Gamma \leqslant \mathrm{PSL}_2(\mathbb{R})$ is a cocompact, torsion-free lattice, hence is isomorphic to the fundamental group of a hyperbolic surface Σ. Since T consists of orientation-preserving isometries, Σ is orientable and $H^2(T'; \mathbb{Z}) \cong \mathbb{Z}$. Hence $\iota^*(\alpha) = 0$ so G_α is virtually split. □

Theorem 4.8. *Let $C = C_{d,a,b}$ and let $2n + 1 = \gcd(2a + 1, 2b + 1)$ for some $n \geqslant 0$. If $n > 0$ and (d, a, b) is neither $(4, 1, 1)$ nor $(7, 4, 1)$, then $\pi_1(\mathbb{P}^2 \backslash C)$ is not CAT(0).*

Proof. Let $T = T_{2,2n+1,d-2}$. Since $2n+1 = \gcd(2a+1, 2b+1)$ and $2(d-1) = (2a+1)+(2b+1)$ we can write

$$2(d - 1) = 2w(2n + 1)$$

for some integer $w > 0$. From (5), the presentation of T is given by

$$T = \langle x, y \mid x^2 = y^{2n+1} = (yx)^{d-2} = 1 \rangle$$

Let $A = x^2$, $B = y^{2n+1}$ and $C = (yx)^{d-2}$, which we can regard as 2-cells for the presentation 2-complex of T. From the Hopf formula for $H_2(T)$ (cf. [Bro12, Theorem 5.3]), one readily computes that $H_2(T) \cong \mathbb{Z}$ and generated by an element Δ given in terms of A, B, C,

$$\Delta = \begin{cases} -(2n+1)(d-2)A - 2(d-2)B + 2(2n+1)C, & d \text{ odd} \\ \frac{-(2n+1)(d-2)}{2}A - (d-2)B + (2n+1)C, & d \text{ even} \end{cases}$$

By the universal coefficient theorem, $H^2(T)$ has rank 1, generated by the dual cocycle Δ^*. By Lemma 4.7, we conclude that the extension in (6) will be virtually split if and only if the extension cocycle evaluates to 0 on Δ.

Let μ be the generator of the central $\mathbb{Z}$ subgroup appearing in sequence (6). In the proof of [AB97, Corollary 1], it is shown that in terms this presentation we have

$$A = x^2 = \mu,\ B = y^{2n+1} = \mu^{-n},\ C = (yx)^{d-2} = \mu^w.$$

Thus, evaluating the extension cocycle on Δ substitutes $A = 1$, $B = -n$, $C = w$, whence we obtain $d/2$ when d is even and d when d is odd. Since $d \geqslant 3$, neither of these vanish, so the extension is not virtually split, and thus not CAT(0). □

Theorem 4.8 shows that there exist infinitely many irreducible plane curves such that the fundamental group of the complement is not CAT(0). On the other hand, it is known that $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$, hence up to quasi-isometry, even these examples resemble CAT(0) groups.

4.3. **Semi-universal deformations of ADE singularities.**
Recall (cf.[Gre07], Chapter II, Cor.1.17), that for a germ X given by $f(x_1, \cdots, x_n) = 0$ with an isolated singularity at the origin, the semi-universal deformation $\widetilde{X} \to \mathbb{C}^\tau$ of f can be given as the map induced by projection $\widetilde{X} \subset \mathbb{C}^n \times \mathbb{C}^\tau \to \mathbb{C}^\tau$ where $\widetilde{X}$ is the hypersurface $F(x,t) = f(x) + \sum_1^\tau t_j g_j(x)$ with $g_j(x)$ being a $\mathbb{C}$-basis of the Tjurina algebra $\mathbb{C}\{x_1, \cdots, x_n\}/\langle f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \rangle$. In the case of a weighted homogeneous singularity $f = 0$, the Tjurina algebra coincides with the Milnor algebra. Moreover, the complement to the discriminant of F (*i.e.* the subset of $\mathbb{C}^\tau$ where $F(x,t) = 0$ is smooth, which is also weighted homogeneous if f is) is a retract of the complement to the corresponding hypersurface in $\mathbb{C}^\tau$.

We consider here semi-universal deformations of germs corresponding to simple singularities of plane curves, which are indexed by a root system R of type A_n, D_n, E_6, E_7, or E_8. Accordingly, we have polynomials $f_R(x)$ and $F_R(x,t)$ as in the preceding discussion. For example, in the E_6 case one has a germ $f_{E_6}(x) = x_1^3 + x_2^4$, with $F_{E_6}(x,t) = x_1^3 + x_2^4 + t_1 + t_2 x_1 + t_3 x_2 + t_4 x_2^2 + t_5 x_1 x_2 + t_6 x_1 x_2^2$. Let $U_R \subset \mathbb{C}^\tau$ be the complement of the discriminant locus of F_R, and let $V_R \subset \mathbb{C}^n \times \mathbb{C}^\tau$ be its pre-image. The projection $\pi_R{:}V_R \to U_R$ is a locally trivial fiber bundle with the fiber diffeomorphic to an affine curve C_R.

Topologically, C_R is homeomorphic to the interior of a surface with boundary having first Betti number $\mu(R)$ and $b(R)$ boundary components, where $\mu(R)$ is the Milnor number and $b(R)$ is the number of branches of singularity corresponding to R. The germs of functions having simple singularities can be chosen to be weighted homogeneous, *i.e.* generic linear combinations of monomials: $x_1^{i_1} x_2^{i_2}$ where $\frac{i_1}{w_1} + \frac{i_2}{w_2} = 1$ for appropriate weights (w_1, w_2), in

which case $\mu(R) = (w_1 - 1)(w_2 - 1)$ (cf. [Dim12]). The number of branches $b(R)$ coincides with the number of irreducible factors of the germ or, alternatively, the number of connected components of the strict transform of the germ in a resolution of its singularity. For example, $\mu(E_6) = (3-1)(4-1) = 6$, and $b(E_6) = 1$.

Setting $g(R) = \mu(R) + b(R) - 1$, we see that $\pi_1(C_R) \cong F_{g(R)}$, the free group on $g(R)$ generators. On the other hand, the base space U_R is biholomorphic to the quotient of the finite affine hyperplane complement corresponding to R by the Weyl group W_R. According to Deligne's work, this quotient is a classifying space for A_R, the Artin group of type R. Since C_R is also aspherical, this implies the total space V_R is aspherical as well. In particular, $\pi_2(U_R) = 1$ and one has a short exact sequence

$$1 \to F_{g(R)} \to \pi_1(V_R) \xrightarrow{(\pi_R)_*} A_R \to 1$$

The locally trivial fiber bundle $\pi_R \colon V_R \to U_R$ induces a geometric monodromy homomorphism $\phi_R \colon A_R \to \mathrm{Mod}(C_R)$, where $\mathrm{Mod}(C_R)$ is the mapping class group of C_R fixing the boundary pointwise. It was shown by Perron-Vannier [PV92] that ϕ_R is injective when R has type A_n or D_n. In contrast, Wajnryb showed that $\ker \phi_R$ is infinite when $R = E_6$, E_7, or E_8. In all cases, the standard Artin generators map to Dehn twists on the surface, hence the image of the geometric monodromy is necessarily infinite. We will leverage these to facts to show that $\pi_1(V_R)$ cannot be CAT(0) in these cases.

4.4. **Wajnryb's theorem and non-CAT(0) families.**

Definition 4.9. Let A be an Artin group. A representation $\phi \colon A \to \mathrm{Mod}(S)$ is called *geometric* if each of the standard Artin generators is sent to Dehn twists about simple closed curves on S.

Theorem 4.10 (Theorem 3, [Waj99])**.** *Let R be the Artin group associated to either E_6, E_7, or E_8. There does not exist an injective geometric representation $\phi \colon A \to \mathrm{Mod}(S)$ for any S.*

Let γ_1, γ_2 be essential simple closed curves on a finite type surface, and let T_{γ_i} be the Dehn twist along γ_i, $i = 1, 2$. We may assume that γ_1 and γ_2 are in general position and have been isotoped to have a minimal number of intersections. The subgroup generated by $\{T_{\gamma_1}, T_{\gamma_2}\}$ is determined as follows

$$\langle T_{\gamma_1}, T_{\gamma_2} \rangle \cong \begin{cases} \mathbb{Z}^2, & |\gamma_1 \cap \gamma_2| = 0 \\ B_3, & |\gamma_1 \cap \gamma_2| = 1 \\ F_2, & |\gamma_1 \cap \gamma_2| \geqslant 2 \end{cases}$$

where B_3 is the 3-strand braid group of the disk. Under a geometric representation, each of the standard Artin generators for the E_6, E_7 or E_8 type must go to a non-separating simple closed curve of S. A regular neighborhood of the collection of curves giving the representation for E_6 must look as in Figure 4.

Wajnryb uses the natural Garside structure on these groups to show that any such representation cannot be injective by finding an explicit element in the kernel of ϕ. Specifically, let $\{a_1, \ldots, a_6\}$ be the standard Artin generators of E_6 as in Figure 4. The Garside element is $\Delta = (a_2 a_4 a_6 a_1 a_3 a_5)^6$. Wajnryb finds a factorization $\Delta = bc$ and shows that

$$\phi(ba_1ca_1ba_1) = \phi(a_1ba_1ca_1b) \tag{7}$$

but that $ba_1ca_1ba_1 \neq a_1ba_1ca_1b$ in A_{E_6}. By a result of van der Lek [vdL83] (see also [Par97]), for any root system R containing E_6 there is an injection A_{E_6} into A_R. Wajnryb then argues

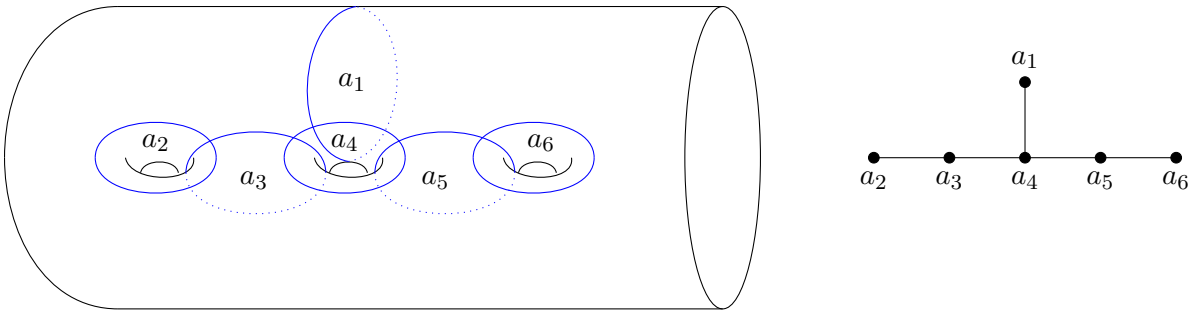


FIGURE 4. The Dynkin diagram of E_6 and the surface with boundary C_{E_6} associated with it, along with the curves which define the geometric representation of A_{E_6}.

that for any geometric representation ϕ of A_R, Equation 7 must still be satisfied, hence ϕ cannot be injective either. This takes care of the cases $R = E_7$ and $R = E_8$. The following appears as Theorem B in the introduction.

Theorem 4.11. *Let R be the root system of type E_6, E_7, or E_8 and let V_R be the universal family associated to the singularity of type R. Then $\pi_1(V_R)$ is not CAT(0).*

Proof. Let $G = \pi_1(V_R)$. The center $Z = Z(A_R)$ of A_R is infinite cyclic and virtually splits off as a direct factor. That is, there exists a finite index subgroup $A' \leqslant A_R$ such that $Z \leqslant A'$ and $A' \cong P' \times Z$. Explicitly, the abelianization map $\alpha\colon A_R \to \mathbb{Z}$ is obtained by sending every positive Artin generator to $1 \in \mathbb{Z}$. Since the center is generated by a positive word, its image under α is a nontrivial positive integer $n \in \mathbb{Z}$. Thus, we may take $A' = \alpha^{-1}(n\mathbb{Z})$, which has finite index in A_R, and contains Z as a retract. In particular, we can take $P' = \ker(\alpha)$ to obtain $A' = P' \times Z$.

If V' is the cover of V_R corresponding to $G' = (\pi_R)_*^{-1}(A')$ we have an extension

$$1 \to F_{g(R)} \to G' \xrightarrow{(\pi_R)*} A' \to 1$$

Since $G' \leqslant G$ has finite index, if G is CAT(0) then G' is too. Under the monodromy action ϕ, the center of A_R acts by a multi twist along the boundary of C_R. In particular, since $Z(A_R)$ is free, there is a splitting $\sigma\colon Z \to G'$ such that $\sigma(Z)$ is central in G'. Since Z is clearly still a retract of G', we obtain a direct product decomposition $G' = H' \times \sigma(Z)$. By a result of Bowers–Ruane [BR96], G' is CAT(0) if and only if H' is.

By construction, H' fits into a short exact sequence

$$1 \to F_{g(R)} \to H' \to P' \to 1$$

We claim that this extension satisfies the hypotheses of Theorem 2.5 and that the restriction of the monodromy ϕ to P' contains the nontrivial infinite kernel found by Wajnryb. This will imply H' is not CAT(0), and thus that G is not CAT(0) either. First, $F_{g(R)}$ has trivial center as $g(R) \geqslant 2$. Secondly, as $P' \cap Z = 1$, P' maps isomorphically onto a finite index subgroup of A_R/Z. The latter is acylindrically hyperbolic by a result of Calvez–Wiest [CW17]. Thus, P' is itself acylindrically hyperbolic and has the LIP property by Proposition 2.7.

Finally, we verify that the restriction of ϕ to P' has infinite kernel. Since $F_{g(R)}$ is torsion-free, any torsion element of H' injects into $P' \leqslant A_R/Z$. On the other hand, by [Bes99,

Theorem 4.5], any torsion element of A_R/Z is conjugate to the image of a positive word. Since $P' = \ker \alpha$, this means it is torsion-free. As A_R is torsion-free, it suffices to show that P' contains an element of $\ker \phi$. Recall from Equation (7) and the discussion above that A_R contains elements a_1, b, c satisfying

$$\phi(ba_1ca_1ba_1) = \phi(a_1ba_1ca_1b) \quad \text{and} \quad ba_1ca_1ba_1 \neq a_1ba_1ca_1b.$$

In particular, this means $g = (ba_1ca_1ba_1)(a_1ba_1ca_1b)^{-1} \in \ker \phi$. On the other hand, g is clearly in the commutator subgroup of A_R, which is just $\ker \alpha = P'$. □

Appendix A. Background from the algebraic geometry of plane curves

A.1. **Fundamental groups of complements.** We give a brief overview of the topology of plane curves for non-experts, including some classical results on the numerical invariants. We also describe a construction that associates with each plane curve, a locally trivial fibration, such that the fundamental group of the total space is an extension of a free group by another free group.

A.1.1. *Pencils of lines.* Let $C \subset \mathbb{P}^2$ be a plane curve and let $O = [a : b : c]$ be a point in $\mathbb{P}^2 \backslash C$. The lines in $\mathbb{P}^2$ containing O are parametrized by a projective line $B = Z(ax+by+cz)$ in the Grassmannian of lines in $\mathbb{P}^2$, which is itself isomorphic to $\mathbb{P}^2$ and classically called *the dual plane* $\mathbb{P}^{2\vee}$. Since each line $L = Z(l_1x + l_2y + l_3z)$ passing through O satisfies $l_1a + l_2b + l_3c = 0$, we can identify the line L with the point $[l_1 : l_2 : l_3] \in B$. To O one associates a map $p : \mathbb{P}^2 \backslash \{O\} \to B$ (called *the projection with center at* O) that assigns to $P \in \mathbb{P}^2 \backslash \{O\}$ the unique line in $\mathbb{P}^2$ containing O and P. If one identifies (non-canonically) B with a line in $\mathbb{P}^2 \backslash \{O\}$ then this map assigns to Q the unique intersection point of the line OP with B. The projection of the complement to O is a holomorphic map that cannot be extended to a map of $\mathbb{P}^2$ but can be extended to the map of the Hirzebruch surface F_1 that is the blow-up $\mathrm{Bl}_O(\mathbb{P}^2)$ of $\mathbb{P}^2$ at O. The fibers of this extension are projective lines, while the fibers of p are affine lines.

For $b \in B$, let L_b be the line passing through O and b. The restriction of this projection to C gives a (ramified) covering $p_C : C \to B$ such that for generic $b \in B$ the corresponding line L_b intersects C at smooth points only and transversally. In this case, the number of preimages is equal to the degree d of C. For arbitrary $b \in B$, the sum of the *intersection multiplicities* of L_b with C is equal to d. The intersection multiplicity is greater than one if either the line L_b contains a singular point of C or it is tangent to C. If the order of tangency of L_b is greater than 2, it is called a *flex* (if the order of tangency is greater than 3, a *higher-order flex*), and if L_b contains more than one point of tangency, it is called a *multiple tangent* (a *bitangent* if this number is 2).

Information about the special fibers of pencils of lines is contained in the classical Plücker formulas (cf. [Kle77]). For example, if C contains as singularities only δ ordinary nodes and κ ordinary cusps (*i.e.* the singularities having local equation $y^2 = x^2$ and $x^2 = y^3$, respectively) and the pencil contains only the lines that either have a tangency of order 2 and are transverse to the lines of the tangent cones of all singular points (such pencils are called *generic*) then the number $d^\vee$ of the lines in the pencil containing fewer than d intersections with C is

$$d^\vee = d(d-1) - 3\kappa - 2\delta \tag{8}$$

It is convenient to describe the singular fibers of non-generic pencils in terms of the dual curve of C, *i.e.* the curve $C^\vee$ in $\mathbb{P}^{2\vee}$ that is the projective closure of the set of points

formed by the lines tangent to C at its smooth points. Singularities of $C^\vee$ correspond to the flexes and multiple tangents of C (cf. [Kle77]) while the ordinary flexes (resp. ordinary bitangents) of C correspond to the ordinary cusps (resp. ordinary nodes) of $C^\vee$. Thus the pencil in $\mathbb{P}^2$ consisting of the lines containing O corresponds to a line $P^\vee$ in $\mathbb{P}^{2\,\vee}$ whose special fibers of the pencil corresponding to the intersection points of $P^\vee$ with $C^\vee$ or a line in $\mathbb{P}^{2\,\vee}$ corresponding to the pencils in $\mathbb{P}^2$ centered at the singular points of C.

The numerical data of non-generic pencils can be obtained using the remaining Plücker formulas for curves with ordinary nodes cusps, flexes, and bitangents (the number of the latter two denoted by ι and τ, respectively) are given by

$$d = d^\vee(d^\vee - 1) - 3\iota - 2\tau, \tag{9}$$

$$\iota = 3d(d-2) - 6\delta - 8\kappa, \tag{10}$$

$$\kappa = 3d^\vee(d^\vee - 1) - 6\tau - 8\iota. \tag{11}$$

The symmetry of the 4 equations in (8)–(11) reflects the duality relation $C^{\vee\,\vee} = C$ (cf. [Kle77]).

Example A.1. Let C be a cubic curve with one node (*e.g.* $y^2z = x^3 - x^2z$) and thus numerical invariants $\delta = 1, \kappa = 0$. Then $d^\vee = 4, \iota = 3, \tau = 0$. Hence, $C^\vee$ is a 3-cuspidal quartic. It has no nodes and, by duality, has one bitangent corresponding to the node of C. There are 3 lines in $\mathbb{P}^2$ connecting the node of C with the intersection point of one of 3 pairs of inflectional tangents of C. A pencil in $\mathbb{P}^{2\,\vee}$ corresponding to either of these 3 lines L_i in $\mathbb{P}^2$, has 4 special fibers containing respectively two tangency points (this line in $\mathbb{P}^{2\,\vee}$ corresponds to the node of C), two cusps (the line corresponding to the intersection point in $\mathbb{P}^2$ of two inflectional lines), one cusp (this fiber corresponds to the intersection point of L_i with remaining inflectional tangent) and finally, the fiber corresponding to the intersection point of L_i with C outside of the node and containing one simple tangency point.

A.1.2. *Braid monodromy.* Let $S \subset B$ be the set of singular values, and let $L = \cup_{s\in S} L_s$ be the set of singular lines. Then $p_C : C\backslash L \to B\backslash S$ is an unramified covering space, and the restriction of $p\colon \mathbb{P}^2\backslash(C \cup L) \to B\backslash S$ (also denoted p) is a locally trivial fibration. Both its base and fibers are complements to a finite set of points in $\mathbb{P}^1$: this is the set $S \subset B$ in the former case and $O \cup p_C^{-1}(b) \subset L_b$, $b \in B\backslash S$ in the latter. Fix a basepoint $b_0 \in B\backslash S$, a point $x_0 \in X_0 = p^{-1}(b_0)$ and let $i_0\colon X_0 \to \mathbb{P}^2\backslash(C \cup L)$ be the inclusion. Since $S \neq \varnothing$, $B\backslash S$ is aspherical, hence the long exact sequence of the fibration determines an extension

$$1 \to \pi_1(X_0, x_0) \xrightarrow{(i_0)_*} \pi_1(\mathbb{P}^2\backslash(C \cup L), x_0) \xrightarrow{p_*} \pi_1(B\backslash S, b_0) \to 1 \tag{12}$$

The identification of $\pi_1(B\backslash S, b_0)$ with the free group can be made by a choice of simple closed curves $\{\gamma_s \mid s \in S\}$ in $B\backslash S$ based at b_0 but otherwise disjoint, and which circle each puncture in S. Similarly, a choice $\{\delta_i \mid 1 \leqslant i \leqslant d\}$ of loops in X_0 based at x_0 give an identification $\pi_1(F_0, f_0)$ with the free group F_d.

The extension can be described as follows. Let $\gamma\colon [0,1] \to B\backslash S$ represent a loop based at b_0 (*i.e.* satisfying $\gamma(0) = \gamma(1) = b_0$). Pulling back $p\colon \mathbb{P}^2\backslash(C \cup L) \to B\backslash S$ via γ yields a punctured $\mathbb{P}^1$-bundle over the interval $[0,1]$ which is trivial since $[0,1]$ is contractible. A trivialization $\Phi\colon \gamma^*(\mathbb{P}^2\backslash(C \cup L)) \to [0,1] \times X_0$ such that $\Phi_0\colon X_0 \to X_0$ is the identity determines an orientation-preserving diffeomorphism $\Phi_1\colon X_0 \to X_0$. We can assume this diffeomorphism is the identity outside of a sufficiently large disk containing $p_C^{-1}(b_0)$ (we may choose x_0 sufficiently close to O so that it also lies outside this disk). The isotopy class of this diffeomorphism is independent of homotopy class of γ rel endpoints, as well as the

choice of trivialization. The group of isotopy classes of diffeomorphisms of the disk with d punctures is the braid group B_d.

Definition A.2. The homomorphism $\beta\colon \pi_1(B\backslash S, b_0) \to B_d$ into the braid group assigning the braid $\beta(\gamma) \in B_d$ corresponding to the class of isotopy of the above diffeomorphism is called *braid monodromy* associated to C.

Example A.3. *Braids corresponding to singular fibers of the pencil.* Each fiber $p^{-1}(s), s \in S$ contains either a singular point of C or a tangency point. Let γ_s be a path in B connecting b_0 and s (avoiding other points in S). Then $p^{-1}(s) \cap C$ contains fewer points than $p^{-1}(b_0)$ and one can show that there is a 1-complex (vanishing fold) such that extension of the diffeomorphism of pairs $I \times (\pi^{-1}(b_0), C \cap p^{-1}(b_0) \to (\mathbb{P}^2, C)$ well defined over I with removed preimage of 1, to the map of $1 \times (\pi^{-1}(b_0), C \cap p^{-1}(b_0)$ contracts this complex to a point.

If s corresponds to a point with multiplicity 2 or a simple tangency point, then this vanishing fold is just a segment that gets contracted to a point as one moves along γ from b_0 to s. In this case, it is called *the vanishing cycle*. Denote it δ_s and let $\gamma(s)$ be a loop that is union of $\gamma_s\backslash D_s \cap \gamma_s$ where D_s is a small disk centered at s, the boundary of D_s oriented counterclockwise, and then returning back to b_0 along γ_s. The braid monodromy assigns to $\gamma(s)$ some power of the Dehn twist about δ_s. More precisely, if the projection looks locally as a projection of $y^k = x^2$ onto x-axis, the braid corresponding to γ_s is the k-th power of the Dehn twist, and the braid is conjugate to a_1^k where a_1 is one the standard generators of the braid group (cf.[Moi81], [CA11]).

Theorem A.4 (Zariski–van Kampen)**.** *Let $C \subset \mathbb{P}^2$ be an algebraic curve, $P \in \mathbb{P}^2\backslash C$ and with the same notation as above, let $\beta\colon \pi_1(B\backslash S, b_0) \to B_d$ be the braid monodromy. Then:*

(a) The split extension (12) is determined by $\varphi\colon \pi_1(B\backslash S, b_0) \to \mathrm{Aut}(F_d)$, where φ is the composition of β and the Artin representation $\rho_A\colon B_d \to \mathrm{Aut}(F_n)$ (cf. [FM11])

(b) Choose $\tilde{b} \in B, \tilde{b} \neq b_0$. Then the group $\pi_1(\mathbb{P}^2\backslash p^{-1}(S \cup \tilde{b}) \cup C)$ has the following presentation ($N = |S|$)

$$\pi_1(\mathbb{P}^2\backslash p^{-1}(S \cup \tilde{b}) \cup C) = \langle t_1, \dots, t_N, x_1, \dots, x_d \mid x_i^{t_j} = \beta_j(x_i) \rangle \tag{13}$$

(here $j = 1, \cdots, N, i = 1, \cdots d$)

(c) (complements to projective and affine curves) If $s \in B\backslash S$ then $\pi_1(\mathbb{P}^2\backslash C)$ (resp. $\pi_1(\mathbb{C}^2\backslash C)$ where $\mathbb{C}^2 = \mathbb{P}^2\backslash p^{-1}(\tilde{b}))$) has the following presentation:

$$\pi_1(\mathbb{P}^2\backslash C) = \{x_1, \dots, x_d | \beta_j(x_i) = x_j, x_1 \cdots x_d = 1\} \tag{14}$$

resp.

$$\pi_1(\mathbb{C}^2\backslash C) = \{x_1, \dots, x_d | \beta_j(x_i) = x_j\} \tag{15}$$

The fundamental group $\pi_1(F_1\backslash C)$ where F_1 is the blow-up of $\mathbb{P}^2$ at P has expression similar to (14) The only change was the replacement of Artin's braid group with the braid group of spheres. Blow-up does not change, of course, the fundamental group.

A.2. **Generic projections.** Let V be a smooth surface in $\mathbb{P}^N$. Let $H \cong \mathbb{P}^{N-3} \subset \mathbb{P}^N\backslash V$ be a linear subspace. The Schubert subvariety B of the Grassmannian of linear $N-2$-subspaces in $\mathbb{P}^N$ containing H can be canonically identified with $\mathbb{P}^2$. Extending the construction in A.1.1, one can consider the map $p : \mathbb{P}^N\backslash H \to B$ assigning to a point P the linear subspace of $\mathbb{P}^N$ spanned by P and H and its restriction $p_V : V \to B$ to V. For a Zariski open subset

in B, the corresponding linear subspaces are transverse to V and intersect V at $d = \deg(V)$ points. The complement is called the branching curve C_V of p_V.

For H in a Zariski open set in the Grassmannian of linear subspaces $\mathbb{P}^{N-3} \subset \mathbb{P}^N$, the branching curve C_V is reduced, and singularities of C_V are ordinary cusps and nodes. Projections corresponding to any subspace H in this set are called *generic*. The singularities of C_V can be described in terms of the ramification curve R (*i.e.* the curve $R \subset V$ formed by critical points of p_V), as the singularities of the image of projection $p_V|_R : R \to B$. Note that $p_V|_R$ is almost never generic as a map of curves (since generic projections of curves have nodes as the only singularities, while cusps of C_V are unavoidable for almost all V).

Branching curves of generic projections form an important class of plane singular curves, and their fundamental groups have been extensively studied. For past results on fundamental groups of the complements to the branching curves to generic projections, as well as more recent calculations, see [FT12] and the references therein.

A.2.1. *Duals of rational nodal curves.* Let $C_d \subset \mathbb{P}^2$ be a rational curve of degree d having only nodes as singularities. Such a curve can be obtained as a generic projection of the smooth rational normal curve in $\mathbb{P}^d$, *i.e.* the image of $\mathbb{P}^1$ under the embedding $x_i = s^i t^{d-i}$, for $[s : t] \in \mathbb{P}^1$. It has $\binom{d-1}{2}$-nodes. It follows from (8) and (9) that the dual curve $C_d^\vee$ has degree $2d-2$, $3d-6$ cusps, and $2(d-2)(d-3)$ nodes.

Theorem A.5. [Zar36] *The fundamental group $\pi_1(\mathbb{P}^2 \backslash C_d^\vee)$ is isomorphic to $B_d(S^2)$, the braid group of sphere on d strands.*

Note that $B_d(S^2) \cong B_d / \langle\!\langle \sigma_1 \cdots \sigma_{d-2} \sigma_{d-1}^2 \sigma_{d-2} \cdots \sigma_1 \rangle\!\rangle$. When $d = 3$, $C_3^\vee$ is a quartic with 3 cusps and no nodes, and $\pi_1(\mathbb{P}^2 \backslash C_3^\vee) \cong B_3(S^2)$ is a finite group of order 12.

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