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Discrete Torsion, Orbifold Elliptic Genera and the Chiral de Rham Complex

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To Robert MacPherson on his 60th birthday

Abstract: Given a compact complex algebraic variety with an effective action of a finite group G, and a class $\alpha \in H^2(G, U(1))$, we introduce an orbifold elliptic genus with discrete torsion α, denoted $Ell^{\alpha}_{orb}(X, G, q, y)$. We give an interpretation of this genus in terms of the chiral de Rham complex attached to the orbifold $[X/G]$. If X is Calabi-Yau and G preserves the volume form, $Ell^{\alpha}_{orb}(X, G, q, y)$ is a weak Jacobi form. We also obtain a formula for the generating function of the elliptic genera of symmetric products with discrete torsion.

1. Introduction

The two-variable elliptic genus (see for example [Kri]) of a compact complex manifold X is a generating function

$$Ell(X, y, q) = \sum_{m,l} c(m, l) q^m y^l \tag{1.1}$$

which captures important topological information about X. For appropriate values of y and q, $Ell(X, y, q)$ specializes to the L, $\widehat{A}$, and χ_y genera respectively. Mathematically, the elliptic genus can be defined as follows. For a holomorphic vector bundle V on X and a formal variable t, let

$$\operatorname{Sym}_t V = 1 + tV + t^2 \operatorname{Sym}^2 V + t^3 \operatorname{Sym}^3 V + \cdot \quad \in K_0(X)[[t]]$$

and

$$\Lambda_t V = 1 + tV + t^2 \Lambda^2 V + t^3 \Lambda^3 V + \cdot \quad \in K_0(X)[[t]]$$

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Let T_X, T_X^* denote the holomorphic tangent and cotangent bundles respectively, and

$$(1.2)\quad \mathcal{E}ll(X,q,y) = y^{-\frac{\dim X}{2}} \otimes_{n\geq 1} (\Lambda_{-yq^{n-1}} T_X^* \otimes \Lambda_{-y^{-1}q^n} T_X \otimes S_{q^n} T_X^* \otimes S_{q^n} T_X)$$

viewed as an element of $K_0(X)[[q]][[y^{\pm\frac{1}{2}}]]$. Then

$$Ell(X,q,y) = \chi(\mathcal{E}ll(X,q,y)).$$

In physics, $Ell(X,q,y)$ is part of the partition function of a two-dimensional conformal field theory with target X (cf. [EOTY]).

In this paper, we will say that X is Calabi-Yau if K_X is trivial - this is of course weaker than the usual mathematical Calabi-Yau condition, but agrees with the physics terminology. When X is Calabi-Yau, $Ell(X,q,y)$ has nice modular properties. Let $\mathbb{H}$ denote the upper half plane. A *weak Jacobi form* of weight $k \in \mathbb{Z}$ and index $r \in \frac{1}{2}\mathbb{Z}$ is a holomorphic function on $\mathbb{H} \times \mathbb{C}$ satisfying the transformation property

$$\phi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{2\pi i \frac{rcz^2}{c\tau+d}} \phi(\tau,z), \quad \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\phi(\tau, z+m\tau+n) = (-1)^{2r(m+n)} e^{-2\pi i r(m^2\tau+2mz)} \phi(\tau,z), \quad (m,n) \in \mathbb{Z}^2$$

that in addition has a Fourier expansion $\sum_{l,m} c_{m,l} y^l q^m$ with nonnegative m (see [EZ]), where $q = e^{2\pi i\tau}, y = e^{2\pi i z}$. It is shown in [BL, Gri] that if X is Calabi-Yau then $Ell(X,q,y)$ is a weak Jacobi form of weight 0 and index $\dim(X)/2$.

In [BL1], the authors introduced a notion of orbifold elliptic genus $Ell(X,G,q,y)$ attached to the global quotient orbifold $[X/G]$, where X is a smooth compact complex manifold, and $G \in Aut(X)$ is a finite group (see also [DLiM] for general reduced orbifolds). A mathematical definition of this genus is given in section 2.1. Roughly, $Ell(X,G,q,y)$ is obtained by adding the contributions of Euler characteristics of bundles analogous to 1.2 over the various fixed-point sets $X^g, g \in G$ of the G–action on X. Furthermore, for each $g \in G$, the contribution takes into account the eigenvalues of g on $TX|_{X^g}$. It is also shown in [BL1] that if X is Calabi-Yau, and G preserves the volume form, then $Ell(X,G,q,y)$ is a weak Jacobi form of weight 0 and index $\dim(X)/2$.

It was observed in [BL] that the ordinary elliptic genus $Ell(X,y,q)$ can be interpreted in terms of an object called the *chiral de Rham complex*, denoted Ω_X^{ch}. The latter is a sheaf of vertex superalgebras attached to any smooth complex manifold X, introduced in [MSV] (for a brief discussion of vertex algebras, see section 3.1). The sheaf Ω_X^{ch} has an increasing filtration $F^\bullet\Omega_X^{ch}$, and also a compatible bigrading by two operators L_0, J_0. One can therefore describe the associated graded sheaf $gr_F\Omega_X^{ch}$ in terms of the bigraded components. One finds that Supertrace$(q^{L_0}y^{J_0}, \Omega_X^{ch})$ is the sheaf 1.2 above. It follows from this observation that

$$Ell(X, y, q) = \text{Supertrace}(q^{L_0}y^{J_0-\dim(X)/2}, H^*(X, \Omega_X^{ch}))$$

It is believed that $H^*(X, \Omega_X^{ch})$ captures some of the information of the two-dimensional quantum field theory on X mentioned above. Ω_X^{ch} also carries a differential $Q_{BRST} \in \text{End}(\Omega_X^{ch})$, $Q_{BRST}^2 = 0$ (which is why it is called a complex). The "de Rham" part of the name comes from the fact that the complex $(\Omega_X^{ch}, Q_{BRST})$ is quasiisomorphic to the holomorphic de Rham complex (Ω_{dR}, ∂).

In [FS], the construction of Ω_X^{ch} was extended to orbifolds (another construction of the chiral de Rham complex for orbifolds was obtained independently by A. Vaintrob). For each $g \in G$, one introduces sheaves $\Omega_X^{ch,g}$ supported on X^g, which are modules over Ω_X^{ch}. Each $\Omega_X^{ch,g}$ carries a canonical $C(g)$–equivariant structure, where $C(g)$ denotes the centralizer of g in G. The sheaves $\Omega_X^{ch,g}$ allow one to interpret some of the "stringy" geometric invariants of the orbifold $[X/G]$. In particular, it is shown in [FS] that

$$Ell(X, G, q, y) = \text{Supertrace}(q^{L_0}y^{J_0-\dim(X)/2}, \bigoplus_{[g]} H^*(X, \Omega_X^{ch,g})^{C(g)}) \tag{1.3}$$

and

$$\bigoplus_{[g]} \mathbb{H}^*(\Omega_X^{ch,g}, Q_{BRST})^{C(g)} \cong \bigoplus_{[g]} H_{dR}^*(X^g/C(g), \mathbb{C}) \tag{1.4}$$

where $[g]$ denotes a set of representatives for the conjugacy classes in G. The object on the right in 1.4, with an appropriate grading and ring structure is called the Chen-Ruan cohomology of $[X/G]$ (see [CR]). The isomorphism 1.4 is an isomorphism of graded vector-spaces.

We now come to *discrete torsion*. This term was introduced in the physics literature to refer to the discovery (see [Va1, Va2]) that an orbifold quantum field theory on $[X/G]$ could be "twisted" by a cocycle $\alpha \in H^2(G, U(1))$. In terms of physics, the path integral can be written as a sum of contributions from sectors parametrized by commuting pairs of elements (g, h) in $G \times G$, and the contribution from the (g, h)–sector is multiplied by the phase $\delta(g, h) = \alpha(g, h)/\alpha(h, g)$. This modification produces a consistent physical theory and leads to α–twisted invariants of the orbifold $[X/G]$. For a mathematical discussion of various aspects of discrete torsion see [AF, R, Ka]. In this paper, we build on the results in [BL1, FS] to give a mathematical treatment of orbifold elliptic genera with discrete torsion $Ell^\alpha(X, G, q, y)$. One way to define this object along the lines of [BL1] is to multiply each contribution in the sum 2.3 by the appropriate phase $\delta(g, h)$. From the point of view of the chiral de Rham complex, this definition can be cast in a manner somewhat closer to the original physics approach as follows. Recall that in 1.3 one uses the $C(g)$–equivariant structure on $\Omega_X^{ch,g}$ to project on the $C(g)$–invariant part of $H^*(X, \Omega_X^{ch,g})$. A cocycle $\alpha \in H^2(G, U(1))$ induces characters $\alpha_g : C(g) \mapsto U(1)$ by $\alpha_g(h) = \delta(g, h)$. The character α_g allows us to twist the $C(g)$ equivariant structure, and taking invariants with respect to this twisted structure projects on a different subspace. We can therefore define

$$Ell^\alpha(X, G, q, y) = \text{Supertrace}(q^{L_0} y^{J_0 - \dim(X)/2}, \bigoplus_{[g]} H^*(X, \Omega_X^{ch,g})^{C(g)_\alpha}) \tag{1.5}$$

where $C(g)_\alpha$ indicates that the twisted action is being used. We show that if X is Calabi-Yau, $Ell^\alpha(X, G, q, y)$ is a Jacobi form of weight 0 and index $\dim(X)/2$. We also show that there is an isomorphism of graded vector spaces

$$\bigoplus_{[g]} \mathbb{H}^*(\Omega_X^{ch,g}, Q_{BRST})^{C(g)_\alpha} \cong \bigoplus_{[g]} H^*_{dR}(X^g/C(g), \mathcal{L}_g^\alpha)$$

where the object on the right denotes the α–twisted Chen-Ruan cohomology of $[X/G]$ valued in the collection of local systems $\mathcal{L}_g^\alpha$, introduced in [R].

An important example of discrete torsion arises in the case of symmetric products (see [Di]). S_N, the symmetric group on N letters, acts on the hyperplane in $\mathbb{R}^N$ given by the equation $x_1 + \cdots + x_N = 0$. This yields an embedding $S_N \hookrightarrow O(N-1)$. Pulling back the double cover $Pin(N-1) \mapsto O(N-1)$ yields a central extension of S_N by $\mathbb{Z}/2\mathbb{Z}$, which we denote $\widehat{S}_N$ - i.e.

$$1 \mapsto \mathbb{Z}/2\mathbb{Z} \mapsto \widehat{S}_N \mapsto S_N \mapsto 1 \tag{1.6}$$

The extension 1.6 is non-split for $N \geq 4$, and therefore yields a non-zero class $\alpha \in H^2(S_N, \mathbb{Z}/2\mathbb{Z})$, which via the inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow U(1)$ can be pushed into $H^2(S_N, U(1))$.

The orbifold elliptic genera of symmetric products can be arranged into remarkable generating functions. It was proved in [BL1] following a physics derivation in [DMVV] that

$$\sum_{N \in \mathbb{Z}_+} p^N Ell_{orb}(X^N, S_N, y, q) = \prod_{n,m,\ell \geq 0} (1 - p^n q^m y^\ell)^{-c(nm,\ell)}.$$

where the $c(m, l)$'s are as in 1.1. In section 4.2 we obtain a generalization of this formula with discrete torsion given by α above, which was originally obtained by Dijkgraaf ([Di]) in the physics literature.

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2. Orbifold elliptic genera

2.1. **The orbifold elliptic genus.** Let X be a complex manifold on which a finite group G acts effectively via holomorphic transformations. Let X^h will be the fixed point set of $h \in G$ and $X^{g,h} = X^g \cap X^h (g, h \in G)$. Let

$$TX|_{X^h} = \oplus_{\lambda(h) \in \mathbf{Q} \cap [0,1)} V_\lambda. \tag{2.1}$$

where the bundle V_λ on X^h is determined by the requirement that h acts on V_λ via multiplication by $e^{2\pi i \lambda(h)}$. For a connected component of X^h (which by abuse of notation we also will denote X^h), the fermionic shift is defined as $F(h, X^h \subseteq X) = \sum_\lambda \lambda(h)$ (cf. [Z], [BD]). Let us consider the bundle:

$$\begin{aligned} V_{h,X^h \subseteq X} := \otimes_{k \geq 1} \Big[(\Lambda^\bullet_{yq^{k-1}} V_0^*) \otimes (\Lambda^\bullet_{y^{-1}q^k} V_0) \otimes (Sym^\bullet_{q^k} V_0^*) \otimes (Sym^\bullet_{q^k} V_0) \otimes \\ \otimes \big[\otimes_{\lambda \neq 0} (\Lambda^\bullet_{yq^{k-1+\lambda(h)}} V_\lambda^*) \otimes (\Lambda^\bullet_{y^{-1}q^{k-\lambda(h)}} V_\lambda) \otimes (Sym^\bullet_{q^{k-1+\lambda(h)}} V_\lambda^*) \otimes (Sym^\bullet_{q^{k-\lambda(h)}} V_\lambda) \big] \Big] \end{aligned} \tag{2.2}$$

Definition 2.1. The orbifold elliptic genus of a G-manifold X is the function on $H \times \mathbf{C}$ given by:

$$Ell_{orb}(X, G, q, y) := y^{-\dim/X/2} \sum_{[g], X^g} y^{F(g, X^g \subseteq X)} \frac{1}{|C(g)|} \sum_{h \in C(h)} L(h, V_{g, X^g \subseteq X})$$

where the summation in the first sum is over all conjugacy classes in G and connected components X^g of an element $g \in [g]$, $C(g)$ is the centralizer of $g \in G$ and

$$L(h, V_{g, X^g \subseteq X}) = \sum_i (-1)^i \mathrm{tr}(h, H^i(V_{g, X^g \subseteq X}))$$

is the holomorphic Lefschetz number.

Using the holomorphic Lefschetz fixed-point formula ([AS]) one can rewrite this definition as follows.

Theorem 2.1. [BL1] Let $TX|_{X^{g,h}} = \oplus W_\lambda$ and let x_λ be the collection of Chern roots of W_λ. Let

$$\theta(z, \tau) = q^{\frac{1}{8}} (2\sin\pi z) \prod_{l=1}^{l=\infty} (1 - q^l) \prod_{l=1}^{l=\infty} (1 - q^l \mathrm{e}^{2\pi \mathrm{i} z})(1 - q^l \mathrm{e}^{-2\pi \mathrm{i} z})$$

where $q = \mathrm{e}^{2\pi \mathrm{i} \tau}$ be the Jacobi's theta function and let

$$\Phi(g, h, \lambda, z, \tau, x) = \frac{\theta(\frac{x}{2\pi i} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x}{2\pi i} + \lambda(g) - \tau\lambda(h))} e^{2\pi i z \lambda(h)}.$$

Then:

$$Ell_{orb}(X, G, z, \tau) = \frac{1}{|G|} \sum_{gh=hg} \prod_{\lambda(g)=\lambda(h)=0} x_\lambda \prod_\lambda \Phi(g, h, \lambda, z, \tau, x_\lambda)[X^{g,h}]. \tag{2.3}$$

The orbifold elliptic genus so defined specializes for $q = 0$ to

$$Ell_{orb}(X, G, 0, y) = y^{-\dim X} \chi_{-y}(X, G)$$

where

$$\chi_y(X, G) = \sum_{[g], X^g} y^{F(g, X^g \subset X)} \sum_{p,q} (-1)^q \dim H^{p,q}(X^g)^{C(g)}$$

On the other hand $\chi_y(X, G)$ is the value of the orbifold E-function

$$E(u, v, G) = \sum_{[g], X^g} (uv)^{F(g, X^g \subset X)} \sum_{p,q} \dim H^{p,q}(X^g)^{C(g)} u^p v^q$$

for $u = y, v = -1$. In paticular $Ell_{orb}(X, G, 0, 1)$ coincides with the orbifold Euler characteristic: $e_{orb}(X, G) = \frac{1}{|G|} \sum_{fg=gf} e(X^{f,g})$.

2.2. **Discrete torsion.**

Definition 2.2. Let $\alpha \in H^2(G, U(1))$, and define

$$\delta(g, h) = \frac{\alpha(g, h)}{\alpha(h, g)}$$

The orbifold elliptic genus with discrete torsion α, written $Ell^{\alpha}_{orb}(X, G, q, y)$, is defined as

(2.4)

$$Ell^{\alpha}_{orb}(X, G, q, y) := y^{-\dim X/2} \sum_{[g], X^g} y^{F(g, X^g \subseteq X)} \frac{1}{|C(g)|} \sum_{h \in C(h)} \delta(g, h) L(h, V_{g, X^g \subseteq X}).$$

As above, using the holomorphic Lefschetz fixed-point formula this can be rewritten as

$$Ell^{\alpha}_{orb}(X, G; y, q) = \frac{1}{|G|} \sum_{gh=hg} \delta(g, h) \prod_{\lambda(g)=\lambda(h)=0} x_\lambda \prod_{\lambda} \Phi(g, h, \lambda, z, \tau, x_\lambda)[X^{g,h}].$$

Such twisted elliptic genus has specialization properties similar to the case $\alpha = 0$. Using Dolbeault cohomology corresponding to the inner local systems L_α defined by α (cf. [R]) one can define twisted E-function:

$$E^{\alpha}(u, v, G) = \sum_{[g], X^g} (uv)^{F(g, X^g \subset X)} \sum_{p,q} \dim H^{p,q}(X^g, L_\alpha)^{C(g)} u^p v^q$$

which for $u = 1, v = -1$ yields:

$$e^{\alpha}(X, G) = \frac{1}{|G|} \sum_{fg=gf} \delta(f.g) e(X^{f,g})$$

The elliptic genus 2.4 satisfies:

$$Ell^{\alpha}(0, y, G) = y^{\frac{\dim X}{2}} E^{\alpha}(y, -1, G)$$

We proceed to investigate the modularity properties of this twisted orbifold elliptic genus.

2.2.1. *Jacobi forms.* Let $\mathbb{H}$ denote the upper half plane. A *weak Jacobi form* of weight $k \in \mathbb{Z}$ and index $r \in \frac{1}{2}\mathbb{Z}$ is a function on $\mathbb{H}\times\mathbb{C}$ satisfying the transformation property

$$\phi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{2\pi i \frac{rcz^2}{c\tau+d}} \phi(\tau, z), \qquad \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\phi(\tau, z+m\tau+n) = (-1)^{2r(m+n)} e^{-2\pi i r(m^2\tau+2mz)} \phi(\tau, z), \quad (m,n) \in \mathbb{Z}^2$$

and has a Fourier expansion $\sum_{l,m} c_{m,l} y^l q^m$ with nonnegative m. Equivalently, we can say that a Jacobi form is an automorphic form for the Jacobi group $\Gamma^J = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ generated by the four transformations:

$$(z,\tau) \to (z+1,\tau),\ (z,\tau) \to (z+\tau,\tau),\ (z,\tau) \to (z,\tau+1),\ (z,\tau) \to (\frac{z}{\tau}, -\frac{1}{\tau}).$$

Theorem 2.2. *Let X be a compact complex manifold of dimension d such that K_X is trivial, G a finite group acting effectively on X, and $\alpha \in H^2(G, U(1))$. Let n denote the order of G in $\mathrm{Aut}(H^0(X, K_X))$. Then $Ell^{\alpha}_{orb}(X,G)$ is a weak Jacobi form of weight 0 and index $d/2$ with respect to subgroup of the Jacobi group Γ^J generated by transformations*

$$(z,\tau) \to (z+n,\tau),\ (z,\tau) \to (z+n\tau,\tau),\ (z,\tau) \to (z,\tau+1),\ (z,\tau) \to (\frac{z}{\tau}, -\frac{1}{\tau}).$$

In particular, if the action preserves holomorphic volume then $Ell_{orb}(X,G)$ is a weak Jacobi form of weight 0 and index $d/2$ for the full Jacobi group.

Proof. We use the notation $Ell^{\alpha}_{orb}(X,G,z,\tau)$ rather than $Ell^{\alpha}(X,G,q,y)$ to emphasize the dependence on τ and z. It is shown in [BL1] that

$$\Phi(g,h,\lambda,z+1,\tau,x) = -\mathrm{e}^{2\pi \mathrm{i}\lambda(h)} \cdot \Phi(g,h,\lambda,z,\tau,x)$$

which implies that

$$\prod_{\lambda} \Phi(g,h,\lambda,z+n,\tau,x_\lambda)[X^{g,h}] = (-1)^{dn} \mathrm{e}^{2n\pi \mathrm{i} \sum \lambda(h)} \prod_{\lambda} \Phi(g,h,\lambda,z,\tau,x_\lambda)[X^{g,h}]$$

Now, $n\sum \lambda(h) \in \mathbb{Z}$ by the assumption that h^n acts trivially on $H^0(X, K_X)$. Thus

$$Ell^{\alpha}_{orb}(X,G,z+n,\tau) = (-1)^{dn} Ell^{\alpha}_{orb}(X,G,z,\tau).$$

The following formulas are also obtained in [BL1]:

$$\Phi(g,h,\lambda,z,\tau+1,x) = \Phi(gh^{-1},h,\lambda,z,\tau,x) \tag{2.5}$$

$$\Phi(g,h,\lambda,z+n\tau,\tau,x) = (-1)^n \mathrm{e}^{-2\pi \mathrm{i} nz - \pi \mathrm{i} n^2\tau} \mathrm{e}^{nx+2\pi \mathrm{i} n\lambda(g)} \cdot \Phi(g,h,\lambda,z,\tau,x) \tag{2.6}$$

$$\Phi(g,h,\lambda,\frac{z}{\tau},\ -\frac{1}{\tau},\frac{x}{\tau}) = e^{\frac{\pi i z^2}{\tau} - \frac{zx}{\tau}} \cdot \Phi(h,g^{-1},\lambda,z,\tau,x). \tag{2.7}$$

Equation 2.5 and $\delta(g,h) = \delta(gh^{-1}, h)$ imply that

$$Ell^{\alpha}_{orb}(X, G, z, \tau + 1) = Ell^{\alpha}_{orb}(X, G, z, \tau).$$

Equation 2.6 implies that

$$Ell^{\alpha}_{orb}(X, G, z + n\tau, \tau) = (-1)^{dn} e^{-2\pi \mathrm{i} dnz - \pi \mathrm{i} dn^2 \tau} Ell^{\alpha}_{orb}(X, G, z, \tau)$$

In order to see how

$$Ell^{\alpha}_{orb}(X, G, z/\tau, -1/\tau) = e^{\frac{\pi i d z^2}{\tau}} Ell^{\alpha}_{orb}(X, G, z, \tau) \tag{2.8}$$

follows, we write

$$\prod_{\lambda(g)=\lambda(h)=0} x_\lambda \prod_{\lambda} \Phi(g, h, \lambda, z, \tau, x_\lambda) = \sum_{\mathbf{k}_\lambda} Q(g, h, z, \tau) {\mathbf{x}_\lambda}^{\mathbf{k}_\lambda}$$

where $\mathbf{k}_\lambda$ are multiindices and $\mathbf{x}_\lambda$ are the corresponding monomials. We thus obtain,

$$\prod_{\lambda(g)=\lambda(h)=0} \frac{x_\lambda}{\tau} \prod_{\lambda} \Phi(g, h, \lambda, \frac{z}{\tau}, \frac{-1}{\tau}, \frac{x_\lambda}{\tau}) = \sum_{\mathbf{k}_\lambda} (\frac{1}{\tau})^{deg(\mathbf{k}_\lambda)} Q(g, h, \frac{z}{\tau}, \frac{-1}{\tau}) {\mathbf{x}_\lambda}^{\mathbf{k}_\lambda}$$

whereas 2.7 implies that

$$\begin{aligned}
\prod_{\lambda(g)=\lambda(h)=0} \frac{x_\lambda}{\tau} \prod_{\lambda} \Phi(g, h, \lambda, \frac{z}{\tau}, \frac{-1}{\tau}, \frac{x_\lambda}{\tau}) &= e^{\frac{\pi i d z^2}{\tau}} \prod_{\lambda(g)=\lambda(h)=0} \frac{x_\lambda}{\tau} \prod_{\lambda} \Phi(h, g^{-1}, \lambda, z, \tau, x) \\
&= \tau^{-dim(X^{g,h})} \sum_{\mathbf{k}_\lambda} Q(h, g^{-1}, z, \tau) {\mathbf{x}_\lambda}^{\mathbf{k}_\lambda}
\end{aligned}$$

Thus for multiindices $\mathbf{k}_\lambda$ such that $deg(\mathbf{k}_\lambda) = dim(X^{g,h})$, we find

$$Q_k(g, h, \frac{z}{\tau}, \frac{-1}{\tau}) = Q_k(h, g^{-1}, z, \tau)$$

Finally, $\delta(g,h) = \delta(h, g^{-1})$ ensures that 2.8 holds. $\square$

3. Discrete torsion and the chiral de Rham complex

Let X be a smooth complex algebraic variety, and G a finite group acting effectively on X. In this section, we briefly review the construction of the chiral de Rham complex of an orbifold introduced in [FS]. Another construction of this object was obtained independently by A. Vaintrob.

3.1. **Vertex algebras and twisted modules.** In this section we will use the language of vertex superalgebras, their modules, and twisted modules. For an introduction to vertex algebras and their modules [FLM, K, FB], and for background on twisted modules, see [FFR, D, DLM, FS].

We recall that a conformal vertex superalgebra is a $\mathbb{Z}_+$–graded super vector space

$$V = \bigoplus_{n=0}^{\infty} V_n,$$

$$V_n = V_n^{\overline{0}} \oplus V_n^{\overline{1}}$$

together with a vacuum vector $|0\rangle \in V_0^{\overline{0}}$, an even translation operator T of degree 1, a conformal vector $\omega \in V_2^{\overline{0}}$ and an even linear map

$$\begin{aligned} Y : V &\to \operatorname{End} V[[z^{\pm 1}]], \\ A &\mapsto Y(A,z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}. \end{aligned}$$

These data must satisfy certain axioms (see [FLM, K, FB]). In what follows we will denote the collection of such data simply by V, and the parity of an element $A \in V$ homogeneous with respect to the $\mathbb{Z}/2\mathbb{Z}$ grading by $p(A)$.

A vector superspace M is called a V–module if it is equipped with an even linear map

$$\begin{aligned} Y^M : V &\to \operatorname{End} M[[z^{\pm 1}]], \\ A &\mapsto Y^M(A,z) = \sum_{n\in\mathbb{Z}} A^M_{(n)} z^{-n-1} \end{aligned}$$

such that for any $v \in M$ we have $A^M_{(n)} v = 0$ for large enough n. This operation must satisfy the following axioms:

- $Y^M(|0\rangle, z) = \mathrm{Id}_M$;
- For any $v \in M$ and homogeneous $A, B \in V$ there exists an element

  $$f_v \in M[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

  such that the formal power series

  $$Y^M(A,z)Y^M(B,w)v, \quad (-1)^{p(A)p(B)} Y^M(B,w)Y^M(A,z)v, \qquad \text{and}$$

  $$Y_M(Y(A,z-w)B,w)v$$

  are expansions of f_v in $M((z))((w))$, $M((w))((z))$ and $M((w))((z-w))$, respectively.

The power series $Y^M(A,z)$ are called vertex operators. We write the vertex operator corresponding to ω as

$$Y^M(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^M z^{-n-2},$$

where L_n^M are linear operators on V generating the Virasoro algebra. Following [D], we call M *admissible* if L_0^M acts semi-simply with integral eigenvalues.

Now let σ_V be a conformal automorphism of V, i.e., an even automorphism of the underlying vector superspace preserving all of the above structures (so in particular $\sigma_V(\omega) = \omega$). We will assume that σ_V has finite order $m > 1$. A vector space M^σ is called a σ_V–*twisted* V–module (or simply twisted module) if it is equipped with an even linear map

$$Y^{M^\sigma} : V \to \operatorname{End} M^\sigma[[z^{\pm\frac{1}{m}}]],$$
$$A \mapsto Y^{M^\sigma}(A, z^{\frac{1}{m}}) = \sum_{n\in\frac{1}{m}\mathbb{Z}} A_{(n)}^{M^\sigma} z^{-n-1}$$

such that for any $v \in M^\sigma$ we have $A_{(n)}^{M^\sigma} v = 0$ for large enough n. Please note that we use the notation $Y^{M^\sigma}(A, z^{\frac{1}{m}})$ rather than $Y^{M^\sigma}(A,z)$ in the twisted setting. This operation must satisfy the following axioms (see [FFR, D, DLM, Li, FS]):

- $Y^{M^\sigma}(|0\rangle, z^{\frac{1}{m}}) = \mathrm{Id}_{M^\sigma}$;
- For any $v \in M^\sigma$ and homogeneous $A, B \in V$, there exists an element

  $$f_v \in M^\sigma[[z^{\frac{1}{m}}, w^{\frac{1}{m}}]][z^{-\frac{1}{m}}, w^{-\frac{1}{m}}, (z-w)^{-1}]$$

  such that the formal power series

  $$Y^{M^\sigma}(A, z^{\frac{1}{m}}) Y^{M^\sigma}(B, w^{\frac{1}{m}}) v, \quad (-1)^{p(A)p(B)} Y^{M^\sigma}(B, w^{\frac{1}{m}}) Y^{M^\sigma}(A, z^{\frac{1}{m}}) v, \qquad \text{and}$$

  $$Y^{M^\sigma}(Y(A, z-w)B, w^{\frac{1}{m}}) v$$

  are expansions of f_v in $M^\sigma((z^{\frac{1}{m}}))((w^{\frac{1}{m}}))$, $M^\sigma((w^{\frac{1}{m}}))((z^{\frac{1}{m}}))$ and $M^\sigma((w^{\frac{1}{m}}))((z-w))$, respectively.
- If $A \in V$ is such that $\sigma_V(A) = e^{\frac{2\pi i k}{m}} A$, then $A_{(n)}^{M^\sigma} = 0$ unless $n \in \frac{k}{m} + \mathbb{Z}$.

The series $Y^{M^\sigma}(A,z)$ are called twisted vertex operators. In particular, the Fourier coefficients of the twisted vertex operator

$$Y^{M^\sigma}(\omega, z^{\frac{1}{m}}) = \sum_{n\in\mathbb{Z}} L_n^{M^\sigma} z^{-n-2},$$

generate an action of the Virasoro algebra on M^σ.

3.2. **The chiral de Rham complex of an orbifold.** For $g \in G$, let X^g denote the fixed-point set of g, and denote by

$$i^g : X^g \hookrightarrow X$$

the inclusion map of X^g into X. The following results were obtained in [FS].

- For each $g \in G$, there exists a sheaf $\Omega_X^{ch,g}$ supported on X^g. When $g = 1$, this is a sheaf of vertex superalgebras, originally introduced in [MSV], and called the chiral de Rham complex. We denote it simply by Ω_X^{ch}. Being a sheaf of vertex superalgebras means that for each open $U \in X$, $\Omega_X^{ch}(U)$ is a vertex superalgebra.
- Let $g, h \in G$, and let $g' = hgh^{-1}$. There exist isomorphisms of sheaves

$$R^h_{g,hgh^{-1}} : \Omega_X^{ch,g} \mapsto h^*\Omega_X^{ch,hgh^{-1}} \tag{3.1}$$

  satisfying

$$R^k_{g',g''} \circ R^h_{g,g'} = R^{kh}_{g,g''}$$

  where $k \in G$ and $g'' = khgh^{-1}k^{-1}$.
- When $g \neq 1$, $\Omega_X^{ch,g}$ is a sheaf of g–twisted modules, meaning that for each g–invariant U, $\Omega_X^{ch,g}(U)$ is a g–twisted $\Omega_X^{ch}(U)$–module. This structure induces a corresponding twisted module structure on $H^*(X, \Omega_X^{ch,g})$.
- $\Omega_X^{ch,g}$ carries a bigrading by two operators L_0^g, J_0^g. This bigrading induces a bigrading on $H^*(X, \Omega_X^{ch,g})$.
- $\Omega_X^{ch,g}$ carries a differential Q^g, such that $(Q^g)^2 = 0$. Furthermore, there exists an inclusion of the de Rham complex of X^g

$$i_*^g(\Omega_{dR}(X^g, d)) \hookrightarrow (\Omega_X^{ch,g}, Q^g)$$

  which is a quasiisomorphism. This implies in particular that

$$\mathbb{H}^*(\Omega_X^{ch,g}, Q^g) \cong H^*_{dR}(X^g, \mathbb{C})$$

  3.1 implies that $C(g)$, the centralizer of g, acts on $(\Omega_X^{ch,g}, Q^g)$, and therefore on its hypercohomology. This gives an isomorphism

$$\mathbb{H}^*(\Omega_X^{ch,g}, Q^g)^{C(g)} \cong H^*_{dR}(X, \mathbb{C})^{C(g)} \cong H^*_{dR}(X^g/C(g), \mathbb{C})$$

  We therefore have

$$\bigoplus_{[g]} \mathbb{H}^*(\Omega_X^{ch,g}, Q^g)^{C(g)} \cong \bigoplus_{[g]} H^*_{dR}(X^g/C(g), \mathbb{C}) \tag{3.2}$$

  The right-hand side is isomorphic as a vector space to the Chen-Ruan orbifold cohomology of $[X/G]$. Furthermore, the operators J_0^g acting on $\Omega_X^{ch,g}$, induce a gradation on the left which coincides with the Chen-Ruan gradation shifted by the fermionic shift (see [CR, Z]). 3.2 is therefore an isomorphism of graded vector spaces.

- There exists an increasing exhaustive filtration on $\Omega_X^{ch,g}$

$$F^0\Omega_X^{ch,g} \subset F^1\Omega_X^{ch,g} \subset F^2\Omega_X^{ch,g} \subset \cdots . \tag{3.3}$$

Let $\overline{\Omega}_X^{ch,g}$ denote the restriction of $\Omega_X^{ch,g}$ to X^g, which inhertis a filtration from 3.3. The bigrading operators J_0^g, L_0^g are compatible with 3.3, and so the associated graded $gr_F(\overline{\Omega}_X^{ch,g})$ can be described in terms of its decomposition into eigenbundles for J_0^g, L_0^g. We have:

$$gr_F(\overline{\Omega}_X^{ch,g}) = \bigotimes_{k\geq 1} \Big(\Lambda^\bullet_{yq^{k-1}} V_0^* \otimes \Lambda^\bullet_{y^{-1}q^k} V_0 \otimes \mathrm{Sym}^\bullet_{q^k} V_0^* \otimes \mathrm{Sym}^\bullet_{q^k} V_0 \otimes$$
$$\bigotimes_{\lambda\neq 0} (\Lambda^\bullet_{yq^{k-1+\lambda(g)}} V_\lambda^* \otimes \Lambda^\bullet_{y^{-1}q^{k-\lambda(g)}} V_\lambda \otimes \mathrm{Sym}^\bullet_{q^{k-1+\lambda(g)}} V_\lambda^* \otimes \mathrm{Sym}^\bullet_{q^{k-\lambda(g)}} V_\lambda) \Big). \tag{3.4}$$

where

$$TX|_{X^g} = \bigoplus V_\lambda.$$

If we now form

$$\mathcal{H}_{orb}(X,G) = \bigoplus_{[g]} H^*(X, \Omega_X^{ch,g})^{C(g)}$$

then as shown in [FS]

$$Ell_{orb}(X,G,q,y) = \mathrm{Supertrace}(q^{L_0} y^{J_0 - \dim(X)/2}, \mathcal{H}_{orb}(X,G)).$$

3.2.1. *Adding discrete torsion to the chiral de Rham complex.* In this section we show how to incorporate discrete torsion in the above setup. Suppose that Y is a G–manifold, and W a G–equivariant sheaf on Y. This means that for each $g \in G$, we are given an isomorphism

$$T_g : W \mapsto g^* W$$

such that

$$T_g T_h = T_{gh}$$

Suppose now that $\chi : G \mapsto \mathbb{C}^\times$ is a character of G. Then

$$T'_g = \chi(g) T_g$$

is a new G equivariant structure on W.

We apply this observation to the sheaves $\Omega_X^{ch,g}$ and $C(g)$ rather than G. A class $\alpha \in H^2(G, U(1))$ yields characters

$$\alpha_g : C(g) \mapsto U(1)$$

defined by $\alpha_g(h) = \delta(g,h)$. We can now twist the $C(g)$–equivariant structure on $\Omega_X^{ch,g}$ described above, given by the $R^h_{g,g'}$, to obtain a new $C(g)$–equivariant structure, denoted $C(g)_\alpha$.The following theorem is an immediate consequence of the above discussion.

Theorem 3.1. *i)*

$$\mathbb{H}^*(\Omega_X^{ch}, Q_{BRST})^{C(g)_\alpha} \cong H^*_{dR}(X^g/C(g), \mathcal{L}_g^\alpha)$$

where the right-hand side is the de Rham cohomology of $[X^g/C(g)]$ *with values in the orbifold local system* $\mathcal{L}_g^\alpha$ *described in* [R]*. Thus we have an isomorphism of graded vector spaces*

$$\bigoplus_{[g]} \mathbb{H}^*(\Omega_X^{ch}, Q_{BRST})^{C(g)_\alpha} \cong H^*_{orb,\alpha}([X/G], \mathbb{C})$$

where the right-hand side is the α*–twisted Chen-Ruan cohomology of* $[X/G]$ *(see* [R]*).*
ii) Let

$$\mathcal{H}^\alpha_{orb}(X,G) = \bigoplus_{[g]} H^*(X, \Omega_X^{ch,g})_\alpha^{C(g)}$$

Then

$$Ell^\alpha_{orb}(X,G,q,y) = \mathrm{Supertrace}(q^{L_0} y^{J_0 - \dim(X)/2}, \mathcal{H}^\alpha_{orb}(X,G))$$

4. Symmetric products and discrete torsion

4.1. **The spin double cover of S_N.** We begin by reviewing discrete torsion for the symmetric group following [Di].

Let S_N denote the symmetric group on N letters. It is well-known (see eg. [Kar]) that for $N \geq 4$

$$H^2(S_N, U(1)) \cong \mathbb{Z}_2. \tag{4.1}$$

which implies that for $N \geq 4$, there is a unique non-trivial central extension of the permutation group

$$1 \to \mathbb{Z}_2 \to \widehat{S}_N \to S_N \to 1. \tag{4.2}$$

The extension $\widehat{S}_N$ can be constructed as follows. S_N acts on the hyperplane in $\mathbb{R}^N$ given by

$$x_1 + \cdots x_N = 0$$

preserving the standard inner product. This yields an embedding

$$S_N \hookrightarrow O(N-1).$$

Now, $O(N-1)$ has a double cover $\mathrm{Pin}(N-1)$. Pulling back this central extension to S_N yields $\widehat{S}_N$. We call the latter the *spin double cover* of S_N.

In terms of generators and relations, $\widehat{S}_N$ can be described as follows. It is generated by elements $1, z, \hat{t}_1, \cdots, \hat{t}_{N-1}$, where z is central, subject to the relations:

$$\begin{aligned} z^2 &= 1, \\ \hat{t}_i^2 &= z, \\ \hat{t}_i\hat{t}_{i+1}\hat{t}_i &= \hat{t}_{i+1}\hat{t}_i\hat{t}_{i+1}, \\ \hat{t}_i\hat{t}_j &= z\,\hat{t}_j\hat{t}_i, \qquad j > i+1. \end{aligned} \tag{4.3}$$

The map $\widehat{S}_N \mapsto S_N$ amounts to sending $\hat{t}_i$ to the transposition t_i interchanging the ith and $i+1$st letters, and sending z to 1. We can think of z as being -1.

4.2. **Generating functions.** Suppose that the elliptic genus of X is given by

$$Ell(X; q, y) = \sum_{m,\ell} c(m,\ell) q^m y^\ell \tag{4.4}$$

As shown in [BL1, DMVV], the generating function of the orbifold elliptic genera of the symmetric products is

$$Z(p,q,y) = \sum_{N \geq 0} p^N Ell_{orb}(X^N, S_N, q, y) = \prod_{n>0,\, m,\ell} (1 - p^n q^m y^\ell)^{-c(nm,\ell)} \tag{4.5}$$

In this section, we obtain a formula for the generating function of elliptic genera of symmetric products with discrete torsion. Let

$$Z^\alpha(p,q,y) = \sum_{N \geq 0} p^N Ell^\alpha_{orb}(X^N, S_N, q, y) \tag{4.6}$$

and let

$$Z_{++}(p,q,y) = \prod_{n>0,\, m,l\geq 0} \frac{\left(1+p^{2n}q^{m-\frac{1}{2}}y^{\ell}\right)^{c(n(2m-1),\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}}$$

$$Z_{+-}(p,q,y) = \prod_{n>0,\, m,l\geq 0} \frac{\left(1-p^{2n}q^{m-\frac{1}{2}}y^{\ell}\right)^{c(n(2m-1),\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}}$$

$$Z_{-+}(p,q,y) = \prod_{n>0,\, m,l\geq 0} \frac{\left(1+p^{2n}q^{m}y^{\ell}\right)^{c(2nm,\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}}$$

$$Z_{--}(p,q,y) = - \prod_{n>0,\, m,l\geq 0} \frac{\left(1-p^{2n}q^{m}y^{\ell}\right)^{c(2nm,\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}} \tag{4.7}$$

Theorem 4.1.

$$Z^{\alpha}(p,q,y) = \frac{1}{2}\left(Z_{++}+Z_{+-}+Z_{-+}+Z_{--}\right). \tag{4.8}$$

We begin by recalling a variation on Lemma 4.5 from [BL1]

Lemma 4.1. *Let $V = V_0 \oplus V_1$ be a super vector space, and A and B two commuting operators acting semisimply on V and preserving the parity decomposition of V. Assume furthermore that B only has non-negative eigenvalues in $\frac{1}{2}\mathbb{Z}$, and that the bigraded pieces $V_{m,l} = \{v \in V | Av = lv, Bv = mv\}$ are finite-dimensional. Let $d(m,l) = sdim(V_{m,l})$, where sdim denotes superdimension. Define the superdimension of V with respect to A, B to be the series*

$$\chi(V)(y,q) = \mathrm{Supertrace}(V, y^A q^B) = \mathrm{tr}(V_0, y^A q^B) - \mathrm{tr}(V_1, y^A q^B) = \sum_{m,l} d(m,l) q^m y^l$$

Let $Sym^N V$ denote the Nth supersymmetric product of V. The operators A and B act on $Sym^N V$, and

$$\sum_N p^N \,\mathrm{Supertrace}(Sym^N V, y^A q^B) = \prod_{m,l} \frac{1}{(1-pq^m y^l)^{d(m,l)}}$$

where the right hand side is expanded in a power series in q and p.

Let $\Lambda^N V$ denote the Nth supersymmetric wedge product of V. Since $\Lambda^N V$ is isomorphic to $Sym^N \overline{V}$, where $\overline{V}$ denotes V with its parity reversed (or directly from the argument in the proof of lemma 4.5 in [BL1]), we obtain the following:

Corollary 4.1. *Let V be as in Lemma 4.1. Then*

$$\sum_N p^N \operatorname{Supertrace}(\Lambda^N V, y^A q^B) = \prod_{m,l} (1 - pq^m y^l)^{d(m,l)}$$

Proof. (Of Theorem 4.1)
Let S_N denote the symmetric group on N letters. We recall that conjugacy classes in S_N are parametrized by partitions of N. The conjugacy class of an element $g \in S_N$ is therefore uniquely determined by the numbers a_j of j–cycles in the cycle decomposition of g. Recall moreover that the centralizer of an element with cycle type $[g] = (1)^{a_1}(2)^{a_2}\cdots(k)^{a_k}$ is

$$\prod_{i=1}^{k} S_{a_i} \ltimes (\mathbb{Z}/i\mathbb{Z})^{a_i}$$

where the $\mathbb{Z}/i\mathbb{Z}$ act by powers of the i–cycles and S_{a_i} permutes the i–cycles among themselves.

Let $c_j \in S_j$ be a j–cycle, and denote $\Omega^{ch,c_j}_{X^j}$ simply by $\Omega^{ch,j}_{X^j}$. Recall that this is a sheaf on X^j supported on X diagonally embedded, whose fibers are twisted modules over the chiral de Rham vertex algebra. Let $\mathcal{H}_{[g]} = H^*(X^N, \Omega^{ch,g}_{X^N})$ and $\mathcal{H}_j = H^*(X^j, \Omega^{ch,j}_{X^j})$, viewed as a super vector space where the parity is given by the sum of the cohomology index and the fermionic charge grading. Furthermore, introduce the operator D which acts on $\mathcal{H}_j$ by multiplication by $-j\dim(X)/2$. We have

$$\Omega^{ch,g}_{X^N} = \boxtimes_{j=1}^{k} (\Omega^{ch,j}_{X^j})^{\boxtimes a_j}$$

and so by the Kunneth formula

$$\mathcal{H}_{[g]} = \bigotimes_{j=1}^{k} \mathcal{H}_j^{\otimes a_j}$$

We have

$$\sum_N p^N Ell^{\alpha}_{orb}(X^N, S_N, q, y) = \sum_N p^N \sum_{[g]\in S_N} \operatorname{Supertrace}(q^{L_0} y^{J_0 + D}, \mathcal{H}^{C(g)_\alpha}_{[g]})$$

where the subscript on $C(g)_\alpha$ indicates that invariants are being taken with respect to the α-twisted action of $C(g)$. If $h \in C(g)$, and T_h denotes the operator of h acting on $\mathcal{H}_{[g]}$ untwisted by α, then the α–twisted action is given by $\delta(g,h)T_h$.

As explained in for instance [Di], the value $\delta(g,h)$ depends on the parity of g and h, where the latter is given by

$$\begin{aligned} p(g) &= \sum_{j=1}^{k}(j-1)a_j \mod 2 \\ &= \sum_{j=1,\, j\,\text{even}}^{k} a_j \mod 2 \end{aligned}$$

$C(g)$ is generated by transpositions $\tau(j)_{ab}$ interchanging two cycles of length j, as well as the j–cycles c_j in the cycle decomposition of g (we use the short-hand notation $c_j \in g$). The result is as follows:

(4.9)
$$\delta(g,\tau(j)_{ab}) = (-1)^{j-1}$$

and if $c_j \in g$, then

(4.10)
$$\delta(g,c_j) = \begin{cases} 1, & \text{if}\, j\, \text{is odd}, \\ (-1)^{p(g)-1} & \text{if}\, j\, \text{is even} \end{cases}$$

It follows from 4.9 and 4.10 that

$$\mathcal{H}_{[g]}^{C(g)_\alpha} = \bigotimes_{j=1\ \text{even}}^{k} \mathrm{Sym}^{a_j}(\mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}_\alpha}) \otimes \bigotimes_{j=1\ \text{odd}}^{k} \Lambda^{a_j}(\mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}_\alpha})$$

The space $\mathcal{H}^{\mathbb{Z}/j\mathbb{Z}_\alpha}$ will depend on how α twists the $\mathbb{Z}/j\mathbb{Z}$–action. For j even, there is only one possibility, and

$$\mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}_\alpha} = \mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}}$$

When j is odd, let

$$\begin{aligned} \mathcal{H}_j^{+} &= \mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}_\alpha}, \ \text{when}\, [g]\, \text{is odd} \\ \mathcal{H}_j^{-} &= \mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}_\alpha}, \ \text{when}\, [g]\, \text{is even} \end{aligned}$$

It was shown in [BL1] that with $\mathcal{H} = \mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}}$ or $\mathcal{H}_j^{+}$

$$\begin{aligned} \mathrm{Supertrace}(q^{L_0}y^{J_0+D},\mathcal{H}) &= \frac{1}{j}\sum_{r=0}^{j-1} Ell(X, q^{\frac{1}{j}}\xi^r, y) \\ &= \frac{1}{j}\sum_{m,l}(\sum_{r=0}^{j-1}\xi^{mr})q^{\frac{m}{j}}y^l \\ &= \sum_{m,l} c(mj,l)y^l q^m. \qquad (4.11) \end{aligned}$$

where $\xi = \exp(2\pi i/j)$. Similarly, using the holomorphic Lefschetz fixed-point formula, one finds

$$\begin{aligned}\text{Supertrace}(q^{L_0}y^{J_0+D}, \mathcal{H}_j^-) &= \frac{1}{j}\sum_{r=0}^{j-1}(-1)^r Ell(X, q^{\frac{1}{j}}\xi^r, y)\\ &= \frac{1}{j}\sum_{m,l}(\sum_{r=0}^{j-1}(-1)^m\xi^{mr})q^{\frac{m}{j}}y^l\\ &= \sum_{m,l} c((m-\frac{1}{2})j, l)y^l q^{m-\frac{1}{2}}. \qquad (4.12)\end{aligned}$$

Let

$$\mathbb{S} = \bigotimes_{j\,\text{odd}} \text{Sym}_{p^j}\,\mathcal{H}^{\mathbb{Z}/j\mathbb{Z}}$$

and let

$$\Lambda^+ \in \bigotimes_{j\,\text{even}} \Lambda_{p^j}\mathcal{H}_j^+ \qquad (4.13)$$

$$\Lambda^- \in \bigotimes_{j\,\text{even}} \Lambda_{p^j}\mathcal{H}_j^- \qquad (4.14)$$

denote the subspaces corresponding to permutations of odd (resp. even) parity. We have that

$$\begin{aligned}\sum_N p^N Ell_{orb}^{\alpha}(X^N, S_n, q, y) &= \text{Supertrace}(q^{L_0}y^{J_0+D}, \mathbb{S}\bigotimes\Lambda^+)\\ &\quad + \text{Supertrace}(q^{L_0}y^{J_0+D}, \mathbb{S}\bigotimes\Lambda^-)\end{aligned}$$

The result now follows from Lemma 4.1, Corollary 4.1, and the observation that

$$\begin{aligned}\text{Supertrace}(q^{L_0}y^{J_0+D}, \Lambda^+) &= \frac{1}{2}\text{Supertrace}(q^{L_0}y^{J_0+D}, \bigotimes_{j\,\text{even}} \Lambda_{p^j}\mathcal{H}_j^+)\\ &\quad - \frac{1}{2}\text{Supertrace}(q^{L_0}y^{J_0+D}, \bigotimes_{j\,\text{even}} \Lambda_{-p^j}\mathcal{H}_j^+)\end{aligned}$$

and

$$\begin{aligned}\text{Supertrace}(q^{L_0}y^{J_0+D}, \Lambda^-) &= \frac{1}{2}\text{Supertrace}(q^{L_0}y^{J_0+D}, \bigotimes_{j\,\text{even}} \Lambda_{p^j}\mathcal{H}_j^-)\\ &\quad + \frac{1}{2}\text{Supertrace}(q^{L_0}y^{J_0+D}, \bigotimes_{j\,\text{even}} \Lambda_{-p^j}\mathcal{H}_j^-).\end{aligned}$$

□

Remark. There is an equivariant version of the theorem 4.1 which is also a twisted form of the product formula for the generating functions for the wreath products (conjectured in [WZ] and proven in [BL1] Remark 4.6. p.341). Let X and G be as above and let

$$Ell(X, G; q, y) = \sum_{m,\ell} c_G(m, \ell) q^m y^\ell \tag{4.15}$$

The wreath product $G \wr S_N$ (consisting of pairs $((g_1, ..., g_N); \sigma), g_i \in G, \sigma \in S_N$ with multiplication: $((g_1, ..., g_N); \sigma_1) \cdot ((h_1, ..., h_N); \sigma_2) = ((g_1 \cdot h_{\sigma_1^{-1}(1)}, ..., g_N \cdot h_{\sigma_1^{-1}(N)}); \sigma_1\sigma_2)$ acts on the symmetric products X^N. The nontrivial class in $H^2(S_N, U(1))$ can be pulled back to the class in $H^2(G \wr S_N, U(1))$ which we denote again as α. Then the generating function $Z^\alpha(X, G, p, q, y) = \sum_{N \geq 0} p^N Ell^\alpha_{orb}(X^N,$ $G \wr S_N, q, y)$ is given by the theorem 4.1 with the coefficients $c(m, l)$ in the formulas 4.6 being replaced by $c_G(m, l)$ from 4.15

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