FREE QUOTIENTS OF FUNDAMENTAL GROUPS OF SMOOTH QUASI-PROJECTIVE VARIETIES

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ABSTRACT. We consider the structure of reducible curves on a projective simply connected surface with irreducible components belonging to a selected subset the effective cone of the surface and which fundamental groups of the complements admit free quotients having rank greater than one. Main result is the following trichotomy depending on the ranks of free (essential) quotients of the fundamental groups with components in the subset of effective cone. A: There can be an infinite number of isotopy classes of curves with classes of components in a selected subset of effective cone and rank of free quotients being below a threshold depending on the subset. B: There are only finitely many isotopy classes of curves with components in selected subset of effective cone admitting surjection onto a free group of rank greater the threshold. C: Moreover, irreducible components of curves admitting an essential surjection onto a free group of rank sufficiently larger than the threshold belong to a pencil of curves having class in the selected subset of the effective cone. Some explicit information on the thresholds for different cases of the trichotomy are discussed.

1. PREFACE

The goal of this note is to describe a result on the structure of the curves $\mathcal{D} \subset V$ on a simply connected projective surface V such that $\pi_1(V \setminus \mathcal{D})$ admits a surjection on a free group of a large rank. We consider the curves $\mathcal{D}$ as elements of *collections* $\mathcal{C}(\Delta)$, parametrized by subsets Δ of the effective cone of V in the sense that the irreducible components of $\mathcal{D}$ are required to have the homology classes in Δ. We make only a technical assumption on singularities of $\mathcal{D}$ at intersections of different components, but not on singularities of irreducible components outside of these intersections and consider surjections $\pi_1(V \setminus \mathcal{D}) \to F_r$ which are essential in the sense that the images in F_r of the elements of $\pi_1(V \setminus \mathcal{D})$ associated with each irreducible component of $\mathcal{D}$ (the meridians) are conjugate to either the chosen generators $g_1, ..., g_r$ of F_r or to their product $g_1 \cdot ... \cdot g_r$. In these circumstances we show that there is a constant $M(\Delta)$ such that existence of surjection $\pi_1(V \setminus \mathcal{D}) \to F_r$ with $r > M(\Delta)$ implies that all components of $\mathcal{D}$ belong to a pencil of curves with fundamental class being a class in Δ. In other words, existence of a surjection of the fundamental group of a quasi-projective surface with compactification V onto F_r with sufficiently large r with components of the divisor at infinity $\mathcal{D} \subset V$ being in Δ implies that the divisor at infinity formed by curves in a pencil *of divisors in* Δ. We expect that a more careful analysis can show that the technical assumptions we made can be eliminated or substantially weaken.

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The main step in the argument is a statement about pencils of curves on V admitting sufficiently many fibers having as irreducible components only the curves from the chosen subset Δ. More precisely, we consider pencils $L \subset \mathbb{P}(H^0(V, \mathcal{O}_V(D)))$ (i.e. $dim L = 1$) admitting $r+1$ distinct divisors $d_1, ...d_{r+1}, d_i \in L$ such that irreducible components of d_i have classes in Δ. We show that existence of such a pencil implies that $r \leq M(\Delta)$, unless all elements of the pencil L are already in Δ i.e. $\mathcal{D}$ is formed by curves belonging to a pencil of curves with generic member in a class from Δ. For any $r > 1$ and a curve $\mathcal{D} = \bigcup_{i=1}^{r+1} d_i$ with $r+1$ components belonging to a pencil L, the rational dominant map onto $\mathbb{P}^1$ corresponding to L induces surjection $\pi_1(V \setminus \mathcal{D}) \to F_r$ and our result shows that given a class Δ of allowed components of $\mathcal{D}$ this is the only way to construct surjections of the fundamental groups of the complements onto F_r with $r > M(\Delta)$. Explicit information about $M(\Delta)$ for $V = \mathbb{P}^2$ and some other surfaces is provided in section (4).

We also show that $\pi_1(V \setminus \mathcal{D})$ has no surjections onto $F_r, r > 10$ for all curves $\mathcal{D}$ having irreducible components in Δ *with only finitely many exceptions depending on* Δ, unless $\mathcal{D}$ is a union of curves belonging to a pencil with generic member in a class from Δ. The number of exceptions in general is growing together with Δ. For some V and $\Delta \in \mathrm{NS}(V)$, absence of surjections onto F_r with finitely many exceptions occurs already for $r \leq 10$ (see section 5). However, for general V, $r \leq 10$ and certain Δ one expects infinitely many curves $\mathcal{D}$ with components in Δ admitting surjections onto F_r and not being a union of members of a pencil with class of generic member in Δ. It is an interesting problem to find the threshold $R(V, \Delta)$ explicitly for specific (V, Δ) with initial steps made in section 4

A precursor of such results is the following statement about arrangements of lines in $\mathbb{P}^2$ which was first shown in [10] (and later improved in [6] and [14]). If an arrangement $\mathcal{A}$ of lines in $\mathbb{P}^2$ is such that there exist a pencil of curves of degree $d \geq 1$ admitting 5 or more elements which are unions of lines and such that the arrangement $\mathcal{A}$ is the union of the lines in these elements of the pencil then $d = 1$ and therefore $\mathcal{A}$ is a central arrangement i.e. is the union of lines all containing a fixed point. (cf. section 4 for discussion of this and other special cases). This implies that $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ has no essential surjections onto $F_r, r \geq 4$ except for the central arrangements. On the other hand there are infinitely many non-central arrangements of lines with $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ having essential surjections onto F_2.

The conditions on the sets Δ we use are as follows. Let $\mathrm{Eff}(V) \subset \mathrm{NS}(V)$ denote the effective cone of V (cf. [9]). We call a subset $\Delta \subset \mathrm{Eff}(V)$ *saturated* if $D \in \Delta$ and divisors C and $D - C$ are both in $\mathrm{Eff}(V)$ implies C and $D - C$ are both Δ. This condition assures that if an "allowed" (i.e. from Δ) class in $\mathrm{NS}(V)$ is represented by a reducible curve its irreducible components are allowed as well. Other conditions of being saturated may lead to different implications on the ranks of free quotients. We work with finite saturated sets. Note that a minimal saturated subset containing a finite set is finite (cf. Lemma 3.2).

Below we consider curves $\mathcal{D}$ on smooth surfaces up to the following equivalence relation: two curves $\mathcal{D}', \mathcal{D}''$ are equivalent if there is equi-singular deformation of $\mathcal{D}'$ into $\mathcal{D}''$. Such deformations do not alter the fundamental groups of the complements and all the finiteness statements are made for such equivalence classes.

We also say that a curve $\mathcal{D}$ is composed of a pencil in a linear system $H^0(V, \mathcal{O}(D))$ if there is a partition of irreducible components of $\mathcal{D}$ into groups such that union of irreducible components in each group is one of the curves in the pencil.

Our main result now can be stated as follows.

Theorem 1.1. *Let V be a smooth simply connected projective surface. Let $\Delta \subset \mathrm{NS}(V)$ be a saturated subset of effective cone i.e. having the property that if $d_1 \in \Delta, d_2 \in \mathrm{Eff}(V)$ are such that $d_1 - d_2$ is effective then $d_2, d_1 - d_2 \in \Delta$. Let $\mathcal{D}$ be a curve such that its irreducible components have classes in Δ. Assume moreover that all singular points belonging to more than one component are ordinary multiple points i.e. locally are transversal intersection of smooth germs. Then there is a constant $M(V, \Delta)$ such that if $\pi_1(V \setminus \mathcal{D})$ admits an essential surjection onto $F_r, r > M(V, \Delta)$ (i.e. a surjection taking each meridian of a component of $\mathcal{D}$ to an element in a conjugacy class of either one r generators of F_r or their product), then irreducible components of $\mathcal{D}$ form elements of a pencil such that the class of its generic element in $\mathrm{NS}(V)$ is a class $\delta \in \Delta$. Moreover, if $r > 10$ then there is only finite number $N(V, \Delta)$ curves $\mathcal{D}$ with components from Δ and not composed of a pencil in $H^0(V, \mathcal{O}(\delta)), \delta \in \Delta$ but admitting surjection $\pi_1(V \setminus \mathcal{D}) \to F_r$.*

The implication of this result on the structure of quasi-projective groups for curves with given classes of components is as follows:

Corollary 1.2. *Given a saturated set Δ of classes in $\mathrm{NS}(V)$ there is trichotomy depending on the ranks of free quotients of fundamental groups $\pi_1(V \setminus \mathcal{D})$ where $\mathcal{D}$ has all its irreducible components belonging to Δ.*

A. There are infinitely many (isotopy classes of) curves admitting surjections $\pi_1(V \setminus \mathcal{D}) \to F_r$ with $r \leq 10$.

B.There are finitely many isotopy classes of curves $\mathcal{D}$ admitting surjections $\pi_1(V \setminus \mathcal{D}) \to F_r, 10 < r < M(V, \Delta)$.

C. If there is surjection $\pi_1(V \setminus \mathcal{D}) \to F_r, r > M(\Delta)$ then $\mathcal{D}$ is composed of curves of a pencil from one of the classes in Δ and $\pi_1(V \setminus \mathcal{D})$ splits as an amalgamated product $H *_{\pi_1(\Sigma)} G$ where Σ is a Riemann surface which is a smooth member of the pencil, H is coming from a finite set of groups associated with the linear system $H^0(V, \mathcal{O}(\delta)), \delta \in \Delta$ and G is an extension:

$$0 \to \pi_1(\Sigma) \to G \to F_r \to 0 \tag{1}$$

The content of the paper is as follows. In section 3, we give the definitions of the classes of curves for which we describe the distributions of the types of free quotients, define the thresholds $M(V, \Delta), N(V, \Delta)$ and $K(V, \Delta)$ and prove the main results of the paper. In particular we obtain the bound on the number of reducible fibers of the pencils, with irreducible components belonging to a saturated set Δ. In section 4 we give details on the free quotients for the curves on $\mathbb{P}^2$ and in section 5 we deal with examples on more general surfaces. More detailed estimates for introduced here thresholds will be given elsewhere.

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2. Summary of Some Notations

Δ is a saturated subset of $\mathrm{NS}(V)$ where V is a smooth projective simply-connected surface.

$M(V, \Delta) \in \mathbb{Z}_+$ (or simply $M(\Delta)$; the same with similar notations below) is the threshold for the ranks of free groups F_r such that beyond it existence of surjection onto free group of $\pi_1(V \setminus \mathcal{D})$, where $\mathcal{D}$ is a curve with classes of components in Δ, implies that all irreducible components of $\mathcal{D}$ belong to a pencil in complete linear system of a curve with class in Δ (cf. theorem 1.1).

$N(V, r, \Delta) \in \mathbb{Z}_+$ is the number of curves $\mathcal{D}$ with irreducible components having classes in a saturated set Δ, which admit a surjection onto a free group $F_r, r > 10$ (cf. theorem 1.1).

$K(V, \Delta)$ is a positive integer such that for pencils in $H^0(V, \mathcal{O}(D))$ where D is such that for any $\delta \in \Delta$ one has $D - K(\Delta)\delta \in \mathrm{Eff}(V)$, there are at most 12 reducible fibers with all irreducible components on Δ (cf. Corollary 3.4).

3. Proof of the main theorem

Definition 3.1. Subset $\Delta \subset \mathrm{Eff}(V)$ is called *saturated* if $d \in \Delta$ and $\delta \in \mathrm{Eff}(V)$ is such that $d - \delta \in \mathrm{Eff}(V)$ implies that $\delta \in \Delta$. A saturated subset of $\mathrm{NS}(V)$ spanned by $d_1, .., d_k, d_i \in \mathrm{Eff}(V)$ is the intersection of all saturated subsets of $\mathrm{Eff}(V)$ each containing all these classes. We denote it by $\Delta(d_1, .., d_r)$.

Lemma 3.2. *Saturated subset spanned by a finite subset of* $\mathrm{Eff}(V)$ *is finite.*

Proof. Let H be an ample divisor. Then for any $\delta \in \Delta(d), d \in \mathrm{Eff}(V)$ one has $(\delta, H) \leq (d, H)$. Therefore $\Delta(d)$ is a discrete subset of a compact set $\{S \in \bar{\mathrm{Eff}}(V) \otimes \mathbb{R} | (S, H) \leq (d, H)\}$ (since, as follows from the Hodge index theorem, $\mathrm{Eff}(V) \otimes \mathbb{R}$ is a cone over a compact set). Hence it is finite. Alternatively, the claim can be derived from [3] Theorem 4.10b. Finally, the set $\Delta(d_1, .., d_r) = \bigcup \Delta(d_i)$ and hence is finite as well. □

The following will be used in an estimate of the number of fibers having classes of components in a fixed saturated subset Δ only, which can appear in pencils on a surface V.

Proposition 3.3. *Let Δ be a saturated subset of* $\mathrm{Eff}(V)$ *and $s = Card\Delta$. Let $\mathcal{F}$ be a curve with irreducible components $f_j, j = 1, ..., \mathcal{J}$ which moves in a pencil with all its reducible fibers being reduced. Let $D = \sum_{i=1}^{s} m_i d_i$ be the class of $\mathcal{F}$ in the free abelian group generated by $d_i \in \Delta \subset$* $\mathrm{NS}(V)$*. We also use the same notations for corresponding classes in* $\mathrm{NS}(V)$*. For $\alpha > \frac{5}{3}$ and all but finitely many $(m_1, ..., m_s) \in \mathbb{Z}^s_{\geq 0}$ one has*

$$\frac{\frac{e(V)}{3} + D^2 + \frac{2}{3}KD}{D^2 + \sum_{j=1}^{\mathcal{J}} e(f_j) + KD} < \alpha \tag{2}$$

($e(f_j)$ denotes topological euler characteristic of component f_j).

Proof. Note that one has

$$e(f_j) = -Kf_j - f_j^2 + \sum_{P \in Sing(f_j)} (2\delta(f_j, P) - bf_j, P) + 1) \tag{3}$$

where P runs through the set $Sing(f_j)$ of singular points of the curve f_j in the class $f_j \in \mathrm{NS}(V)$ which we denote by the same letter as the curve, and $\delta(f_j, P), b(f_j, P)$ are respectively the δ invariant and the number of branches of f_j at P (cf. [13]).

Since each summand in summation in (3) is non-negative, the denominator in (2) satisfies:

$$D^2+\sum_{j=1}^{\mathcal{J}} e(f_j)+KD \ge D^2-\sum m_iKd_i-m_id_i^2+KD=D^2-\sum m_id_i^2$$

Since we assume that the pencil consists of only reduced members, in the decomposition $D=\sum_{i,d_i^2\ge 0} m_id_i+\sum_{i,d_i^2<0} m_id_i$, for coefficients of $d_i, d_i^2<0$ one has $m_i=1$. Indeed, two irreducible curves with negative self-intersection which appears in reducible member $\mathcal{F}$ of the pencil more than once must coincide, since they cannot be deformations on each other. Therefore $D^2-\sum m_id_i^2=\sum(m_i^2-m_i)d_i^2+2\sum m_im_jd_id_j>0$. and hence the inequality (2) would follow from

$$(\alpha-1)D^2-\alpha\sum m_id_i^2-\frac{2}{3}KD>\frac{e(V)}{3}. \tag{4}$$

To show (4), let $D=\sum m_id_i, d_i\in\Delta$ be a divisor which is a reducible member of a pencil (i.e. $dimH^0(V,\mathcal{O}(D))\ge 2$). By Riemann-Roch

$$-DK=2(dimH^0(\mathcal{O}(D))-dimH^1(\mathcal{O}(D))+dimH^2(\mathcal{O}(D))-2\chi(V)-D^2.$$

Asymptotics of the cohomology of the nef divisors (cf. the proof of Theorem 1.4.40 [9] and [8]) implies that for all but finitely many $(m_1,....,m_s)$, i.e. those in the compact subset of $\overline{\mathrm{Eff}(V)}\subset \mathrm{NS}(V)\otimes\mathbb{R}$ where $dimH^1(\mathcal{O}(D))$ exceeds the dimension of $H^0(\mathcal{O}(D))$, one has $\chi(\mathcal{O}(D))\ge 0$. For those $(m_1,....,m_s)$,

$$-DK\ge -2\chi(V)-D^2. \tag{5}$$

Hence for all but finitely many $(m_1,...,m_s)\in\mathbb{Z}^s_{\ge}$ we have the following inequality for the left hand side of (4):

$$(\alpha-1)D^2-\alpha\sum m_id_i^2-\frac{2}{3}KD\ge(\alpha-\frac{5}{3})D^2-\alpha\sum m_id_i^2-\frac{4}{3}\chi(V) \tag{6}$$

We claim that since $\alpha>\frac{5}{3}$, for all but finitely many m_i one has

$$(\alpha-\frac{5}{3})D^2-\alpha\sum m_id_i^2>\frac{e(V)}{3}+\frac{4}{3}\chi(V) \tag{7}$$

Indeed, as was already mentioned, in the decomposition $D=\sum_{i,d_i^2\ge 0} m_id_i+\sum_{i,d_i^2<0} m_id_i$, since we assume that the pencil consists of only reduced members and hence for $d_i, d_i^2<0$ one has $m_i=1$, and for left hand side of (7) one has

$$(\alpha-\frac{5}{3})D^2-\alpha\sum m_id_i^2=$$

$$\alpha\sum_{i,d_i^2\ge 0}(m_i^2-m_i)d_i^2-\frac{5}{3}\sum_{i,d_i^2<0} m_i^2d_i^2+2(\alpha-\frac{5}{3})\sum_{i<j} m_im_jd_id_j\ge\frac{e(V)}{3}+\frac{4}{3}\chi(V)>0$$

This is satisfied for all $(m_1,,,.m_s)\in\mathbb{Z}^s_{\ge}$ but a finite set since exceptions are given by solutions of the opposite inequality which belong to a compact subset of $(\mathbb{R}_{\ge})^s$.

Combined together, (3) and (7) show the Proposition.

□

Proof. (of theorem 1.1) Let $\mathcal{D}$ be a curve as in theorem 1.1. Assume that $\mathcal{D}$ is not composed of curves in a pencil in a linear system $H^0(V,\mathcal{O}(\delta)), \delta\in\Delta$. Note that without this assumption, for a pencil of curves in a class $\delta\in\Delta$, one can construct a curve $\mathcal{D}$, which is a union of an arbitrary large number N of members of this pencil and for this curve the fundamental a group of the complement has essential surjection onto F_{N-1} (cf. proof of Corollary 1.2 C).

If there is a surjection $\pi_1(V \setminus \mathcal{D}) \to F_r, r \geq 2$ it follows from [1] that there is a surjective holomorphic map with connected fibers $V \setminus \mathcal{D} \to C \setminus R$, where C is a smooth curve and $R \subset C$ is a finite set containing $r+1$ points such that $Card R \geq r+1$. Since we assume that $\pi_1(V \setminus \mathcal{D}) \to F_r$ is essential it follows that $Card R = r+1$. The generic curve in this pencil does not belong to Δ since $\mathcal{D}$ is not composed of a pencil as above and in particular each fiber over a point in R is reducible since its class is the class of a generic member in the pencil.

This map extends to a map having indeterminacy points at a subset of V of codimension two. More specifically, the map is well defined outside of a finite subset $B \subset V$ which is a subset of the set of intersections of components of $\mathcal{D}$ i.e. subset of the union of $d' \cap d''$ where d', d'' run through the set of pairs of irreducible components of $\mathcal{D}$. Moreover, since we assume that intersections of components of $\mathcal{D}$ are transversal, this map extends to a holomorphic map $\Phi : \tilde{V} \to \mathbb{P}^1$ of the blow up of V at the indeterminacy points on V. One has $\pi_1(V) = \pi_1(V \setminus B) \to \pi_1(C)$ and since we assume $\pi_1(V) = 0$ this shows that $C = \mathbb{P}^1$. $\Phi^{-1}(R) \subset \tilde{V}$ can be identified with $\mathcal{D}$ due to our assumption on intersection of components on V and surjection of the fundamental group being essential. Outside of a finite subset $B' \subset \mathbb{P}^1$ the smooth fibers of holomorphic map Φ are diffeomorphic and one has inclusion $R \subseteq B'$.

Assumption of transversality of components of $\mathcal{D}$ also implies that $\mathrm{Card} B = (\Phi^{-1}(b) \cdot \Phi^{-1}(b))$. For each $b' \in B'$ and any $p \in \mathbb{P}^1 \setminus B'$, let $e_{rel}(b') = e(\Phi^{-1}(b')) - e(\Phi^{-1}(p))$ be the relative euler characteristic of the fiber at b'. Since the assumption of the theorem on meridians implies that the multiplicities of all components of the pencil Φ are equal to 1, it follows from the additivity of the euler characteristic (cf. also [7]) that

$$e(V) + \mathrm{Card} B = 2e(\Phi^{-1}(p)) + \sum_{b' \in B'} e_{rel}(b') = 2e(\Phi^{-1}(p)) + \sum_{b' \in B'} \mu(b') \tag{8}$$

where $p \in \mathbb{P}^1 \setminus B'$ and $\mu(b')$ is the sum of the Milnor numbers of the singularities in the fiber over b' of Φ. It follows by the adjunction applied on V, that for $p \in \mathbb{P}^1 \setminus B'$ one has $e(\Phi^{-1}(p)) = -(KD + D^2)$ where $D \subset V$ is the union of components of $\mathcal{D}$ which $\Phi|_{V \setminus B}$ takes to a point $b \in R$. Note that for any $b' \in B' \setminus R$ one has $e_{rel}(b') > 0$ and $Card B = D^2$. Therefore

$$e(V) + D^2 \geq -2(KD + D^2) + \sum_{b' \in R} e_{rel}(b') \tag{9}$$

For any $b' \in R$, such that the class of fiber $\Phi^{-1}(b')$ is $D = \sum m_i d_i$, one has

$$e_{rel}(b') \geq \sum -m_i d_i (K + d_i) - \sum \frac{m_i(m_i - 1)}{2} d_i^2 - \sum_{i<j} m_i m_j d_i d_j + D(K + D) \tag{10}$$

$$= -\sum m_i K d_i + \sum d_i^2 (-m_i - \frac{m_i(m_i-1)}{2} + m_i^2) + \sum_{i<j} d_i d_j (2m_i m_j - m_i m_j) + \sum m_i K d_i$$

$$= \sum d_i^2 \frac{m_i(m_i-1)}{2} + \sum d_i d_j m_i m_j = \frac{D^2 - \sum m_i (d_i^2)}{2} = \frac{D^2 + KD + \sum m_i e(d_i)}{2}$$

Selecting $b' \in R$ for which $\frac{D^2-\sum m_i(d_i^2)}{2}$ is the smallest one obtains

$$e(V)+D^2 \geq -2(KD+D^2)+r\frac{D^2+KD+\sum m_ie(d_i)}{2} \tag{11}$$

Hence the last inequality and the Proposition (3.3) imply that

$$r \leq 2\frac{e(V)+3D^2+2KD}{D^2+KD+\sum m_ie(d_i)} < 6\alpha \tag{12}$$

For $\alpha = \frac{5}{3}$ we obtain that, with only finitely many exceptions $\Xi = \{D|D = \sum m_iD_i\}$, a pencil in a linear system D will have at most 10 reducible fibers with all components in Δ. Since each linear system may have only a finitely many isotopy classes of pencils (it is bounded by the number of strata in a stratification of the set of pair $(l, Disc)$ where l is a line and $Disc$ is the discriminant in the complete linear system $\mathbb{P}(H^0(\mathcal{O}(D)))$, this may create only a finite set of pencils in $H^0(\mathcal{O}(D))$ with $D \in \Xi$ having as irreducible components the curves in Δ. This shows the theorem. □

The inequalities considered in the proof above imply the following.

Corollary 3.4. *Let $\Delta \subset \mathrm{NS}(V)$ denote a saturated subset. Assume that either Δ contains a class d such that $d^2 > 0$ or that one has $d^2 < 0, Kd < 0$ for all classes in Δ. Then there exists a constant $K(V,\Delta) \in \mathbb{Z}_+$ such that for a pencil of curves having class $D \in \mathrm{NS}(V)$ satisfying inequality $D > d$ for all $d \in K(V,\Delta)\Delta$ where*

$$K(V,\Delta)\Delta = \{d \in \mathrm{NS}(V)|d = \sum m_id_i, d_i \in \Delta, m_i > K(V,\Delta)\}$$

the number $r(V)$ of reducible fibers with components in Δ is bounded by 12.

Remark 3.5. Recall that we are considering only the pencils subject to condition on meridians stated in the theorem 1.1. It excludes the pencils with all components of reduced fibers being in Δ having only classes d with $d^2 < 0$ in $D = \sum m_id_i, m_i > 1$. Proposition 5.12 below shows that there are pencils having arbitrary large number of reducible components with negative self-intersections and positive intersection with canonical class (albeit on different surfaces)

Proof. Consider the inequality (4) for $\alpha = 2$ i.e.

$$D^2 - 2\sum m_id_i^2 - \frac{2}{3}KD > \frac{e(V)}{3} \tag{13}$$

We want to show that there is $K(\Delta)$ such it holds for $D = \sum m_id_i$ satisfying $m_i > K(V,\Delta)$ for all i

Let $d_1 \in \Delta$ be class such that $d_1^2 > 0$. Let $K(V,\Delta)$ be the maximum of the roots of polynomials $f_i(m)$ or 1 where

$$f_1(m) = m^2d_1^2 - 2md_1^2 - \frac{2}{3}mKd_1 + \frac{2}{3}\sum_{j, d_j^2 \leq 0, Kd_j > 0} Kd_j - \frac{e(V)}{3}$$

and

$$f_i(m) = m^2d_i^2 - 2md_i^2 - \frac{2}{3}mKd_i$$

for each $i > 1$ such that $d_i^2 > 0$.

Then for $D = \sum m_i d_i, m_i \geq K(V,\Delta)$ for each i with $d_i^2 > 0$ one has $f_i(m_i) > 0$ and hence

$$D^2 - 2\sum m_i d_i^2 - \frac{2}{3}KD - \frac{e(V)}{3} =$$

$$\sum_{i,d_i^2>0} m_i^2 d_i^2 + 2\sum_{i,j} m_i m_j d_i d_j - 2\sum_{i,d_i^2>0} m_i d_i^2 + \sum_{i,d_i^2<0} (m^2 - 2m) d_i^2$$

$$-\frac{2}{3}\sum_{i,d_i^2>0} m_i K d_i - \frac{2}{3}\sum_{i,d_i^2<0} m_i K d_i - \frac{e(V)}{3} \geq$$

$$\sum_{i,d_i^2>0} m_i^2 d_i^2 - 2\sum_{i,d_i^2>0} m_i d_i^2 - \frac{2}{3}\sum_{i,d_i^2>0} m_i K d_i - \frac{2}{3}\sum_{j,d_j^2<0,Kd_j>0} K d_j - \frac{e(V)}{3} \geq$$

$$\sum_i f_i(m_i) > 0$$

The first inequality used that $m_i = 1$ for curves with $d_i^2 < 0$ since multiplicities of components are equal to 1 and positivity of other dropped terms. Therefore the inequality (4) is satisfied for $\alpha = 2$ and hence the inequality (12) with $\alpha = 2$ implies that a pencil of curves in $H^0(V, \mathcal{O}(D))$ has at most 12 reduced fibers with components having classes only in Δ.

□

Finally we will show the Corollary 1.2.

Proof. (of Corollary 1.2) The implications A and B are an immediate consequence of the theorem. We will show that the fundamental groups of the complements to a union of $r+1$ members of a pencil have the form described in C. Consider the map $\pi : V \setminus \mathcal{D} \to \mathbb{P}^1 \setminus \mathcal{R}, Card\mathcal{R} = r+1$ corresponding to the pencil and let $\mathcal{R} = R \bigcup S$ where R (resp. S) are the images of singular (resp. smooth) fibers of π. Let D_1 be a disk in $\mathbb{P}^1$ containing all critical values of π outside of R and let D_2 be a disk in intersecting D_1 at one point and containing $\mathcal{R}$. Let $H = \pi_1(\pi^{-1}(D_1))$. The fundamental group of $\pi|_{\pi^{-1}(D_2)}$ is isomorphic to extension (1) since over D_2 the map π is a locally trivial fibration and $\pi_2(D_2 \setminus \mathcal{R}) = 0$. Finally $V \setminus \mathcal{D}$ can be retracted onto a union if preimages of D_1 and D_2 and van Kampen theorem gives a presentation as the amalgamated product with $\Sigma = \pi^{-1}(D_1 \cap D_2)$ i.e. the complement in a generic fiber, i.e. a closed Riemann surface, to the set of base points of the pencil which is the extension of π to $\tilde{V}$. Since the set of isotopy classes of pencils in one of linear system $H^0(V, \mathcal{O}(\delta))$ is finite and for a fixed pencil each subgroup H is determined by subset R of the total set of critical points of S with reducible preimages, the finiteness claim follows. □

4. Pencils on $\mathbb{P}^2$

We will consider concrete examples of estimates of type of fibers pencils on surfaces for various choices of the types Δ of components of reducible curves $\mathcal{D} \subset V$. In many cases the bound $M(\Delta)$ can be made more explicit.

4.1. **General estimates.** In the case $\Delta = [1] \subset \mathbb{Z} = Pic(\mathbb{P}^2)$ one has $M(\Delta) \le 3$ (cf. [10],[14],[6]). Indeed, as was shown in these an references, a pencil of curves can have at most 4 fibers which are unions of lines unless this is a pencil of lines. If $\mathcal{A}$ is an arrangement of lines such that one has essential surjection $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}) \to F_r$, in the sense that it takes each meridian to either to conjugate to one of the generators g_i of F_r or to a conjugate of $g_1, ..., g_r$ then the holomorphic map Φ described in the proof of the Theorem 1.1 has a union of reducible fibers coinciding with $\mathcal{A}$ (i.e. any line of $\mathcal{A}$ belongs to one of the fibers of Φ). If $r > 3$ then the pencil must be a pencil of lines and $\mathbb{P}^2 \setminus \mathcal{A}$ is fibered over $\mathbb{P}^1$ with r points removed and fiber isomorphic to $\mathbb{C}$. Hence $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}) = F_r$.

There are pencils of curves of arbitrary large degrees d containing arrangements of lines with 3 fibers which are union of lines (for example the curves C_d given by equation $\lambda(x^d - y^d) + \mu(y^d - z^d) = 0$ and hence finiteness of the number of pencils of curves for which a union of reducible fibers is union of lines and admits surjection $\pi_1(\mathbb{P}^2 \setminus C_d) \to F_r$ may take place only for $r \ge 3$. There is only one known pencil of curves with 4 fibers which are unions of lines (pencil of cubics with union of reducible fibers being 12 lines containing 9 inflection points of a smooth cubic). If no other pencils with 4 fibers being a unions of lines then $M(\Delta) = 3$.

Now consider the case $\Delta_k = \{[1], ..., [k]\} \in Pic(\mathbb{P}^2)$. Theorem (1.1) yields the following:

Corollary 4.1. *There exists a function $k \to M_k = M(\Delta_k) \in \mathbb{Z}_+$ such that a curve $\mathcal{C}$ having the degree of all components at most k and such that there is surjection $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to F_r, r > M_k$ is composed of a pencil of curves of degree k.*

Corollary 3.4 shows that the pencils of curves of degree d, have independent of d or k the number of fibers which are unions of irreducible curves of degree at most k provided $d >> k$. Proposition below makes it more explicit. It shows that the number $\rho_{d,k}$ of the reducible fibers with degrees of components at most k (*we call such pencils k-reducible*) in a pencils of curves of degree $d \ge 2k$ is at most 11. In other words:

$$M_k \ge \rho_{d,k}, \quad \rho_{d,k} \le 11 \ (d \ge 2k)$$

Moreover, the constant 10 in the theorem 1.1 can be decreased for $\mathbb{P}^2$ to 5.

Proposition 4.2. *Assuming $d = nk + d_0 \ge 2k$ (or equivalently $n \ge 2$), one has the following universal bounds*

$$\rho_{d,k} \le \begin{cases} 6 & \textit{if } k = 2, \\ 8 & \textit{if } k = 3, \\ 9 & \textit{if } k = 4, 5, \\ 10 & \textit{if } 6 \le k \le 11, \\ 11 & \textit{otherwise.} \end{cases} \tag{14}$$

Proof. For the pencil of curves of degree $d = nk + d_0$ one has at least n components of degree k and one component of degree d_0 and hence in notations used in inequality (12) one has $d^3 - 3d + \sum m_i e(d_i) \ge n(3k - k^2) + 3d_0 - d_0^2 = (d - d_0)(d + d_o - k)$. Therefore we obtain from (12):

$$\rho_{d,k} \le 2\frac{e(V) + 3D^2 + 2KD}{D^2 + KD + \sum m_i e(d_i)} \le \frac{6(d-1)^2}{(d - d_0)(d + d_0 - k)} \tag{15}$$

The condition that a constant α is an upper bound of the right hand term in (15) is equivalent to positivity of the function

$$\begin{aligned} h(k,n,\alpha) &= nk\big((n-1)k+2d_0\big)\alpha - 6(nk+d_0-1)^2 \\ &= (\alpha-6)k^2n^2 + \big(12+\alpha(2d_0-k)\big)kn - 6. \end{aligned} \tag{16}$$

For $\alpha = 12$ (and $n \geq 2$) one has:

$$h(k,n,\alpha) \geq 6 \times 4k^2 + (12-12k)k \times 2 - 6 > 0$$

for $k \geq 1$. The rest of of inequalities (14) follows by direct verification.

The second part of the Proposition follows from inequality (16) as well. □

Corollary 4.3. *For $r > 5$ there are only finitely curves, with components of degree at most k and not composed of a pencil, which may admit surjections $\pi_1(\mathbb{P}^2 \setminus \mathcal{D}) \to F_r$*

Proof. Indeed for $\alpha > 6$ and fixed k the function $h(k,n,\alpha)$ in (16) takes only finitely many negative values. □

Corollary 4.4. *The number of reducible curves $\rho_{d,d-1}(\mathbb{P}^2)$ in a primitive base-component-free pencil of degree d is at most $3(d-1)$ and there exist pencil of curves of degree d with $3(d-1)$ reducible fibers. In particular $M(k) \geq 3(k+1)$.*

Proof. It is an immediate consequence of the bound

$$\rho_{d,k} \leq \left[\frac{3(d-1)^2}{e_{rel,k}}\right] \leq \left[\frac{6(nk+d_0-1)^2}{nk((n-1)k+2d_0)}\right],$$

applied to the particular case $k = d-1$, that is, $n = 1$ and $d_0 = 1$. The existence is a consequence of the example of a pencil due to Ruppert. □

4.1.1. *Ruppert's Example.* For the sake of completeness we will briefly discuss the sharpness of the lineal bound given in Corollary 4.4. Ruppert described in [12] the existence of a pencil of curves of any degree d with exactly $3(d-1)$ reducible fibers. Consider the net $\mathcal{N}$ in $\mathbb{P}^2$ given by the following curves $\mathcal{C}_\lambda$ of degree d defined by the equation:

$$F_\lambda(x_0,x_1,x_2) = \lambda_0 x_0(x_1^{d-1} - x_2^{d-1}) + \lambda_1 x_1(x_2^{d-1} - x_0^{d-1}) + \lambda_2 x_2(x_0^{d-1} - x_1^{d-1})$$

for any $\lambda = [\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2$. One has the following properties:

1. The curves $\mathcal{C}_{[1:0:0]}$, $\mathcal{C}_{[0:1:0]}$, and $\mathcal{C}_{[0:0:1]}$ are products of d lines $x_i(x_j^{d-1} - x_k^{d-1})$, $\{i,j,k\} = \{0,1,2\}$.
2. The generic member of $\mathcal{N}$ is smooth. In order to check this note that

$$\mathcal{C}_{[1:0:0]} \cap \mathcal{C}_{[0:1:0]} \cap \mathcal{C}_{[0:0:1]} = \\ \{P_0 = [1:0:0], P_1 = [0:1:0], P_2 = [0:0:1], Q_{i,j} = [1:\zeta^i:\zeta^j]\},$$

   where $\zeta^{d-1} = 1$ are the $(d-1)^2 + 3 \leq d^2$ base points of this net. By direct calculation of the Jacobian of F_λ, one can check that the base points are only singular points of $\mathcal{C}_\lambda$ for a finite number of values of λ and hence by Bertini's Theorem, the generic member is smooth.

(3) The curve $\mathcal{C}_\lambda$ is reducible if λ satisfies $S(\lambda) = 0$ where $S(\lambda)$ is the degree $3(d-1)$ polynomial
$$S(\lambda) = (\lambda_0^{d-1} - \lambda_1^{d-1})(\lambda_1^{d-1} - \lambda_2^{d-1})(\lambda_2^{d-1} - \lambda_0^{d-1}).$$
In other words, the net $\mathcal{N}$ intersects the discriminant variety $\mathcal{D}$ in its locus of reducible curves and the intersection splits as a product of $3(d-1)$ lines.

(4) A pencil in $\mathcal{N}$ is given as $\mathcal{P} = \{\mathcal{C}_\lambda \in \mathcal{N} \mid L(\lambda) = 0\}$, where L is a linear form. If $L(\lambda)$ is in general position with respect to $S(\lambda)$, then $L(\lambda)$ defines a pencil with exactly $3(d-1)$ reducible fibers.

(5) Moreover, if $\mathcal{C}_\lambda$ is a generic point of $\mathcal{N} \cap \mathcal{D}$, then $\mathcal{C}_\lambda$ is the union of a line and a smooth curve of degree $(d-1)$.

4.2. **Examples of pencils with a maximal number of 2-reducible curves.** Let

$$\rho_k(\mathbb{P}^2) := \max\{\rho_{d,k}(\mathbb{P}^2) \mid d \geq 2k\}. \tag{17}$$

It follows from [10],[14] that $\rho_1(\mathbb{P}^2) = 4$ and the arrangement of 12 lines containing 9 inflection points of a smooth cubic provides an example of a pencil with 4 fibers which are unions of lines. Our purpose in this section will be to study $\rho_2(\mathbb{P}^2)$.

4.2.1. *The bound $\rho_{d,2}$.* By Corollary 4.4 we know that $\rho_{3,2}(\mathbb{P}^2) = 6$ and 15 shows that $\rho_{d,2}(\mathbb{P}^2) \leq 6$ for all $d \geq 4$. It is the purpose of this section to make this into an equality by constructing a pencil of quartics with six 2-reducible curves.

Consider a pencil of conics Λ in general position and three lines L_1, L_2, L_3 such that there exist three conics $C_1, C_2, C_3 \in \Lambda$ such that L_i is tangent to C_j and C_k with $\{i,j,k\} = \{1,2,3\}$. This can be achieved for instance with the pencil $\Lambda = \{\alpha(x^2 - z^2) + \beta(y^2 - z^2)\}$, the lines
$$L_1 = \sqrt{2}x + i\sqrt{2}y + \sqrt{3}z, \quad L_2 = 2x + iy + \sqrt{3}z, \quad L_3 = \sqrt{2}x + \sqrt{2}y - 3z$$
and the conics
$$C_1 = x^2 + 2y^2 - 3z^2, \quad C_2 = 2x^2 + y^2 - 3z^2, \quad C_3 = 2x^2 - y^2 - z^2.$$
Let κ denote the Kummer cover of order two associated with the abelian $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover ramified along L_1, L_2, and L_3 (i.e. associated with surjection $\pi_1(\mathbb{P}^2 \setminus \bigcup_1^3 L_i) \to \mathbb{Z}_2^3/\mathbb{Z}_2$ sending the meridian of L_i to i-th component of $\mathbb{Z}_2^3$). Note that $\Lambda' = \kappa^*(\Lambda)$ becomes a pencil of quartics intersecting transversally at the 16 points in the preimage of the base points of Λ and also that $\kappa^*(C_i)$, $i = 1,2,3$ is a union of two conics. Finally, note that Λ contains 3 singular fibers C_i', $i = 1,2,3$ which are products of two lines. Hence $\kappa^*(C_i')$ is also a product of two conics intersecting transversally.

Additivity of euler characteristic or the main result of [7] allows to relate the Euler characteristic of the surface V, the Euler characteristic of the generic fiber $e_{\hat{\varphi}}(t_0)$ (a Riemann surface of genus $\binom{4-1}{2} = 3$), and the relative Euler characteristic of the singular fibers e_{rel} as follows:
$$e(X) = 3 + |B| = 3 + 16 = e(\mathbb{P}^1)e_{\hat{\varphi}}(t_0) + 6e_{rel} + n = 2 \cdot (-4) + 6 \cdot 4 + n,$$
where n is the relative Euler characteristic of the remaining singular fibers. Hence $n = 3$, which amounts for the number of additional nodal quartics in the pencil Λ'.

Proposition 4.5. *The pencil Λ' above is a primitive base-component-free pencil of quartics with six 2-reducible members. Therefore $M_2 = \rho_2(\mathbb{P}^2) = 6$.*

5. Completely reducible fibers of pencils on surfaces in $\mathbb{P}^3$

The purpose of the remaining section is to exhibit examples of pencils on surfaces with a large number of completely reducible divisors as well as bounds which follow from the calculations in section 3

We shall start with the case $V = \mathbb{P}^1 \times \mathbb{P}^1, \Delta = \{(1,0),(0,1),(1,1)\}$. The left hand side of inequality (12) gives the following upper bound for the number of reducible fibers in pencils on $\mathbb{P}^1 \times \mathbb{P}^1$ of curves of bidegree $(m,n), m \geq n, (m,n) \neq 1$ the following:

$$3 \times \frac{2+3mn-(m+n)}{mn-n}$$

which does not exceed 12 (and in fact does not exceed 10 with possible exceptions for bidegree $(m,n), 6 \geq m \geq n$.

We will show that $M(\mathbb{P}^1 \times \mathbb{P}^1, \Delta) \geq 4$, $M(S, \Delta_1) \geq 5$ for cubic surfaces and that $M(X, \Delta)$ can be arbitrarily large for general surfaces in $\mathbb{P}^3$ for appropriate saturated sets Δ or Δ_1 on respective surfaces.

5.1. **Generalized Hesse arrangements on $\mathbb{P}^1 \times \mathbb{P}^1$.** The purpose of this section is to exhibit a example of a pencil of curves on $\mathbb{P}^1 \times \mathbb{P}^1$ with 4 completely reducible fibers, showing that Hesse is not the only such pencil on a rational surface.

5.1.1. *A Special Configuration of Points.* Consider 9 points on a smooth cubic $\mathcal{C} \subset \mathbb{P}^2$ satisfying Pascal's Theorem as in Figure 1:

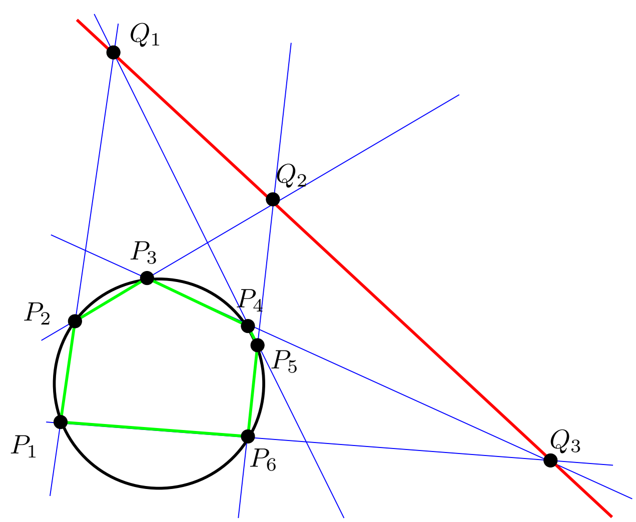

Figure 1. Pascal Point Configuration

Note that such a configuration of points would have to satisfy the following relations in the Picard group of the cubic:

$$\begin{aligned} P_1+P_2+Q_1&=0, & P_4+P_5+Q_1&=0,\\ P_2+P_3+Q_2&=0, & P_5+P_6+Q_2&=0,\\ P_3+P_4+Q_3&=0, & P_1+P_6+Q_3&=0. \end{aligned} \tag{18}$$

In other words $P_i + P_{i+1} + Q_j = 0$, $i \in \mathbb{Z}_9$ and $\pi(i) = j$, where $\pi : \mathbb{Z}_9 \to \mathbb{Z}_3$. By Pascal's Theorem

$$\sum_i P_i = 0, \quad \sum_i Q_i = 0. \tag{19}$$

We also ask for three additional relations involving the diagonals:

$$\begin{array}{c} P_1 + P_4 + Q_2 = 0, \\ P_2 + P_5 + Q_3 = 0, \\ P_3 + P_6 + Q_1 = 0. \end{array} \tag{20}$$

Definition 5.1. Any configuration of 9 points on a smooth cubic satisfying (18), (19), and (20) will be called a *special Pascal configuration* of points.

Lemma 5.2. *For any special Pascal configuration of points, the points Q_i are necessarily inflection points.*

Proof. By symmetry, it is enough to show $3Q_1 = 0$. Note that $0 = (P_1+P_4+Q_2)+(P_3 + P_6 + Q_1)$ (from (20)) and $0 = (P_3 + P_4 + Q_3) + (P_1 + P_6 + Q_3)$ (from (18)). Subtracting both relations one obtains

$$0 = (P_1+P_4+Q_2)+(P_3+P_6+Q_1)-(P_3+P_4+Q_3)-(P_1+P_6+Q_3) = Q_2+Q_1-2Q_3.$$

Finally subtracting the previous equality to $0 = Q_1 + Q_2 + Q_3$ (from (19)) one obtains the desired relation $3Q_1 = 0$. ☐

Proposition 5.3. *For any smooth cubic $\mathcal{C}$ there is a family of special Pascal configurations of points parametrized by $\mathcal{C}$.*

Proof. In order to construct a list of such 9 points one needs to choose three aligned inflection points, say Q_1, Q_2, Q_3, and an extra point on the cubic, say P_1. The remaining P_i are obtained from P_1 and Q_j. ☐

Generically, the three lines defined by (20) intersect in three double points, the conic defined in (19) is smooth, and intersects the line also defined in (19) transversally.

Consider the pencil of cubics generated by $\mathcal{C}_1 := \mathcal{L}_{12} \cup \mathcal{L}_{34} \cup \mathcal{L}_{56}$ and $\mathcal{C}_2 := \mathcal{L}_{23} \cup \mathcal{L}_{45} \cup \mathcal{L}_{16}$, where $\mathcal{L}_{ij}$ is the line passing through P_i and P_j. Note that the original smooth cubic $\mathcal{C}$ belongs to such a pencil and so does $\mathcal{C}_3 := \mathcal{L}_{14} \cup \mathcal{L}_{25} \cup \mathcal{L}_{36}$ and the union of the conic $\mathcal{Q}$ passing through $P_1, ..., P_6$ and the line $\mathcal{L}$ passing through Q_1, Q_2, Q_3. In other words, one can find equations such that $C_3 = C_1 - C_2$ and $QL = C_1 + C_2$.

Proposition 5.4. *After blowing up the base points, the pencil of cubics described above induces an elliptic surface which is generically of type $I_1 + I_2 + 3I_3$.*

Proof. The existence of I_2 and $3I_3$ is given by hypothesis, then by a standard Euler characteristic computation, there should be an additional fiber of type I_1. ☐

Example 5.5. Equations for Figure 2 can be given as:

$$\begin{array}{l} C_1 = (2y + (-2\sqrt{3} + 2)z)(\sqrt{3}x + y + \sqrt{3}z)(-\sqrt{3}x + y + \sqrt{3}z) \\ C_2 = (2y + 2z)(\sqrt{3}x + y - \sqrt{3}z)(-\sqrt{3}x + y - \sqrt{3}z) \\ C_3 = (\sqrt{3}x - y - (2 - \sqrt{3})z)(\sqrt{3}x + y + (2 - \sqrt{3})z)y \\ Q = 3x^2 + 3y^2 - 3z^2 + 2(2 - \sqrt{3})yz \\ L = z \end{array}$$

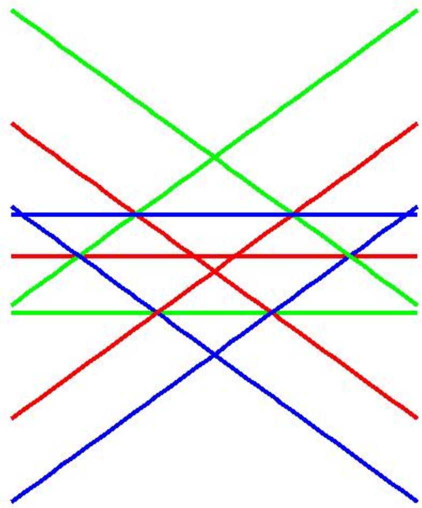

FIGURE 2. Special Pascal Point Configuration of type $I_1 + I_2 + 3I_3$

Proposition 5.6. *There are only two possible degenerations of the previous generic pencil:*

(1) A surface of type $I_2 + 2I_3 + IV$
(2) A surface of type $4I_3$.

Proof. By hypothesis, we know that three singular fibers are products of lines, hence of type I_3 or IV, and a fourth singular fiber contains a line, hence it is of type I_2, I_3, III, or IV. Therefore, numerically, there can only be three possibilities: $4I_3$, $I_2 + 2I_3 + IV$, and $3I_3 + III$. A surface of type $4I_3$ corresponds with the Hessian pencil, which comes from the choice of P_1 as an inflection point. A surface $I_2 + 2I_3 + IV$ appears when the three lines in $\mathcal{C}_3$ are concurrent. Finally, the surface $3I_3 + III$ does not exist according to Miranda's list of rational elliptic surfaces (cf.[11, p.197, item 92.]). $\square$

Example 5.7. The special Pascal configuration of type I_2+2I_3+IV can be realized as the set of zeroes of:

$$
\begin{aligned}
&C_1 = (2y - \sqrt{3}z)(\sqrt{3}x + y + \sqrt{3}z)(-\sqrt{3}x + y + \sqrt{3}z)\\
&C_2 = (2y + \sqrt{3}z)(\sqrt{3}x + y - \sqrt{3}z)(-\sqrt{3}x + y - \sqrt{3}z)\\
&C_3 = (3x - \sqrt{3}y)(3x + \sqrt{3}y)y\\
&Q = x^2 + y^2 - z^2\\
&L = z
\end{aligned}
$$

5.1.2. *Generalized Hesse Arrangements on* $\mathbb{P}^1 \times \mathbb{P}^1$. Consider a double cover δ of $\mathbb{P}^2$ ramified along a smooth conic which is bitangent to $\mathcal{Q}$. The rational surface which realizes this covering is a ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$. Any irreducible component in the preimage of a line in $\mathbb{P}^2$ by δ has bidegree $(1,1)$, $(1,0)$, or $(0,1)$ according to the relative position of the ramification locus and the line.

Definition 5.8. We say a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is *completely reducible* if it is a union of irreducible components all being in the set Δ consisting of 3 classes: $(1,1)$, $(1,0)$, or $(0,1)$.

Theorem 5.1. *There exist pencils on $\mathbb{P}^1 \times \mathbb{P}^1$ with four completely reducible fibers.*

Proof. Consider a special Pascal configuration of type $I_1 + I_2 + 3I_3$ or $I_2 + 2I_3 + IV$ and a double cover δ of $\mathbb{P}^2$ ramified along a smooth conic, which is bitangent to $\mathcal{Q}$. Then the pencil of cubics described above induces a pencil of curves of genus four on $\mathbb{P}^1 \times \mathbb{P}^1$. The preimage of the I_2-fiber becomes two $(1,1)$-curves as preimage of the conic $\mathcal{Q}$ and one more $(1,1)$-curve as a preimage of $\mathcal{L}$. The preimage of the I_3-fibers is a union of three $(1,1)$-curves. $\square$

Example 5.9. Using the special Pascal configuration of type I_2+2I_3+IV provided in Example 5.7 and the covering $\delta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ given by: $\delta([u,v],[s,t]) = [2(ut+vs), us - vt, us + vt]$, which ramifies along $\{x^2 + 4y^2 = 4z^2\}$ (a conic which is bitangent to $\mathcal{Q} = \{x^2 + y^2 = z^2\}$). Note that:

$$\begin{array}{rcl}
\delta^*(C_1) & = & \left(6ut + 6vs - (3+\sqrt{3})us - (3-\sqrt{3})vt\right) \\
 & & \left(6ut + 6vs + (3+\sqrt{3})us + (3-\sqrt{3})vt\right)\left(vt - (7-4\sqrt{3})us\right) \\
\delta^*(C_2) & = & \left(6ut + 6vs - (3-\sqrt{3})us - (3+\sqrt{3})vt\right) \\
 & & \left(6ut + 6vs + (3-\sqrt{3})us + (3+\sqrt{3})vt\right)\left(vt - (7+4\sqrt{3})us\right) \\
\delta^*(C_3) & = & \left(6ut + 6vs - \sqrt{3}us + \sqrt{3}vt\right)\left(6ut + 6vs + \sqrt{3}us - \sqrt{3}vt\right)(us - vt) \\
\delta^*(Q) & = & \left(2ut - (1-\sqrt{-3})vs\right)\left(2ut - (1+\sqrt{-3})vs\right) \\
\delta^*(L) & = & us + vt
\end{array}$$

Corollary 5.10. *In notations of the Corollary 3.4 one has the bound $K(D = (d,1), \Delta, (\mathbb{P}^1 \times \mathbb{P}^1) \geq K(D = (3,3), \Delta, \mathbb{P}^1 \times \mathbb{P}^1) \geq 4$.*

Moreover, primitive base-component-free pencils $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ of bidegree $(3,3)$ and pencils confirming the equality $K((3,3), \Delta, \mathbb{P}^1 \times \mathbb{P}^1) = 4$ are not unique.

5.2. **Completely reducible fibers on a cubic surface.** Let S be a smooth cubic surface in $\mathbb{P}^3$. The subsets $\Delta_1 \subset \mathrm{NS}(S)$ consisting of the classes of 27 lines (generating the closure of the effective cone (cf. [4] p.485, section 9.1) is saturated.

Proposition 5.11. *Let $S \subset \mathbb{P}^3$ be a smooth cubic as above, then $M(\Delta_1) \geq 5$*

Proof. The pencil of planes in $\mathbb{P}^3$ containing a fixed line induces a base point free pencil of residual for this line plane quadrics with 5 reduced fibers each consisting of 2 lines (cf. [4], section 9.1). Hence $M(\Delta_1) \geq 5$.

On the other hand it follows from Proposition (3.3) that $M(\Delta) < 6$ (since $d_i \in \Delta_1$ a lines, one has $\sum m_i e(d_i) = 2HD$).

$\square$

5.3. **Completely reducible fibers of pencils on surfaces of higher degree.**

Proposition 5.12. *For any positive integer d there is a surface $S_d \subset \mathbb{P}^3$ of degree d and a pencil on it containing at least d completely reducible curves.*

Proof. The following is a well-known fact about how to construct surfaces containing a large number of lines. Consider $f(x,y)$ and $g(z,t)$ two homogeneous polynomials of degree d with no multiple roots, then the surface

$$S_{f,g} = \{[x : y : z : t] \in \mathbb{P}^3 \mid f(x,y) = g(z,t)\}$$

contains at least d^2 lines, namely all the lines $L_{i,j},\ i,j = 1,...,d$ joining a point $P_i = [x_i : y_i : 0 : 0]$ and a point $Q_j = [0 : 0 : z_j : t_j]$ where $f(x_i : y_i) = g(z_j, t_j) = 0$.

The pencil of hyperplanes containing the line $L = \{x = y = 0\}$ induces a pencil of curves on $S_{f,g}$. Given any point P_i, the hyperplane $H_i = \{y_i x = x_i y\}$ containing P_i has to contain the lines $L_{i,1}, ..., L_{i,d}$. Therefore this pencil contains at least d completely reducible fibers. □

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