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FACTORS OF JACOBIANS AND ISOTRIVIAL ELLIPTIC SURFACES

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Abstract. We show that the rank of the Mordell-Weil group of an isotrivial elliptic surface over $\mathbb{C}(t)$ can be calculated as the number of isogeny factors which are elliptic curves in the jacobian of the cyclic cover of a projective line associated to the elliptic surface. We illustrate this method by calculating the ranks in several examples, some of which recover already known results, and discuss relation between open questions on factors of jacobians and elliptic surfaces.

1. Introduction

In papers [3] and [15] we developed a method for calculation of the ranks of Mordell-Weil groups of isotrivial complex elliptic threefolds which yields an expression for these ranks in terms of the Albanese variety of cyclic covers of the base of the elliptic fibration. In many cases this leads to explicit values of the rank (cf. [3],[15]) since the structure of Albanese variety, always depending on the singularities of the discriminant of elliptic fibration, is often rather simple even for discriminants with quite complicated singularities. In present note we illustrate a similar approach to the study of Mordell-Weil ranks of isotrivial *elliptic surfaces.* The upshot is a relation between the Mordell Weil rank of an elliptic surface with generic fiber isomorphic to an elliptic curve E and the isogeny factors isomorphic to E in the Jacobian of the appropriate cyclic cover of the base of elliptic fibration. More precisely we have the following.

Theorem 1.1. *Let $\mathcal{E} \to \mathbb{P}^1$ be an isotrivial elliptic surface over $\mathbb{C}$. Denote by E a generic fiber of this fibration and let $\Gamma = AutE$. Denote by C_Γ the cyclic cover of $\mathbb{P}^1$ branched over the zero set of the discriminant of $\mathcal{E}$ over which the pullback of $\mathcal{E}$ is biholomorphic to a direct product. Let $Jac(C_\Gamma)$ be the Jacobian of C_Γ and let*

$$r = \{max\ k | Jac(C_\Gamma) \sim_\Gamma E^k \times A\} \tag{1}$$

(here $\sim_\Gamma$ denotes equivariant isogeny of abelian varieties with Γ-action). If E has a complex multiplication then the rank of Mordell-Weil of $\mathcal{E}$ satisfies:

$$rkMW(\mathcal{E}) = 2r \tag{2}$$

Otherwise this rank is r.

Approach to the study of isotrivial families via covering space over which family trivializes in the case of surfaces was used in the past (e.g. [8] [1]). The advantage of the case of surfaces over high dimensional elliptic fibrations is that the ranks of elliptic surfaces were the objects of intense scrutiny for a long time (cf. [17]). The theorem 1.1 allows to understand the values of the Mordell-Weil ranks from a different perspective. As examples we recover several results of Usui (cf. [24]), Shioda (cf. [21]) and others, in particular a calculation of the maximal known at the moment rank of elliptic surfaces (i.e.68). Our calculations of Mordell-Weil ranks in these

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[1] I am grateful to R. Kloosterman for pointing out this reference; see also his thesis www.math.hu-berlin.de/ klooster/publ.php

examples depends on the description of the Jacobian of Fermat curves due to Koblitz (cf. [10] cf. also [1]) [2] and hyperelliptic curves given in [6] (cf. also [22]). Understanding products of elliptic curves which appear as factors of Jacobians is an interesting problem (cf. [4]) and in fact one can use the theorem 1.1 to obtain for some cyclic covers the multiplicity of a curve in the isogeny decomposition of the Jacobian (cf. 4.8) using available information on the Mordell-Weil ranks.

The content of the paper is as follows. In section 2 we recall definitions and introduce notations. Section 3 contains a proof the theorem 1.1 and in section 4 we discuss examples illustrating approach to the ranks of elliptic surfaces using Jacobians. Concluding section 5 contains a discussion of related problems.

2. Preliminaries

By elliptic surface we mean a smooth projective surface $\mathcal{E}$ together with a morphism $\pi : \mathcal{E} \to C$ where C is a smooth curve whose generic fiber is a genus one curve and which moreover is endowed with a section $s_0 : C \to \mathcal{E}$. Section s_0 allows to give to fibers of π the structure of elliptic curve. An elliptic surface is called isotrivial (resp. trivial) if the j-invariant of a generic fiber over $c \in C$ is a constant function of c (resp. $\mathcal{E}$ is birational to the surface $E \times C$ for some elliptic curve E over $\mathbb{C}$). Below $E_{\mathcal{E}}$ denotes the elliptic curve which is a smooth fiber of an isotrivial surface $\mathcal{E}$ (subscript will be omitted when the choice of $\mathcal{E}$ is clear from context). We refer for the basics of the theory of elliptic surfaces to the surveys [17] or [5, Ch.1]. For additional material related to this discussion see [15].

Recall that the Mordell-Weil group of $\mathcal{E}$ (denoted $MW(\mathcal{E})$) is the group of sections $s : C \to \mathcal{E}$ of π with the group structure given by addition of $s_1(c), s_2(c) \in E_c$ where E_c is the fiber of $\mathcal{E}$ over c with the group structure existing on any smooth curve of genus one after choice of $s_0(c)$ as the zero. This group of sections is finitely generated, *unless* $\mathcal{E}$ is trivial but in latter case the group of sections modulo the subgroup of constant sections of $C \times E$ given by $s_e : c \to e, e \in E$ (the Chow trace) is still finitely generated (cf. [12], [17]). The Mordell-Weil group in the case when $\mathcal{E}$ is trivial, is the quotient of the group of sections by the subgroup of constant sections. The morphism π gives to $\mathcal{E}$ the structure of elliptic curve over C and from this view point $MW(\mathcal{E})$ is just the group of points of elliptic curve over the function field $\mathbb{C}(C)$ (again if $\mathcal{E} \neq E \times C$ and the quotient by the subgroup $E(\mathbb{C})$ otherwise).

Since $MW(\mathcal{E})$ is a finitely generated abelian group, it is isomorphic to $\mathbb{Z}^r \oplus Tor$ where $r \in \mathbb{Z}^{\geq 0}$ and Tor is a finite abelian group. The integer r is called the rank of the elliptic surface. The rank of $\mathcal{E}$ has the following expression: (Shioda-Tate formula)

$$r = rkNS(E) - 2 - \sum_{v \in \Delta_\pi} (m(F_v) - 1) \tag{3}$$

where $NS(\mathcal{E})$ is the Neron-Severi group of $\mathcal{E}$, Δ_π is the set of points in C over which the fibers of π are singular and $m_v(F)$ is the number of irreducible component in $\pi^{-1}(v)$. Most calculations of the ranks are based on a use of (3). Note that set $\Delta_\pi \subset C$ consists of the points at which the discriminant vanishes (the latter is an element of $H^0(C, \mathcal{L}^{12})$ for some line bundle on C cf. [5, th.1.4.1].

In many cases, the ranks and Mordell-Weil groups of elliptic surfaces are known. However it seems is unknown if there is a universal bound (cf. [17]). The largest known rank 68 of elliptic surfaces is achieved by $y^2 = x^3 + t^{360k} - 1$ (cf. [21] and the section 4.5 below). Over

[2] interestingly, Shioda's calculation in these example depends on properties of Delsarte surfaces closely related to Fermat surfaces.

function fields of characteristic $p > 0$, the ranks are unbounded for both isotrivial (cf. [18]) and non-isotrivial cases (cf. [23]).

We shall need the following description of isotrivial surfaces in terms of trivial ones which we shall briefly sketch (cf. [5, 1.4.2] and references there).

Proposition 2.1. *Let $\pi : \mathcal{E} \to C$ be an isotrivial fibration with generic fiber E. Let $\Gamma = AutE$ be the automorphism group of E (i.e. a cyclic group of order $2, 4$ or 6). Then there is a curve C_Γ and a covering map $\pi_\Gamma : C_\Gamma \to C$ with the covering group Γ and ramification set supported at Δ_π such that one has birational isomorphism:*

$$\mathcal{E} \times_C C_\Gamma = C_\Gamma \times_{\mathbb{C}} E_{\mathcal{E}} \tag{4}$$

Proof. As in [15] one can use the results in [13] and [11] to deduce that there is a Γ-covering $C' - S \to C - \Delta_\pi$ such that C' is a smooth projective curve, S is a finite subset of C' and $\pi^{-1}(C - \Delta_\pi) = E \times (C' - S)/\Gamma$ where the quotient on the right is taken for the diagonal action of Γ. Using the identification $E \times (C' - S)/\Gamma \times_{C - \Delta_\pi} C_\Gamma - S = E \times (C' - S)$ the birational equivalence (4) is clear.

Alternatively, since the j-invariant i.e. the map $C - \Delta_\pi \to H/PSL_2(\mathbb{Z})$ is constant (here H is the upper half-plane) the monodromy representation $\pi_1(C - \Delta_\pi) \to Aut^+ H_1(E, \mathbb{Z})$ factors through $\pi_1(C - \Delta_\pi) \to Aut(E)$ (cf. [5, p.40]). Hence the pullback of $\mathcal{E}$ on the covering of $C - \Delta_\pi$ corresponding to the latter homomorphism of the fundamental group yields a family with constant j-invariant and trivial holonomy i.e. the direct product. □

Finally recall that the Jacobian of a curve C can be characterized as an abelian variety universal with respect to holomorphic maps into abelian varieties A i.e.

$$Mor(C, A) = Hom(Jac(C), A) \tag{5}$$

(on the left is the group of maps up to a translation by a point in A). Moreover this correspondence is compatible with holomorphic maps of C and in particular if Γ is a subgroup of $Aut(C)$ acting on A then for the group of Γ-maps one has:

$$Mor_\Gamma(C, A) = Hom_\Gamma(Jac(C), A) \tag{6}$$

where the subscript indicates equivariant maps.

3. Proof of theorem 1.1

In this section we shall prove the theorem 1.1. Let $\mathcal{E}$ be isotrivial but non trivial elliptic surface. Let $s : C \to \mathcal{E}$ be a point of $\mathcal{E}$ over $\mathbb{C}(C)$. The map $C_\Gamma \to \mathcal{E} \times_C C_\Gamma$ given by $\tilde{c} \to (s(\pi_\Gamma(\tilde{c})), \tilde{c})$, ($\tilde{c} \in C_\Gamma$) yields the lift $\tilde{s}$ of s which is a section of the trivial (cf. (4)) elliptic surface $C_\Gamma \times E_{\mathcal{E}}$. Unless s has order 2 in $MW(\mathcal{E})$, $\tilde{s}$ is not constant since otherwise $\tilde{s}(\tilde{c}) = (\tilde{c}, e)$ and $e \in E_{\mathcal{E}}$ must be Γ-invariant i.e. is a 2-torsion point. Let $Sec'(C_\Gamma \times E)$ be a subgroup of the group of sections of $C_\Gamma \times E$ isomorphic to $MW(C_\Gamma \times E)$. [3] It can be defined using a splitting of the sequence

$$0 \to E \to Sec(C_\Gamma \times E) \to MW(C_\Gamma \times E) \to 0 \tag{7}$$

For example, a map $Sec(C_\Gamma \times E) \to E$ given by sending $\tilde{s}$ to $\tilde{s}(\tilde{c})$ for a point $\tilde{c} \in C_\Gamma$ yields such a split. One has the following isomorphisms of groups in which the first one is obvious while the

[3] or, if $\Gamma = \mathbb{Z}_2$ or $\mathbb{Z}_4$, rather a maximal 2-torsion free subgroup among the subgroups with the following property: the only constant Γ-invariant (for diagonal action of Γ) element in $Sec'(C_\Gamma \times E)$ is the zero section.

second is a consequence of the universal property of the Jacobian with respect to the maps from the curve to abelian varieties (cf.(5)):

$$Sec'(C_\Gamma \times E) = Mor(C_\Gamma, E) = Hom(Jac(C_\Gamma), E) \tag{8}$$

The lift of s induces the equivariant map $Jac(C_\Gamma) \to E$ with respect to the natural action of $\Gamma \subset AutE$ on E. Vice versa, equivariant map from Jacobian of C_Γ to E induces the Γ-equivariant map of C_Γ which viewed as a section of $C_\Gamma \times E$ descents to a section of $\mathcal{E}$. Hence

$$MW(\mathcal{E}) = Hom_\Gamma(Jac(C_\Gamma), E) \tag{9}$$

Next let $Jac(C_\Gamma) \sim_\Gamma E^r \times A$ and A is not Γ-isogenous to $E \times A'$. By Poincare reducibility theorem (cf. [16]) the latter is equivalent to $Hom_\Gamma(A, E) = 0$ and hence $Hom_\Gamma(Jac(C_\Gamma), E) = Hom(E^r, E) = End(E)^r$. The latter has rank $2r$ if E has complex multiplication since $rkEnd(E) = 2$ in this case. Otherwise $rkHom(E^r, E) = r$.

Note that the above argument shows that calculation of the rank in theorem 1.1 also holds if $\mathcal{E}$ is trivial.

4. Elliptic surfaces related to Fermat curves

4.1. **Cyclic covers of $\mathbb{P}^1$.** In most examples considered below the curves over which the elliptic surfaces becomes trivial and Jacobians of which according to the theorem 1.1 determine the Mordell Weil groups are quotients of Fermat curve. In particular the Jacobians of these curves are subvarieties of Jacobians of Fermat curves. We recall results from [10] describing the factors of Jacobians of Fermat curves.

Let F_N be the curve given by the equation:

$$x^N + y^N + z^N = 0 \tag{10}$$

The one dimensional components of $Jac(F_N) = H^0(F_N, \Omega^1_{F_N})^*/H_1(F_N, \mathbb{Z})$ correspond to one dimensional subspaces of $H^0(F_N, \Omega^1_{F_N})$ generated by forms

$$\omega_{r,s,t} = \frac{x^{Nr-1}y^{Ns-1}dx}{y^{N-1}} \quad (Nr, Ns, Nt \in \mathbb{Z}^+, r+s+t=1) \tag{11}$$

Note that $\omega_{r,s,t}$ spans an eigenspace for transformation induced by $(x, y, z) \to (\zeta_N x, y, z)$ (resp. transformation induced by $(x, y, z) \to (x, \zeta_N y, z))$ where $\zeta_N = exp(\frac{2\pi\sqrt{-1}}{N})$ corresponding to the eigenvalue: $exp(2\pi\sqrt{-1}r)$ (resp. $exp(2\pi\sqrt{-1}s)$).

The curves which are the isogeny components of $Jac(F_N)$ all appear as the factors of the abelian varieties denoted as $J_{[r,s,t]}$ where $[r, s, t]$ is the orbit of the triple for the action defined below. $J_{[r,s,t]}$ are all of CM type and as such correspond to the cyclotomic fields $\mathbb{Q}(\zeta_M)$ (cf. [19], [14]) with $M|N$. The CM type of $J_{[r,s,t]}$ is given by the subset of $Gal(\mathbb{Q}(\zeta_M)/\mathbb{Q}) = (\mathbb{Z}/M\mathbb{Z})^*$ defined as follows:

$$H_{r,s,t} = \{u \in (\mathbb{Z}/M\mathbb{Z})^* | < ur > + < us > + < ut >= 1\} \tag{12}$$

(here $< \cdot >$ denotes the least non negative rational residue modulo 1). The abelian varieties $J_{[r,s,t]}$ are labeled by the orbits of the following action of $H_{r,s,t}$ on triples (r, s, t): $u(r, s, t) = (< ur >, < us >, < ut >)$. Each of $J_{[r,s,t]}$ is isogenous to the product $E^{CardH_{r,s,t}}$ for an appropriate CM curve E.

Proposition 4.1. *The abelian varieties $J_{[r,s,t]}$ having the curve E_0 with j-invariant zero as an isogeny component are given in the following table:*

(13)

M	Mr	Ms	Mt	$CardH_{r,s,t}$	*label*
3	1	1	1	1	$3(i)$
6	1	1	4	1	$6(i)$
6	1	2	3	1	$6(ii)$
12	1	1	10	2	$12(i)$
12	1	2	9	2	$12(ii)$
12	1	3	8	2	$12(iii)$
12	1	4	7	2	$12(iv)$
12	1	5	6	2	$12(v)$
15	1	2	12	4	$15(i)$
15	1	4	10	4	$15(ii)$
18	1	3	14	3	$18(i)$
21	1	4	16	6	$21(i)$
21	1	8	12	6	$21(ii)$
24	1	1	22	4	$24(i)$
24	1	4	19	4	$24(ii)$
24	1	5	18	4	$24(iii)$
24	1	6	17	4	$24(iv)$
24	1	7	16	4	$24(v)$
24	1	10	13	4	$24(vi)$
24	1	11	12	4	$24(vii)$
30	1	5	24	4	$30(i)$
30	1	10	19	4	$30(ii)$
39	1	16	22	12	$39(i)$
48	1	22	25	8	$48(i)$
60	1	10	49	8	$60(i)$

Proof. Abelian varieties $J_{[r,s,t]}$ admitting the curve E_0 as isogeny component admit the automorphism of order 6 and hence $\mathbb{Q}(\zeta_6)$ is a subfield of $End(J_{[r,s,t]}) \otimes \mathbb{Q} = \mathbb{Q}(\zeta_M)$. Therefore $3|M$. The table (13) is the part of the table from [10] corresponding to M with this divisibility condition. □

Proposition 4.2. *Let C_{6m} be the cyclic cover of $\mathbb{P}^1$ which is a compactification of the curve $s^6 = t^{6m} - 1$. Denote by T the automorphism of Jacobian induced by the automorphism of C_{6m} given by $(s,t) \to (\zeta_6 s, t)$. Let $\mathcal{S}(E_0)$ be the set of ordered triples $\frac{6m}{M}(r,s,t)$ (where $M|6m$) such that $\frac{r}{M} = \frac{1}{6}$ and such that no two triples belong to the same orbit of $H_{r,s,t}$. Then the maximal number of E_0-factors in $Jac(C_{6m})$ on the tangent space at identity of which T acts as multiplication by ζ_6 is equal to*

$$\sum_{(r,s,t)\in\mathcal{S}(\mathcal{E}_0)} CardH_{r,s,t} \tag{14}$$

Proof. C_{6m} is the quotient of F_{6m} by the action of the group of roots of unity generated by $(x,y,z) \to (\zeta_m x, y, z)$. Hence $Jac(C_{6m})$ is component of the product of varieties $J_{[r,s,t]}$ such that $6m(r,s,t) \in (\mathbb{Z}^+)^3$ and such that $6r \in \mathbb{Z}^+$. The equivariance condition restricts r further to $r = \frac{1}{6}$. The claim follows. □

The following examples were obtained by Shioda and Usui (cf. [24]) using a different method which we shall derive from Prop. 4.2.

Example 4.3. Elliptic surface $\mathcal{E}_6 : y^2 = x^3 + t^6 - 1$ The curve C_6 is the Fermat curve F_6 and we count the number of E_0-components in the isogeny class of $Jac(F_6)$. This is the number of triples in the table (13) with $M|6$, and $r = \frac{1}{6}$. We obtain the following triples (Mr, Ms, Mt):

$$(1,1,4),(1,4,1),(1,2,3),(1,3,2). \tag{15}$$

Hence $Jac(C_6) \sim E_0^4 \times A$ where A does not have E_0 isogeny components and hence $rkMW(\mathcal{E}_6) = 8$ (cf. [24] where the corresponding lattice given by the height pairing is identified with the lattice E_8).

Example 4.4. Consider the elliptic surface $\mathcal{E}_9 : y^2 = x^3 + t^9 - 1$. $rkMW(\mathcal{E}_9)$ depends on the number of E_0 components of $u^6 = t^9 - 1$. This curve is the quotient of F_{18} by the action of $\mu_3 \times \mu_2$ given by multiplication of coordinates. Hence the E_0 components of $Jac(F_{18}/\mu_3 \times \mu_2)$ correspond to triples (a, b, c) such that,

a) $a + b + c = 18$ and
b) $3|a, 2|b$
c)$M|18$

The triples satisfying these conditions are:

$$3(i) : (6,6,6), 6(i) : (3,12,3), 6(ii) : (3,6,9), 6(ii)* : (9,6,3), 18(i) : (3,14,1). \tag{16}$$

The equivariance conditions yield that $a = 3$. This is satisfied by: $6(i), 6(ii), 18(i)$. $CardH_{r,s,t}$ in the first two cases is 1 and in the case $18(i)$ it is 3. Hence Jacobian contains E_0^5 and hence $rkMW(\mathcal{E}_9) = 10$ (cf. [24]).

Example 4.5. Consider the surface $\mathcal{E}_{360} : y^2 = x^3 + t^{360} - 1$. Shioda's calculation yields $rkMW(\mathcal{E}_{360}) = 68$. To see this from the viewpoint of the theorem 1.1 we need to calculate the number of (equivariant) E_0-factors of $Jac(C_{360})$. They correspond to the triples (r, s, t) such that $M|360$ and one of components (r, s, t) is $\frac{1}{6}$. Note that this implies that $6|M$. The triples satisfying these condition are given by the following list. Below $(R, S, T) = 360(r, s, t)$; each triple in the list comes from the triple $\frac{1}{(g.c.d(R,S,T))}(R, S, T)$ appearing in the table with $M = \frac{R+S+T}{g.c.d(R,S,T)}$; we indicate its label in table (13) in front and add asterisk if it is obtained by a permutation of a triple in (13).

$$6(i) : (60,60,240), 6(i)* : (60,240,60), 6(ii) : (60,120,180), 6(ii)* : (60,180,120), \tag{17}$$

$$12(ii) : (60,30,270), 12(ii)* : (60,270,30), 18(i) : (60,20,280), 18(ii)* : (60,280,20), \tag{18}$$

$$24(ii) : (60,15,285), 24(ii)* : (60,285,15), 30(i) : (20,60,288), 30(i)* : (20,288,60), \tag{19}$$

$$60(i) : (60,6,294), 60(i)* : (60,294,6) \tag{20}$$

However $60(i), 60(i)*$ give the same abelian 4-fold since $(60, 6, 294) \cdot 7^2 = (60, 294, 6)$ ($H_{[1,10,49]} = \{7^i 31^j | i, j \in \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\}$ cf. [10]). Similarly $(60, 15, 285) \cdot (7 \cdot 13) = (60, 285, 15)$ (($H_{[1,4,19]} = \{7^i 13^j | i, j \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\}$) i.e. $J_{[60,285,15]}$ coincides with $J_{[60,15,285]}$. There are no other repetitions in abelian varieties corresponding to triples (17)-(20) as follows by direct calculation using data on $H_{r,s,t}$ from [10]. The dimensions of $J_{[r,s,t]}$ above are as follows. For $6(i), 6(i)*, 6(ii), 6(ii)*$ the dimension is equal to 1, for $12(ii), 12(**)$ it is 2, for $18(i), 18(ii)*$ it is 3, for $24(ii), 24(ii)*$ it is 4, for $30(i), 30(i)*$ it is 4 and for $60(i), 60(i)*$ it is 8. Taking into account that, as was mentioned, $J_{[1,10,49]} = J_{[1,49,10]}$ and $J_{[4,19,1]} = J_{[4,1,19]}$ we obtain:

$$\sum CardH_{r,s,t} = 1 + 1 + 1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 4 + 8 = 34 \tag{21}$$

Hence the total rank is $2 \times 34 = 68$

Example 4.6. The case of surfaces $y^2 = x^3 + t^{360k} - 1$ can be analyzed similarly. We need to know the number of factors of the Jacobian of $s^6 = t^{360k} - 1$ which are μ_6-equivalent to the curve with j-invariant zero with given automorphism of order 6. Those are all the factors of the Jacobian of Fermat curve (10) with $N = 360k$. The equivariance conditions on (r, s, t) to appear as the factor in the Jacobian of the curve $s^6 = t^{360k} - 1$ is $Mu \cdot (\frac{N}{M}) = \frac{N}{6}$ i.e. $Mu = \frac{M}{6}$ (where $u = r, s, t$) which is the same as in the case $k = 1$. Hence the collection of the varieties $J_{[r,s,t]}$ which are the product of the curves E_0 is independent of k i.e. $rkMW$ by theorem 1.1 is independent of k as well.

Example 4.7. In [6] the authors show that the hyperelliptic curve:

$$H : y^2 = x(x-1)(x-\lambda)(x-\mu)(x-\nu) \tag{22}$$

where $\mu = \nu\frac{1-\lambda}{1-\nu}$ and λ, μ, ν are pairwise distinct, different from 0 and 1 has the Jacobian isogenous to the product of two copies of the curve:

$$E : y^2 = x(x-1)(x-\Lambda) \tag{23}$$

where Λ is a solution of

$$\nu^2\lambda^2\Lambda^2 + 2\nu\mu(-2\nu + \Lambda)\Lambda + \mu^2 = 0. \tag{24}$$

Then the elliptic surface

$$E \times H/\mu_2 \tag{25}$$

where the diagonal action of μ_2 is via multiplication by -1 on the first factor and via hyperelliptic involution on the second has the following ranks:

$$rkMW = \begin{cases} 2 & \text{if } E \text{ is without CM} \\ 4 & \text{if } E \text{ has CM} \end{cases} \tag{26}$$

This provides isotrivial elliptic surface with arbitrary j-invariant and positive Mordell-Weil rank. Many examples with large $rkMW$ can be constructed using examples of hyperelliptic curves with split Jacobian given in [22]

Example 4.8. In work [2] the authors calculated the rank of elliptic surface $y^2 = x^3 - 27(t^{12} - 11t^6 - 1)$ is equal to 18. For the curve $u^6 = t^{12} - 11t^6 - 1$ this translates into $Jac = E_0^9 \times A$ with $A \nsim E_0 \times A'$ for any A'.

5. Abelian varieties, families with higher dimensional bases and some questions

5.1. **Isotrivial abelian varieties.** Using results from [15] theorem 1.1 can be extended to the case of abelian varieties over $\mathbb{C}(t)$:

Theorem 5.1. *Let $\mathcal{A} \to \mathbb{P}^1$ be isotrivial family of abelian varieties over $\mathbb{C}$ with a simple generic fiber. Fix a projective embedding of $\mathcal{A}$, denote by A a generic fiber of this fibration and let Γ be the automorphism group of A preserving the polarization. Denote by C_Γ the cover of $\mathbb{P}^1$ branched over the zero set of the discriminant of $\mathcal{A}$ over which the pullback of $\mathcal{A}$ is biholomorphic to a direct product (cf.* [15]*, section 2.1 references there). Let $Jac(C_\Gamma)$ denote the Jacobian of C_Γ and let*

$$r = \{max\ k | Jac(C_\Gamma) \sim_\Gamma A^k \times B\} \tag{27}$$

(here $\sim_\Gamma$ denotes equivariant isogeny of abelian varieties with Γ-action). Then

$$rkMW(\mathcal{A}) = r\dim_{\mathbb{Q}} End(A) \otimes \mathbb{Q} \tag{28}$$

Example 5.2. Let $\mathcal{E}$ be an isotrivial elliptic surface with fiber being the curve E_0 and such that $rkMW(\mathcal{E}) > 0$. Let $\mathcal{A}_n = \mathcal{E} \times_{\mathbb{P}^1} ... \times_{\mathbb{P}^1} \mathcal{E}$ (n-fold product). Then $rkMW(\mathcal{A}_n) = \gamma \cdot n \ \ \gamma \in \mathbb{Z}$.

Indeed, $\mathcal{A}_n$ becomes trivial over the same cover C of $\mathbb{P}^1$ as A_1, and the argument used in the proof of 1.1 yields that $rkME(\mathcal{A}_n) = rdimHom(E_0, E_0^n) = 2nr$ where r is the multiplicity of E_0 in $Jac(C)$.

5.2. **Remarks on isotrivial elliptic threefolds.** The following relation, shown for example in [7], between the Mordell-Weil ranks of elliptic surfaces and ranks of Mordell-Weil groups of threefolds was used in [3] in order to obtain restrictions on the fundamental groups of the complements to discriminants.

Proposition 5.3. *Let $\mathcal{E} \to \mathbb{P}^2$ be an elliptic threefold, Let L be a generic line in $\mathbb{P}^2$ and $\mathcal{E}|_L$ is the restriction of $\mathcal{E}$ on L. Then $rkMW(\mathcal{E}) \le rkMW(\mathcal{E}|_L)$.*

In particular universal bounds on ranks of elliptic surfaces over $\mathbb{C}(t)$ yield bounds on ranks of n-folds (over $\mathbb{P}^{n-1}$). The relation between the fundamental groups and ranks from [3] is the following:

Theorem 5.4. *Let D be the zero set of the discriminant of elliptic threefold $\mathcal{E} \to \mathbb{P}^2$. Assume that D is irreducible and that the only singularities of C are either ordinary nodes or ordinary cusps. Then*

$$rkMW(\mathcal{E}) = rk\pi_1(\mathbb{P}^2 - D)'/\pi_1''(\mathbb{P}^2 - D) \tag{29}$$

(right hand side is the quotient of the commutator of the fundamental group by the second commutator)

Known bounds on the right hand side in (29), coming from various interpretations, (cf. [3], [9]) are linear in degree d of D. This leads to the following question:

Question 5.5. Let $f(t)$ be a polynomial of degree $d \equiv 0(mod\ 6)$ and $C_d(f)$ be the cyclic cover of $\mathbb{P}^1$ given by equation the $u^6 = f(t)$. Consider the elliptic surface $\mathcal{E}_f$ given by

$$y^2 = x^3 + f(t) \tag{30}$$

Does exist $\epsilon > 0$ and a positive constant α such that for the rank of $\mathcal{E}_f$ one has

$$rkMW(\mathcal{E}_f) \le \alpha d^{1-\epsilon} \tag{31}$$

By the theorem 1.1 this is equivalent to the following:

Question 5.6. Does there exist a bound on the number of isogeny component of $Jac(C_d(f))$ isomorphic to E_0 of the form $\alpha d^{1-\epsilon}$?

Note that for the curve $C_d(f)$ in (5.5) one has $g(C_d(f)) = \frac{5}{2}d - 5$ i.e. $\epsilon \ge 0$ in (31). As was mentioned known examples in characteristic zero obeys bound (31) with $\epsilon = 1$. It would be interesting to know what is ϵ in positive characteristic.

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