ALEXANDER POLYNOMIALS OF ELLIPTIC CONFIGURATIONS

A.LIBGOBER

Abstract. We calculate the Alexander polynomials for several examples of arrangements of lines with multiple points belonging to an elliptic curve extending the classical configurations of points and lines, including Hesse-Salmon's, $(36_{16}, 192_3)$, etc.

To E.Artal-Bartolo on occasion of his 60th birthday.

1. Introduction.

One of the main open problems in the study of the Alexander polynomials of plane algebraic curves is to decide for which $p(t)$ there exists a curve C whose Alexander polynomial is equal to $p(t)$. Though many constraints are known, they have not been sufficient to tell what are possible multiplicities of non-trivial factors or even if these multiplicities can be arbitrarily large (cf. [15]). In the special case of reducible curves with all components having degree 1, i.e. arrangements of lines, these polynomials were discussed in the literature as the characteristic polynomials of the Milnor fibers of arrangements and the same realization problem exists in this context as well. The interest in the Alexander polynomials stems from the fact that they provide information about the fundamental group of the complement $\pi_1(\mathbb{P}^2 \setminus C)$ beyond the abelianization of the latter, i.e. $H_1(\mathbb{P}^2 \setminus C)$ [1].

In this paper, we consider the arrangements that are configurations (p_λ, l_π) of p points, each having the multiplicity λ and l lines each containing π points of this multiplicity, associated one way or another with elliptic curves. Elliptic configurations, i.e. such that the points of fixed multiplicity are positioned on an elliptic curve, were considered for a long time from combinatorial point of view (cf. [7], [9]) and many are dated back to 19 century. In this short note, we consider several families of elliptic arrangements studied by Cayley, Hesse, and Salmon but also their natural extensions and calculate for them the Alexander polynomials showing that they have only a single root $t = 1$ [2] except for those classical ones, for which the Alexander polynomials are already known.

The method used here for the calculation of the Alexander polynomials goes back to [13] and is based on the interpretation of the multiplicities of its roots as the superabundances of the linear systems of curves defined explicitly in terms of the singularities of the curve. Despite being known for a long time this method was not used much with the exception of the work [10] where it was applied to the plane curves with ordinary cusps. We recall it in the next section.

[1] see [4] for somewhat different motivations and the calculation having overlap with the arrangements considered in the note

[2] with multiplicity being equal to the number of the irreducible components of the arrangement minus 1

Most calculations follow from the vanishing theorems [2] which can be readily applied since in the case of the ordinary curve singularities the ideals determining the linear systems are the powers of the defining ideal of the reduced set of multiple points. In a few cases, these vanishing theorems are not applicable and, as result, the Alexander polynomials contain other factors besides the powers of $(t-1)$. For them, we use explicit resolutions, though other, even faster, methods can be used. It would be interesting to understand better the possible values of superabundances of linear systems of curves of arbitrary degree determined by ideals in the local rings of singular points of plane curves in general and not just the values of superabundances of the linear systems for which the combinations of the degrees of the curves in the linear systems and the local conditions appear in the the problem of realization of the Alexander polynomials as described in the next section.

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2. The Alexander polynomial of plane curves with ordinary singularities.

In this section, we recall the relevant definitions and make explicit the relation between the Alexander polynomial and the superabundances of the linear systems defined by singularities.

2.1. **Alexander polynomials and fundamental groups of the complements.** Let $C \subset \mathbb{P}^2$ be a plane of degree d given by $f(x,y,z)=0$. The same equation defines an affine surface which is a cone in $\mathbb{C}^3$ with non-isolated singularity at the origin and along the lines corresponding to the singular points of C. Let $M(C) \subset \mathbb{C}^3$ be the surface given the equation $f(x,y,z)=1$. It is diffeomorphic to the Milnor fiber of the singularity at the origin and the restriction to $M(C)$ of the quotient map $\mathbb{C}^3 \setminus 0 \to \mathbb{P}^2$ by the $\mathbb{C}^*$-action induces the d-fold cyclic unbranched cover of $\mathbb{P}^2 \setminus C$. In particular, one has the exact sequence:

$$0 \to \pi_1(M(C)) \to \pi_1(\mathbb{P}^2 \setminus C) \to \mathbb{Z}/d\mathbb{Z} \to 0 \tag{1}$$

Here the right homomorphism is given by the linking number of $\gamma \in \pi_1(\mathbb{P}^2 \setminus C)$ with C, i.e. the intersection number of a disk D with boundary γ, intersecting C transversally at smooth points. (this intersection number is well-defined modulo $d = \deg C$). The identification of the Milnor fiber and the cyclic cover is equivariant in the sense that the monodromy of Milnor fiber corresponds to the deck transformation of this cover induced by the positively oriented meridian of C (cf. [15] Sect. 3.1).

Definition 2.1. The Alexander polynomial $\Delta_C(t)$ of C is the characteristic polynomial of the action of the generator of $\mathbb{Z}/d\mathbb{Z}$ on the abelianization of the kernel of the map $\pi_1(\mathbb{P}^2 \setminus C) \to \mathbb{Z}/d\mathbb{Z}$ given by the linking number. Alternatively, interpreting this kernel as the fundamental group of the Milnor fiber of the cone over C, the Alexander polynomial is the characteristic polynomial of the monodromy acting on the homology $H_1(M(C), \mathbb{Q})$ of the Milnor fiber.

If C is irreducible, then $H_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}/d\mathbb{Z}$, the right homomorphism in (1) is the abelianization of $\pi_1(\mathbb{P}^2 \setminus C)$ and $H_1(M(C), \mathbb{Z}) = \pi_1(\mathbb{P}^2 \setminus C)'/\pi_1(\mathbb{P}^2 \setminus C)''$ i.e. the quotient of the commutator of the fundamental group by the second commutator. For a reducible curve C with irreducible components of degrees $d_1, \cdots, d_r, r > 1$, $H_1(\mathbb{P}^2 \setminus C, \mathbb{Z}) = \mathbb{Z}^r/(d_1, \cdots, d_r)$ (here $(d_1, \cdots, d_r) \in \mathbb{Z}^r$). Hence the left term in (1) is the extension of the commutator $\pi_1(\mathbb{P}^2 \setminus C)'$ by a free abelian group having rank $r - 1$. This implies that the multiplicity of the root $t = 1$ in the Alexander polynomial of C is $r - 1$.

Any root of $\Delta_C(t)$ is a root of unity of degree $d = \deg C$ since the covering group acting on the surface $f(x, y, z) = 1$ is cyclic of order d. The degrees of these roots of unity depend on the local type of the singularities. One has the following divisibility result.

Theorem 2.2. *(cf.* [12],[15]*). Let Sing be the set of singularities of C. For each $P \in Sing$ let $\Delta_P(t)$ denote the characteristic polynomial of the monodromy of the singularity of C at P i.e. the Alexander polynomial of the link of the singularity of C at P (cf.* [19]*). Then $\Delta_C(t)$ divides $\prod_{P \in Sing} \Delta_P(t)$. In other words, the roots of the global Alexander polynomial are all among the roots of unity of degree $d = \deg C$ that are roots of one of the local Alexander polynomials of singularities of C.*

In particular the roots of the Alexander polynomial of an arrangement of d lines with singularities of multiplicity 2 or m are roots of unity of degree m provided m divides d and if $m \nmid d$, then $\Delta_C(t) = (t - 1)^{d-1}$.

2.2. **Alexander polynomials and position of singularities.** To determine the polynomial $\Delta_C(t)$ from the data given by local polynomials $\Delta_P(t)$ it remains to determine the multiplicity of a root of the latter as a root of $\Delta_C(t)$. This was done in [13] (cf. also [15]). The starting point is the following observation (cf. [12] and also [15], Chapter 3): the roots (counted with multiplicity) of the characteristic polynomial of the deck transformation (or the monodromy) acting on $H_1(M(C), \mathbb{Q})$ are the eigenvalues of the action on H_1 of an extension of the deck transformation to an automorphism of a resolution of singularities of a compactification $\widetilde{\overline{M(C)}}$ of the Milnor fiber. Such a compactification $\overline{M(C)}$ can be viewed as the hypersurface in $\mathbb{P}^3$ given by the equation $u^d = f(x, y, z)$ (cf. [15] Sect 3.5.1) [3]

Theorem 2.3. *Let ω_d be a root of unity of degree d that is a root of $\Delta_C(t)$. Its multiplicity coincides with the multiplicity of the eigenvalue ω_d of the action on $H_1(\widetilde{\overline{M(C)}}) = H^3(\widetilde{\overline{M(C)}})$ induced by the biholomorphic automorphism of $\widetilde{\overline{M(C)}}$ given by the deck transformation $u \to e^{\frac{2\pi i}{d}} u$ acting on the compactification in $\mathbb{P}^3$ given by the equation $u^d = f(x, y, z)$.*

Hence the calculation of $\Delta_C(t)$ is reduced to a calculation of the multiplicities of the eigenvalues of the deck transformation acting on the Hodge group $H^{1,2}$ of a resolution of a compactification of $M(C)$. Such an eigenvalue has the form $e^{2\pi i\kappa}$ where κ is an element in the set of rational numbers attached to a germ of one of the singular points of C and called *a constant of quasi-adjunction*. Moreover, the

[3]This hypersurface classically had been referred to as "a cyclic multiple plane" since it is a model of a cyclic branched cover $\overline{M(C)} \to \mathbb{P}^2$ (branched over C) given by the restriction of $(u, x, y, z) \to (x, y, z)$.

multiplicity of $e^{2\pi i\kappa}$ acting on $H^{1,2}(\widetilde{M(C)})$ is given in terms of the ideals attached to a constant of quasi-adjunction κ in the local rings of each $P \in Sing$.

Recall the definition of constants of quasiadjunction following [13]. Let $P = (0,0) \in \mathbb{C}^2$ be a singular point of the germ $h(x,y) = 0$. For $\phi \in \mathcal{O}_P$ consider the function of $n \in \mathbb{Z}_{\geq 1}$:

$$\psi_\phi(n) = \min\{k \in \mathbb{Z}_{\geq 0} | z^k\phi \in Adj(z^n = h(x,y))\} \tag{2}$$

where Adj denotes the adjoint ideal of the singularity $z^n = h(x,y), h(x,y) \in \mathcal{O}_P$ at the origin. [4] Then there exists a rational number κ_ϕ such that for $n \in \mathbb{Z}_{\geq 1}$

$$\psi_\phi(n) = \lfloor \kappa_\phi n \rfloor \tag{3}$$

and $\kappa \in \mathbb{Q}$ is called a constant of quasi-adjunction if there exist $\phi \in \mathcal{O}_P$ such that $\kappa = \kappa_\phi$. It can be shown that $e^{2\pi i\kappa_\phi}$ is a root an eigenvalue of the monodromy of the Milnor fiber of the singularity of h at P. More precisely, the set of constants of quasi-adjunction attached to singularity $h(x,y) = 0$ coincides the the part of the Steenbrink spectrum of h in the interval $(0,1)$ (cf. [17]) and with the set of jumping numbers of the divisor $h = 0$ (cf. [15]) [5]

Let κ be one of constants of quasi-adjunction. Define the ideal sheaf $\mathcal{A}_\kappa$ as follows. The support of $\mathcal{O}_{\mathbb{P}^2}/\mathcal{A}_\kappa$ is a subset of $Sing(C)$ and its stalk at $P \in Sing(C)$ is given by:

$$\mathcal{A}_{\kappa,P} = \{\phi \in \mathcal{O}_P | \kappa_\phi < \kappa\} \tag{4}$$

With these notations set up, we have the following

Theorem 2.4. *(cf.* [13]*) An eigenvalue of the deck transformation acting on* $H^{1,2}(\widetilde{M(C)})$ *has the form* $e^{2\pi i\kappa}$ *where* κ *is one of the constants of quasiadjunction of one of the singularities of* C*. The multiplicity of this eigenvalue is equal to* $\dim H^1(\mathbb{P}^2, \mathcal{A}_\kappa(d-3-\kappa d))$*.*

Note that this gives complete determination of the Alexander polynomials in terms of the ideals of quasi adjunction $\mathcal{A}_\kappa$. Indeed, any eigenvalue of the deck transformation acting on H^3 is either an eigenvalue on $H^{1,2}$ or its conjugate and the multiplicities of both coincide. Let us put $\Delta_C^{1,0}(t) = \prod(t - e^{2\pi i\kappa})^{\dim H^1(\mathbb{P}^2, \mathcal{A}_\kappa(d-3-\kappa d))}$ and denote by $\Delta_C^{0,1}(t)$ the characteristic polynomial of the action of the deck transformation on $H^{2,1}$ i.e. the polynomial with the roots being the conjugates of those

[4] For a modern textbook exposition of the theory of adjoints we refer to [11] vol II, p. 179, but briefly, the classical viewpoint (as was practiced for small dimensions and mild singularities in the first half of 20th century, cf. [25], and recast in a more contemporary language following [3]) is as follows. The adjoint ideal $\mathcal{A}(F) \subset \mathcal{O}_F$ of a hypersurface $F \subset \mathbb{P}^{n+1}$ is defined as the annihilator of the cokernel of $\tau : \alpha_*(\omega_{\tilde{F}}) \to \omega_F$ where $\alpha : \tilde{F} \to F$ is a resolution of singularities, ω_F and $\omega_{\tilde{F}}$ are the dualizing sheaves and τ is the trace map (cf. [3]). The stalk of the sheaf $\mathcal{A}(F)$ at a point $P \in F$ in a chart $U \subset \mathbb{P}$ can be described as the restriction to $F \cap U$ of the ideal consisting of $\{\phi \in \mathcal{O}_U \subset \mathcal{O}_P\}$ such that for any top degree form η on smooth locus of F, the pullback of $\phi\eta$ to $\tilde{F}$ extends over the exceptional locus of α. $\mathcal{A}(F)$ is independent of a choice of resolution $\tilde{F}$ and $h^{2,0}(\tilde{F}) = H^0(\omega_{\tilde{F}})$ can be identified with $H^0(\pi^{-1}(\mathcal{A}_F(d-n-2)))$ where $\pi : \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_F$ is the restriction map. For surfaces, $\chi(\mathcal{A}(F)(d-4)) - 1$ is the expected (or "postulated") dimension of the space of adjoints coinciding with the arithmetic genus of $\tilde{F}$, $H^0(\mathbb{P}^3, \mathcal{A}(F)(d-4))$ is the actual dimension i.e. the geometric genus $p_g(\tilde{F}) = H^0(\tilde{F}, \omega^2_{\tilde{F}})$ and the irregularity of $\tilde{F}$ (i.e. $h^{1,0}(\tilde{F})$) is the superabundance of the adjoints of degree $d-4$.

[5] In particular the set of constants of quasi-adjunction is finite.

of $\Delta_C^{1,0}$ with the same multiplicities. It follows from the above discussion that:

$$\Delta_C(t) = \Delta_C^{1,0}(t)\Delta_C^{0,1}(t)(t-1)^{r-1} \tag{5}$$

where, as above, r is the number of irreducible components of C. In fact (cf. [13], Theorem 5.1), this factorization can be extended to a direct sum decomposition of the Alexander module $G/G' \otimes \mathbb{R}, G = \mathrm{Ker}(\pi_1(\mathbb{P}^2 \setminus C) \to \mathbb{Z}/d\mathbb{Z})$ due to the semi-simplicity of the action of the deck transformation.

Now let us calculate the constants of quasi-adjunction and the multiplicities of the roots of the Alexander polynomial in the case when C is a union of d lines i.e. an arrangement. In this case, the singularities are ordinary ones having $x^m = y^m$ as a local equation.

Proposition 2.5. *(cf.* [14]*, Thm.4.1). The constants of quasi-adjunction of the ordinary plane curve singularity P of multiplicity m are $\frac{s}{m}, 1 \leq s \leq m-2$. The ideal of quasi-adjunction $\mathcal{A}_{\frac{s}{m},P}$ is $\mathcal{M}^{m-s-1}$ where $\mathcal{M}$ is the maximal ideal of the local ring $\mathcal{O}_P$ of the singular point. Let C be an arrangement of d distinct lines having the ordinary singularities of multiplicity 2 and m as the only singular points and such that m divides the degree of C. Let S be the set of singular points of C. Then the only roots of the global Alexander polynomial $\Delta_C(t)$ are $e^{2\pi i\frac{s}{m}}$ or $t = 1$ and the multiplicity of the root $e^{2\pi i\frac{s}{m}}$ in $\Delta^{1,0}(C)$ is equal to* $\dim H^1(\mathbb{P}^2, \mathcal{A}_S^{m-s-1}(d-3-\frac{ds}{m}))$.

Proof. We specialize the calculation made in [13] Section 5, in the case of germs generic for their Newton polygon. The adjoint ideal of singularity $z^n = x^m + y^m$ is generated by monomials ϕ such that $xyz\phi$ is strictly above the Newton boundary of this polynomial i.e. $na+nb+mc = mn$ (cf. [18]) Hence for $\phi = x^a y^b, a, b \in \mathbb{Z}_{\geq 0}$ one has $\psi_\phi(n) = \min\{k \in \mathbb{Z}_{\geq 0} | m(k+1) > mn - n(a+1) - n(b+1)\} = \left\lfloor n(1 - \frac{(a'+b')}{m}) \right\rfloor$ where $a+1 = a' \geq 1, b+1 = b' \geq 1$ and $a'+b' < m$. If $a'+b' \geq m$ then $\psi_\phi(n) = 0$. In other words the constants of quasi-adjunction are $\kappa = 1 - \frac{a'+b'}{m}$ and for $\kappa = \frac{s}{m}$ one has $\kappa_\phi < \frac{s}{m}$ if and only if $1 - \frac{a'+b'}{m} \leq \frac{s}{m}$ i.e. $m - s - 2 < a + b$ i.e. $\phi \in \mathcal{M}^{m-s-1}$ i.e. the ideal of quasi-adjunction of $\frac{s}{m}$ is as claimed in the statement. The statement about the multiplicity of a root in the global Alexander polynomial is immediate from the above discussion.

□

3. Superabundance of the powers of ideal sheaves of complete intersections of plane curves.

3.1. **A vanishing theorem.** First, recall the following general result of Bertram-Ein-Lazarsfeld, which we will need only in the zero-dimensional case.

Proposition 3.1. *(cf.* [2]*) Let $X \subset \mathbb{P}^r$ be a smooth projective variety cut out scheme theoretically by the hypersurfaces of degrees $d_1 \geq d_2 \geq \cdots \geq d_m$. Then*

$$H^i(\mathbb{P}^r, \mathcal{I}_X^a(k)) = 0 \text{ for } i \geq 1 \text{ provided } k \geq ad_1 + \cdots + d_e - r \tag{6}$$

This immediately implies:

Proposition 3.2. *Let $\mathcal{A}$ be an arrangement of d lines such that all multiple points have multiplicity either m or 2 and that the set S of points of multiplicity m is a complete intersection of curves of degrees $d_1 \geq d_2$.*

a) Regularity k of $\mathcal{I}_S^{m-s-1}$ [6] satisfies:

$$k \geq (m-s-1)d_1 + d_2 - 1 \tag{7}$$

b) If m divides d, the ideal sheaf $\mathcal{I}_S^{m-s-1}$ is k regular and the following inequality

$$k \geq d - 2 - \frac{ds}{m} \tag{8}$$

takes place, then the multiplicity of the root $e^{\frac{2\pi is}{m}}$ in the Alexander polynomial of $\mathcal{A}$ is zero [7]. In particular, the latter is the case for

$$d - 3 - \frac{ds}{m} \geq (m-s-1)d_1 + d_2 - 2 \tag{9}$$

Proof. a) is a restatement of Prop. 3.1 and b) is a corollary of a) and Prop. 2.5. □

3.2. **Explicit resolutions.** There are several ways to calculate non-vanishing groups (cf. for example [21]) but we will give (not the shortest) calculation via explicit resolution.

Proposition 3.3. *Let C be a smooth elliptic curve and P_{3k} be a subset which is a complete intersection of C and a curve C_k of degree d. Let $\mathcal{I}_k$ be the ideal sheaf of the reduced sub-scheme P_{3k} of $\mathbb{P}^2$. Then*

$$H^1(\mathbb{P}^2, \mathcal{I}_k(d)) = \begin{cases} 0 & for\ k < d \\ \mathbb{C} & for\ d{=}k \end{cases} \tag{10}$$

and

$$H^1(\mathbb{P}^2, \mathcal{I}_k^2(d)) = \begin{cases} 0 & for\ 2k < d \\ \mathbb{C}^2 & for\ d{=}6, k{=}3 \end{cases} \tag{11}$$

Proof. The set P_{3k} is a complete intersection of the cubic C and a curve C_k of degree k since $\dim H^0(\mathbb{P}^2, \mathcal{I}_k(k)) = \frac{(k+1)(k+2)}{2} - 3k > 0$ for $k \geq 3$. Hence we have the resolution:

$$O \to \mathcal{O}_{\mathbb{P}^2}(-3-k) \xrightarrow{fg} \mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2}(-k) \xrightarrow{(f,g)} \mathcal{I}_k \to 0 \tag{12}$$

where f, g are equations of the curves C and C_k respectively. After taking tensor product with $\mathcal{O}_{\mathbb{P}^2}(d)$ we obtain:

(13)

$$0 \to H^1(\mathbb{P}^2, \mathcal{I}_k(d)) \to H^2(\mathbb{P}^2, \mathcal{O}(d{-}3{-}k)) \to H^2(\mathbb{P}^2, \mathcal{O}(d{-}3)) \oplus H^2(\mathbb{P}^2, \mathcal{O}(d{-}k)) \to H^2(\mathcal{I}_k(d)) \to 0$$

By Serre duality $H^2(\mathbb{P}^2, \mathcal{O}(d-3-k)) = 0$ for $d > k$. For $d = k$ this group is isomorphic to $\mathbb{C}$ and the following it groups are vanishing. This shows (10).

For the case $H^1(\mathbb{P}^2, \mathcal{I}_k^2(d))$ we will use Taylor resolution (cf. [6], Ex..17.11)

(14)

$$0 \to \mathcal{O}(-6{-}2k) \xrightarrow{F_2} \mathcal{O}(-6{-}k) \oplus \mathcal{O}(-3{-}2k) \oplus \mathcal{O}(-6{-}2k) \xrightarrow{F_1} \mathcal{O}(-6) \oplus \mathcal{O}(-3{-}k) \oplus \mathcal{O}(-2k) \xrightarrow{F_0} \mathcal{I}_k^2 \to 0$$

[6] recall that regularity of a sheaf $\mathcal{F}$ on $\mathbb{P}^2$ is the smallest $k \in \mathbb{Z}$ such that $H^i(\mathbb{P}^2, \mathcal{F}(k-i)) = 0$, for $i \geq 1$

[7] If m does not divide d then none of $e^{2\pi i \frac{s}{m}}$ is a root of the global Alexander polynomial as follows from Theorem 2.2

where the maps F_0 and F_2 are given by:

$$F_0 \begin{array}{lll} (1,0,0) & \to & f^2 \\ (0,1,0) & \to & fg \\ (0,0,1) & \to & g^2 \end{array} \quad and \quad F_2 : 1 \to \begin{pmatrix} -g \\ f \\ 1 \end{pmatrix}^t \tag{15}$$

The matrix of the map of F_1 is

$$\begin{pmatrix} g & 0 & g^2 \\ -f & -g & 0 \\ 0 & f & -f^2 \end{pmatrix}$$

It is immediate that these maps make this sequence exact. For $d > 0$ we take the tensor product of $\mathcal{O}(d)$ and the sequence (14):

$$0 \to \mathcal{O}(-6-2k+d) \xrightarrow{F_2} \mathcal{O}(-6-k+d) \oplus \mathcal{O}(-3-2k+d) \oplus \mathcal{O}(-6-2k+d) \xrightarrow{F_1} \tag{16}$$

$$\mathcal{O}(-6+d) \oplus \mathcal{O}(-3-k+d) \oplus \mathcal{O}(-2k+d) \xrightarrow{F_0} \mathcal{I}_k^2(d) \to 0$$

Let $\mathcal{F}_d$ be the cokernel of the map corresponding to F_2 in (16). We have the following split of (16) into short exact sequences;

$$0 \to \mathcal{O}(-6-2k+d) \xrightarrow{F_2} \mathcal{O}(-6-k+d) \oplus \mathcal{O}(-3-2k+d) \oplus \mathcal{O}(-6-2k+d) \to \mathcal{F}_d \to 0 \tag{17}$$

and

$$0 \to \mathcal{F}_d \to \mathcal{O}(-6+d) \oplus \mathcal{O}(-3-k+d) \oplus \mathcal{O}(-2k+d) \xrightarrow{F_0} \mathcal{I}_k^2(d) \to 0 \tag{18}$$

The cohomology sequence corresponding to (18) gives:

(19)

$$H^1(\mathbb{P}^2, \mathcal{O}(-6+d)) \oplus H^1(\mathbb{P}^2, \mathcal{O}(-3-k+d)) \oplus H^1(\mathbb{P}^2, \mathcal{O}(-2k+d)) \to H^1(\mathbb{P}^2, \mathcal{I}_k^2(d))$$

$$\to H^2(\mathbb{P}^2, \mathcal{F}_d) \to H^2(\mathbb{P}^2, \mathcal{O}(-6+d)) \oplus H^2(\mathbb{P}^2, \mathcal{O}(-3-k+d)) \oplus H^2(\mathbb{P}^2, \mathcal{O}(-2k+d))$$

One has

$$H^1(\mathbb{P}^2, \mathcal{O}(l)) = 0, (l \in \mathbb{Z}) \qquad H^2(\mathbb{P}^2, \mathcal{O}(-3+l)) = \begin{cases} 0 & for\ l > 0 \\ \mathbb{C} & for\ l = 0 \end{cases} \tag{20}$$

Hence

$$H^1(\mathbb{P}^2, \mathcal{I}_k^2(d)) = H^2(\mathbb{P}^2, \mathcal{F}_d) \quad for\ d > 2k-3, k \geq 3 \tag{21}$$

On the other hand the sequence (17) gives:

$$H^2(\mathbb{P}^2, \mathcal{O}(-6-2k+d)) \xrightarrow{F_2} \tag{22}$$

$$H^2(\mathbb{P}^2, \mathcal{O}(-6-k+d)) \oplus H^2(\mathbb{P}^2, \mathcal{O}(-3-2k+d)) \oplus H^2(\mathbb{P}^2, \mathcal{O}(-6-2k+d)) \to H^2(\mathbb{P}^2, \mathcal{F}_d) \to 0$$

The map F_2 composed with projection on the last component is the identity as follows from (15) and hence is injective and the middle term is zero for $d > 2k$. For $d = 6, k = 3$ the middle term is $\mathbb{C}^2$ which proves the Proposition. □

If $\mathcal{I}$ is the ideal of the complete intersection of two smooth cubics then (14) becomes:

$$0 \to \mathcal{O}(-12) \to \mathcal{O}(-9)^2 \oplus \mathcal{O}(-12) \to \mathcal{O}(-6)^3 \to \mathcal{I}^2 \to 0$$

with maps given by replacing f, g by the equation of cubic curves giving the complete intersection.

4. Explicit formulas for the Alexander polynomials.

Now we will examine some explicit examples of elliptic arrangements $(3k_m, mk_3)$ and their Alexander polynomials.

Example 4.1. Hesse configuration. This is the arrangement formed by 12 lines containing 9 inflection points of a smooth cubic curve. It is a configuration $(9_4, 12_3)$ the Alexander polynomial is $(t+1)^2(t^2+1)^2(t-1)^{11}$. Indeed, the ideal of quasi-adjunction corresponding to $\kappa = \frac{1}{4}$ is $\mathcal{I}_S^2$ where S is the set of quadruple points and the multiplicity of the corresponding factor t^2+1 is $\dim H^1(\mathbb{P}^2, \mathcal{I}_S^1(12-3-\frac{12}{4})) = 2$.

Example 4.2. Hesse-Salmon configuration $(12_4, 16_3)$ (cf. [5])

Recall the construction. Let a_1, a_2, a_3 be a triple of points forming a complete intersection of a smooth cubic E and a line. Each of the pencils of lines with the base point being at one of the points $a_i, i = 1, 2, 3$, contains 4 lines tangent to E. It follows that these 12 tangency points belong to 16 lines. Indeed, in the group structure on points of E, with zero, as usual, being one of the inflection points, one has the relation $a_1 + a_2 + a_3 = 0$. Moreover, since for each contact point $P_i^j, j = 1, \cdots, 4$ of a tangent to E line through a_i one has the relation $a_i + 2P_i^j = 0$ and $P_i^j - P_i^1$ is a point of order 2 on E. If P and Q are the contact points of tangents through a_1, a_2 then, the third intersection point R of a line through P and Q, since $P + Q + R = 0$ satisfies $a_3 + 2R = 0$ i.e. is a contact point of a tangent through a_3. Moreover, it follows that the group $\mathbb{Z}_2^2$ acts on the set of 12 tangency points by translation without fixed points and with 3 orbits. The triples of collinear contact points correspond to the solutions to the equation $x_1 + x_2 + x_3 = 0$ in $\mathbb{Z}_2^2$. To any pair, $x_1, x_2 \in \mathbb{Z}_2^2$ corresponds unique solution x_3 and hence we have 16 collinear triples of contact points ie. we obtain configuration $(12_4, 16_3)$.

We claim that the Alexander polynomial is $(t-1)^{15}$. Indeed, it follows from the above description of 12 points in terms of the group structure of E, that their sum is zero and therefore they form a complete intersection with E and, more precisely, they form a union of 4 lines intersecting pairwise outside of contact points. Possible constants of quasi-adjunction are $\frac{s}{4}, 1 \leq s \leq 2$. The multiplicity of the factor t^2+1 corresponding to $\frac{1}{4}$ is $\dim H^1(\mathbb{P}^2, \mathcal{I}_S^2(9)) = 0$ (the inequality (9) is equality in this case; $\mathcal{I}_S$ is the ideal sheaf of the set of 12 points of multiplicity 4) and the multiplicity of the factor $t+1$ is $\dim H^1(\mathbb{P}^2, \mathcal{I}_S(5)) = 0$ (this set of 12 points has the superabundance equal to 1 for quartics, by Cayley-Bacharach theorem) and the claim follows.

Example 4.3. Configuration $(36_{16}, 192_3)$. Consider the union of the Hasse-Salmon arrangements in the previous example corresponding to the 12 triples of collinear inflection points on an elliptic curve E. The set S of contact points of the tangents in all 12 arrangements coincides with the set of points of order 6 on E.

The lines in this configuration correspond to unordered distinct triples of points in $\mathbb{Z}_6$ such that reduction modulo 2 gives three distinct points in $\mathbb{Z}_3$. The first point can be any of the 36 points or order 6, the second can be any of 32 points outside of a coset modulo 3 of the first selected point and these choices determine the third one. Considering these as unordered triples gives $\frac{36\times 32}{6} = 192$ lines. The same count shows that there are 16 lines through each point. More directly, by the definition we have a union of 12 groups of 16 lines totaling 192 lines. This has the same numerology as in [20].

Note that a set of $3N$ points on E is zero if and only if this set is a complete intersection of E and a curve of degree N. and the sum of all points in $\mathbb{Z}_k^2$ is zero. In particular, the latter set is a complete intersection. Indeed, if k is odd (resp. k is even), then the sum of the points on each line is zero (respectively a point of order 2, independent of a line). In particular, if $n = 3k$ and k is odd (resp. even), the set of points of order n is a union of n (resp. $\frac{n}{2}$) complete intersections of E and curves of order k (resp. curves having order $2k$ if k is even).

The multiplicity of the root $e^{\frac{2\pi is}{m}}$ in the Alexander polynomial is equal to $\dim H^1(\mathbb{P}^2, \mathcal{I}_S^{15-s}(189-12s))$ which is zero by the inequality (9) in Prop. 3.2 since $189-12s \geq (15-s)12+1$ is satisfied. Hence the Alexander polynomial is $(t-1)^{191}$.

Example 4.4. Configurations $(3n_{n^2}^2, n_3^4)$. The construction of Hesse-Salmon can be generalized to the following family. Start with 3 collinear points (x_1, x_2, x_3) and consider the set of solutions to $x_i + ny_i^j = 0$. There are $3n^2$ solutions corresponding to $i = 1, 2, 3, j = 1, \cdots, n^2$. Then $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$ are on a line if and only if $y_1 + y_2 + y_3 = 0$ has a solution and the number of solutions is n^4. Hence we obtain the arrangement of n^4 lines, with $3n^2$ multiple points. Each line contains 3 points and hence through each point pass n^2 lines i.e. we have the configuration: $(n_3^4, 3n_{n^2}^2)$. The number of double points in this arrangement of lines is:

$$\frac{n^4(n^4-1)}{2} - 3n^2\frac{n^2(n^2-1)}{2} = n^4(\frac{n^4-1-3n^2+3}{2}) = \frac{n^4(n^2-1)(n^2-2)}{2}$$

(for $n = 2$ i.e. in Hesse-Salmon case, one gets 48 double points). Complete intersection property follows from the argument in Example 4.3. Let us calculate the Alexander polynomial of the arrangement $(3n_{n^2}^2, n_3^4)$. Let $N = n^2$. The multiplicities of the roots coincide with $\dim H^1(\mathbb{P}^2, \mathcal{I}_S^{N-s-1}(N^2 - 3 - N^2(\frac{s}{N}))$. Since $N^2-3-Ns \geq N(N-s-1)+1 = N^2-Ns-N+1$ we obtain that these dimensions are zeros i.e. the Alexander polynomial is $(t-1)^{n^4-1}$.

Taking the union of such arrangements corresponding to all 12 lines containing the flexes we obtain the configuration $(9n_{4n^2}^2, 12n_3^4)$. For $n = 2$ this is the configuration 4.3. Here the multiple points correspond to the $9n^2$ points of order $3n$.The lines in this configuration correspond to the unordered triples that form a solution to $y_1 + y_2 + y_3 = 0$ which reductions modulo n are distinct. Again, this description in terms of the group structure shows the multiple points form a complete intersection. The Alexander polynomial in this case is $(t-1)^{12n^4-1}$.

Example 4.5. Sextactic points. In [23] the authors considered the configuration $(27_9, 81_3)$. with multiple points formed by *sextactic* points (a subset of the set of 27 points of order 6 that is complement to the subset of points of order 3) i.e. a subarrangement of the Example 4.3. This set of 27 points forms a complete intersection of curves of degrees $3, 9$. The multiplicities of the roots of the Alexander polynomial are $\dim H^1(\mathcal{I}_S^{8-s}(81-3-81\frac{s}{9}))$. Applying Prop. 9 we see that $78-9s \geq (8-s)9+1 = 73-9s$ and hence the Alexander polynomial is $(t-1)^{80}$.

Remark 4.6. It was shown in [16] that a pencil of curves of degree $d \geq 3$ cannot have more than 5 fibers which are unions of d lines (and later that no more than 4 by the second named author of [16] and J.Stipins [24], [22]). The only known pencil of curves of degree d having 4 elements with all irreducible components being lines, is the Hesse configuration $(12_3, 9_4)$. Since a pencil of curves of degree d with 4 reducible fibers forming an arrangement $\mathcal{A}$ induces the map $\mathbb{P}^2 \setminus \mathcal{A}$ onto the

complement in $\mathbb{P}^1$ to 4 points $\mathcal{P}$, and since the Alexander polynomial of $\mathbb{P}^1 \setminus \mathcal{P}$ for the map $\pi_1(\mathbb{P}^1 \setminus \mathcal{P})$ onto $\mathbb{Z}$ sending each loop about a point of $\mathcal{P}$ to the generator of $\mathbb{Z}$ has as a factor (t^2+1), it follows that the Hesse Salmon arrangement is not formed by completely reducible fibers of a pencil of curves of degree 4.

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Department of Mathematics, University of Illinois, Chicago, IL 60607
Email address: libgober@uic.edu