

COMPLEMENTS TO AMPLE DIVISORS AND SINGULARITIES.

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ABSTRACT. The paper reviews recent developments in the study of Alexander invariants of quasi-projective manifolds using methods of singularity theory. Several results in topology of the complements to singular plane curves and hypersurfaces in projective space extended to the case of curves on simply connected smooth projective surfaces.

1. INTRODUCTION

These notes review interactions between singularity theory and the study of fundamental groups and more generally the homotopy type of the complements to divisors on smooth projective varieties. The main question considered here is how the local topology of singularities as well as their global geometry affect the topology of the complement. Several surveys updating the state of the subject at respective points in time were written over the years (cf. [54]) but most often focusing on specific situations: complements to plane curves, arrangements of lines or hyperplanes etc. reflecting that earlier studies of the complements were mainly focused on the case of plane curves. Below we consider the complements to divisors D on smooth projective surfaces X and their fundamental groups, sometimes indicating how a generalization to the case of homotopy types of the complements in manifolds of dimension greater than two looks like, but mostly referring to other publications for additional details on homotopy invariants beyond fundamental groups. An earlier appearances of the studies of the complements in the context of general pairs (X, D) and their fundamental groups can be traced to the 80s. Some results on the topology of the complements in such set up did appear in [104], [92] [181]. A much earlier, beautiful results, especially those showing the role of the abelian varieties in the subject were obtained by Italian school (cf. [43] for a modern exposition).

The invariants of the fundamental groups with known strong relation to singularity theory are the Alexander type invariants, introduced in [132] and called their *characteristic varieties*. The connections besides singularity theory run through the knot theory, the Hodge theory of quasi-projective varieties, study of elliptic fibrations, symplectic geometry to mention a few.

There are three major approaches to the study of characteristic varieties of fundamental groups. One is topological, allowing their calculation in terms of a presentation of the fundamental group via generators and relations, obtained typically using braid monodromy. The other one is geometric, going through a study of homology of the abelian covers and eventually leading to determination of characteristic varieties in terms of local type of singularities and dimensions of the linear systems determined by the divisor and the local type of singularities. Finally, one can calculate the characteristic varieties using Deligne extensions of bundles endowed with a flat connection. Whole theory is a combination of methods and ideas from all these areas.

Many results related to the discussion of this paper are presented in volume [45] where for the most part the case of plane curves was considered. The exposition which follows,

describes a generalization to the context of the complements to divisors on smooth simply connected surfaces. A very fruitful approach to a study of the complements to divisors is via resolutions of singularities, reducing the case when a divisor has arbitrary singularities to the case of divisors with normal crossings. In this way one replaces the complexity of the divisor by complexity of compactification and the complexity of individual components. The goal here is rather to study how complexity of singularities affects fundamental groups of the complements. Trying to make this paper more independent, we included some basic material which is scattered through existing literature and for which we could not find good references (e.g. theory of branched covers, the relation between quasi-adjunction and multiplier ideals etc.). We also survey several results on the fundamental groups which appear in the last 10-20 years providing an overview of the new results in this area. Several results here are new or did not appear in the literature: they include the divisibility of Alexander polynomials of complements on simply-connected surfaces, extending the case of plane curves (cf. Theorem 3.3), calculation of characteristic varieties in terms of classes of irreducible components in Picard group and invariants of quasi-adjunction of singularities (cf. Theorem 4.18) and others.

The content of the paper is as follows. In section 2 we discuss an analog of classical method of Van Kampen (cf. [214]) to obtain presentations of the fundamental groups of the complements to divisors on smooth surfaces in terms of mapping class group valued monodromy associated to a divisor. We also review conditions on a divisor which allow to deduce that the fundamental group of the complement is abelian. In section 3 we firstly extend the theory of Alexander invariants of plane algebraic curves (cf. [129], [143]) to the complements of curves on smooth projective surfaces (for an earlier work cf. [62]). In particular we obtain a result unifying the divisibility theorems in the case of plane curves, showing the divisibility of global Alexander polynomials respectively in terms of local Alexander polynomials and the Alexander polynomials at infinity. Many results depend on some sort of positivity assumptions of the components which suggest an interesting problem understanding the fundamental groups and its invariants when positivity is lacking. Theory of Alexander invariants is closely related to the study of homology of abelian covers. In section 3.3 we present basic definitions and then describe approaches enumerating covers either in terms of subgroups of fundamental groups or in terms of eigensheaves of direct images of the structure sheaf. The most interesting results about Alexander invariants are obtained through interaction of topological and algebro-geometric view points. The last part of this section deals with multivariable Alexander invariants from topological view point. We included a brief discussion of multivariable Alexander invariants for quasi-projective invariants in higher dimensions including recent results on propagation (cf. [159] for another recent overview of this and related aspects). Section 4 discusses a calculation of characteristic varieties in terms of superabundances of the linear systems associated with a divisor on a smooth projective surface using ideals of quasi-adjunction of singularities of the divisor. The ideals of quasi-adjunction, defined in terms of branched covers of the germs of divisors, can be viewed as the multiplier ideals which received much attention over last 20-30 years. The role of these ideals in the study of the fundamental groups is to specify the linear system which dimensions determine the characteristic varieties and hence allow to give their geometric description. The section contains also another description of characteristic varieties using the Deligne's extension and ends with a brief review of the relations between the characteristic varieties and other invariants studied in Singularity theory, including Bernstein-Sato polynomials and Hodge decomposition of characteristic varieties. Section 5 mostly is based on recent preprint [47] which describes the results

on distribution of Alexander type invariants when complexity (in appropriate sense) of the divisor increases. We describe the finiteness results when one searches for fundamental groups of the complements with large free quotients. The last section discusses several recent calculations of the fundamental groups of the complements. In the 80s scarcity of examples of quasi-projective groups and fundamental groups of the complements was viewed as impediment to development of general theory. In recent years this problem was amply addressed and we present some of the most consequential results.

In these notes, we tried at least to direct a reader to the most important recent developments but nevertheless several important topics were not covered here. Those missing include the relation between the Alexander invariants and the Mordell-Weil groups of isotrivial fibrations (cf. [146]), Chern numbers of algebraic surfaces and arrangements of curves (cf. [183]), free subgroups of the fundamental groups (cf. [72]), virtual nilpotence of virtually solvable quasi-projective groups (cf. [10]), singularities of varieties of representations of the fundamental groups (cf. [120]), the complements to symplectic curves (cf. [24],[23], [97]) among others.

The theory described below appears to be far from completion. Many interesting problems remain very much open (some are mentioned throughout the text) and a thorough understanding of the fundamental groups or homotopy type of quasi-projective varieties is still out of reach.

Finally, I want to thank Alex Degtyarev as well as the referee of this paper for reading the final version of the text and very helpful comments.

2. BRAID MONODROMY, PRESENTATIONS OF FUNDAMENTAL GROUPS AND SUFFICIENT CONDITIONS FOR COMMUTATIVITY

2.1. Braid monodromy presentation of fundamental groups. In the case of plane curves, Zariski-van Kampen method (cf. [219], [214]) is the oldest tool for finding presentations of the fundamental groups of the complements. A convenient way to state the theorem is in terms of braid monodromy. Its systematic use was initiated in [164] and in such form admits a natural generalization to the complements to divisors on arbitrary algebraic surface which we describe in this section. Braid monodromy became an important tool in symplectic geometry (cf. [24]). A good exposition of braid monodromy of curves on ruled surfaces can be found in [57], Section 5.1.

Let X be a smooth projective surface and let D be a reduced divisor on X . To describe a presentation of $\pi_1(X \setminus D, p)$, $p \in X \setminus D$ we make several choices, on which the presentation will depend.

- Select a pencil ¹ of hyperplane sections of $X \subset \mathbb{P}^N$, generic for the pair (X, D) . Its base locus is a generic codimension 2 subspace $P \subset \mathbb{P}^N$ and we can consider the projection with the center at P , i.e. the map $\mathbb{P}^N \setminus P \rightarrow \mathbb{P}^1$ sending to $p \in \mathbb{P}^N \setminus P$ to the hyperplane containing P and p . Its restriction to X produces a regular map $\pi : X \setminus X \cap P \rightarrow \mathbb{P}^1$. Denoting by \tilde{X} the blow up of the surface X at the base locus $P \cap X$ of the pencil, we obtain a regular map $\tilde{\pi} : \tilde{X} \rightarrow \mathbb{P}^1$. Assuming that P was selected so that $D \cap P = \emptyset$ and still denoting by D its preimage in \tilde{X} we obtain the map $\tilde{\pi} : \tilde{X} \setminus D \rightarrow \mathbb{P}^1$. Seifert-van Kampen theorem implies that $\pi_1(X \setminus D) = \pi_1(\tilde{X} \setminus D)$ and so we can do calculations on \tilde{X} .

¹i.e. a family of divisors parametrized by \mathbb{P}^1

- Let $B = \{b_1, \dots, b_k\} \subset \mathbb{P}^1$ be the set consisting of the critical values of $\tilde{\pi}$ ² and the images of the fibers of π , either containing a singular point of D or containing a point of D which is critical point of restriction $\pi|_D$.
- Let $\Omega \subset \mathbb{P}^1$ be a subset, containing B and isotopic to a disk in \mathbb{P}^1 , and let $b_0 \in \partial\Omega$ be a point on the boundary of Ω .
- Let $\partial B_\epsilon(p)$ be the boundary of a small ball $B_\epsilon(p)$ in X ³ centered at a point $p \in X \cap P$ or, equivalently, the boundary of a small regular neighborhood of the exceptional curve E_p in \tilde{X} contracted to $p \in X$. The map $\tilde{\pi}$ restricted to $\partial B_\epsilon(p)$ is the Hopf fibration $\partial B_\epsilon = S^3 \rightarrow \mathbb{P}^1 = S^2$. Using its trivialization over Ω , we define a section over $\Omega \setminus B$: $s_p : \Omega \setminus B \rightarrow \tilde{\pi}^{-1}(\Omega \setminus B)$.
- Let $F_{b_i}, i = 0, 1, \dots, k$ be the fiber of $\tilde{\pi}$ over b_i . The curves $F_{b_i}, i = 1, \dots, k$ either have singularities at critical points of π or contain singular points of D or have non-transversal intersections with D , while F_{b_0} is smooth closed Riemann surface having genus $g = \frac{F_{b_0}(F_{b_0}+K)}{2} + 1$ where K is the canonical divisor of X .
- For any $p \in P \cap X$, let $\bar{F}_{b_0}^\circ$ be the surface with one connected boundary component obtained by removing from F_{b_0} its intersection with the above regular neighborhood of E_p . Denote by $\mathcal{M}(\bar{F}_g^\circ, [d]) = \text{Diff}^+(F_{b_0} \setminus (F_{b_0} \cap B_\epsilon(p)), [F_{b_0} \cap D])$ the mapping class group of the Riemann surface with boundary with d marked points (cf. [88]) i.e. the group of isotopy classes of orientation preserving diffeomorphisms taking the subset $[d]$ of cardinality d into itself and constant on the boundary of the Riemann surface.

Definition 2.1. *The braid monodromy of the pair (X, D) (for selected pencil on X) is the monodromy map*

$$(1) \quad \mu : \pi_1(\Omega \setminus B, b_0) \rightarrow \mathcal{M}(\bar{F}_g^\circ, [d])$$

obtained by

a) selecting a loop (denoted in b) and c) below as γ) for each homotopy class in $\pi_1(\Omega \setminus B, b_0)$,

b) a trivialization of the locally trivial fibration $\pi^{-1}(\gamma) \rightarrow \gamma$ i.e. a differentiable map $\pi^{-1}(b_0) \times [0, 1] \rightarrow \pi^{-1}(\gamma)$ inducing a diffeomorphism of the fiber over $t \in [0, 1]$ onto the fiber over the image of t in parametrization $[0, 1] \rightarrow \gamma$ of the loop.

c) assigning to γ the diffeomorphism of $\bar{F}_g^\circ = \pi^{-1}(b_0)$ sending a point $q \in \pi^{-1}(b_0)$ to the point $q' \in \pi^{-1}(b_0)$ to which the trivialization mentioned in b) takes the end point $q \times 1$ of the segment $q \times [0, 1] \subset \pi^{-1}(b_0) \times [0, 1]$ in $\pi^{-1}(\gamma)$.

One verifies that, though the diffeomorphism in c) depends on both, the loop γ in a) and the trivialization in b), its class in the mapping class group does not depend on these choices.

Recall that the mapping class group $\mathcal{M}(\bar{F}_g^\circ, [d])$ acts on $\pi_1(\bar{F}_g^\circ \setminus [d], q)$ (here q is the base point which we assume is on the boundary of \bar{F}_g°). For example in the case $g = 0$ the group $\mathcal{M}(\bar{F}_0^\circ, [d])$ is the Artin's braid group on d -strings i.e. the group of orientation

²those are absent in the classical case on pencils of lines $X = \mathbb{P}^2$ of Zariski-van Kampen theorem.

³we assume that there are no vanishing cycles corresponding to critical points of π and no points of D inside this ball

preserving diffeomorphisms of a 2-disk Δ , constant on the boundary and taking into itself a given subset of Δ of cardinality d . It has a well known presentation:

$$(2) \quad \langle \sigma_1, \dots, \sigma_{d-1}, \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \sigma_i \sigma_j = \sigma_j \sigma_i, 1 < \mid i - j \mid > \rangle$$

Note that the center of (2) is generated by $[\sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)\dots(\sigma_{d-1}\dots\sigma_1)]^2$ (cf. [98] Sect. 4.3). The action on the free group $\pi_1(\Delta \setminus [d], p)$ is given by

$$(3) \quad \sigma_i(t_i) = t_{i+1}, \sigma_i(t_{i+1}) = t_{i+1}^{-1} t_i t_{i+1}, \sigma_i(t_j) = t_j, \quad j \neq i, i+1^4$$

(which is the canonical action of the mapping class group on the fundamental group for appropriate choice of generators t_i of the latter). This way in the case of $X = \mathbb{P}^2$ one obtains the monodromy with the values in the Artin's braid group, the case described in [164]. The homomorphism (1) in [164] is described in a more combinatorial form, as a product of collection of braids. The ordered collection of factors in this product is the collection of braids corresponding to so called ‘‘good ordered system of generators’’ of the free group $\pi_1(\Omega \setminus B, b_0)$ (cf. [164] for details).

To define the final ingredient for our presentation of $\pi_1(X \setminus D)$, we consider the gluing map of the boundaries of $\pi^{-1}(\Omega)$ and $\pi^{-1}(\mathbb{P}^1 \setminus \Omega)$ which can be viewed as a map $\Phi : \pi^{-1}(\partial\Omega) \rightarrow \pi^{-1}(\partial(\mathbb{P}^1 \setminus \Omega))$, both spaces being locally trivial fibrations over $\partial\Omega = S^1$, preserving the set $D \cap \pi^{-1}(\partial\Omega)$ and commuting with projection onto S^1 . Such map takes the loop $s_p(\partial\Omega)$ (as above, s_p is a section of restriction of the Hopf bundle over \mathbb{P}^1) to the loop $S^1 \rightarrow S^1 \times (F_{b_0} \setminus [d]) \rightarrow F_{b_0} \setminus [d]$ and hence determines a conjugacy class in the fundamental group of its target. We shall denote this class $\rho_{X,D}$. In the case of plane curve of degree d transversal to the line at infinity and pencil of lines, complement to the base point is the total space of line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$, the gluing map Φ induced by positive generator of $\pi_1(GL_2(\mathbb{C}))$ which shows that $\rho_{X,D} = \gamma_1 \cdot \dots \cdot \gamma_d$ is the product of standard ordered system of generators of fundamental group of the complement in generic fiber to the intersection of this fiber with the curve.

The mapping class group valued monodromy determines the fundamental group as follows:

Theorem 2.2. *One has the isomorphism:*

$$(4) \quad \pi_1(X \setminus D) = \pi_1(C_0 \setminus C_0 \cap D) / \{(\mu(\gamma_j)\alpha_i)\alpha_i^{-1}, \rho_{X,D}\}$$

Theorem 2.2 reduces a calculation of the fundamental group to the calculation of the braid monodromy and the element $\rho_{X,D}$. The literature on calculations of braid monodromies of curves is very large and is very hard to review. We refer to [164] and [166] where explicit expressions were obtained for the braid monodromy of smooth plane curves, branching curves of generic projections of smooth surfaces in \mathbb{P}^3 , generic arrangements of lines and branching curves of generic projections of various embeddings of quadric. The survey [209] and the book [57] also are good references for more recent developments. We refer to the former for examples of calculations of fundamental groups using van Kampen method and references to other works on calculation of braid monodromy and the latter for computer use in calculations of braid monodromy and the fundamental groups.

Besides the fundamental group, the braid monodromy defines the homotopy type of the complement (cf. [135] for precise statement). It is however an open problem, if the homotopy type of the complement $X \setminus D$ is determined by the fundamental group and the topological Euler characteristic of the complement (cf. [135] for a discussion of this problem). Considering dependence of the braid monodromy on the curve and numerous

⁴this implies that $\sigma_i^{-1}(t_i) = t_i t_{i+1} t_i^{-1}$, $\sigma_i^{-1}(t_{i+1}) = t_i$.

choices made in its construction, in [16] the authors found conditions implying that the homeomorphism type of the triples (\mathbb{P}^2, L, C) , where C is a plane curve and L is one of the lines of the pencil used to construct the braid monodromy (the line at infinity), determines the braid monodromy. Braid monodromy is an essential tool in showing the existence of symplectic singular curves not isotopic to algebraic ones (cf. [24] and [165]).

2.2. Abelian fundamental groups. The question “whether the fundamental group of the complement to a nodal curve is abelian” was known as “Zariski problem” since it was realized that Severi’s proof of irreducibility of the family of plane curves with fixed degree and the number of nodes is incomplete (cf. [194]). Zariski derived commutativity of the fundamental groups of the complements to nodal curves using that irreducibility implies existence of degeneration of a nodal curve to a union of lines without points of multiplicity greater than two. Once one has degeneration, the relation between fundamental groups of the complements to a curve and to its degenerations (i.e. that given a degeneration $C_0 = \lim C_t$ one has surjection $\pi_1(\mathbb{P}^2 \setminus C_0) \rightarrow \pi_1(\mathbb{P}^2 \setminus C_t)$ which is a consequence of definition of the braid monodromy and presentation (4)) implies the commutativity. Severi statements (and with it the Zariski proof [221]) was eventually validated (cf. [111]). A proof of commutativity, based on connectedness theorem, was found prior to this by W.Fulton ([93]) for algebraic fundamental group and by P.Deligne [66] in topological case.

The central result on commutativity of fundamental groups of complements to divisors is due to Nori with the key step being a generalization of Lefschetz hyperplane section theorem (cf. [174]). See [94] Chapter 5 and [127] Chapter 3.

Theorem 2.3. (*Nori’s weak Lefschetz theorem*). *Let U be a connected complex manifold of dimension greater than one and let $i : H \rightarrow U$ be the embedding of a connected compact complex-analytic subspace defined by a locally principal sheaf of ideals. Let $q : U \rightarrow X$ be a locally invertible map to a smooth projective variety, $h = q \circ i$, and $R \subset X$ be a Zariski closed subset. Assume that $\mathcal{O}_U(H)|_H$ is ample. Then*

A: $G = \text{Im} \pi_1(U \setminus q^{-1}(R)) \rightarrow \pi_1(X \setminus R)$ is a subgroup of a finite index.

B: If $q(H) \cap R = \emptyset$ then $\pi_1(H) \rightarrow \pi_1(X \setminus R)$ is a subgroup of a finite index.

C: If $\dim X = \dim U = 2$ then index of subgroup G of $\pi_1(X \setminus R)$ is at most $\frac{(\text{Div}(h))^2}{H^2}$ where the divisor in numerator is the first Chern class of $h_\mathcal{O}_H$, the Cartier divisor on X corresponding to the divisor H on U (cf. [174], 3.16 for details).*

If q is embedding and H is reduced, this becomes Zariski-Lefschetz hyperplane section theorem (cf. [220], [114]). One has to note a subtlety in the finiteness of index in A and the index bound in C (cf. [110]). Typically $\pi_1(q(H))$ is much bigger than $\pi_1(H)$: for example if $H \rightarrow q(H)$ is normalization and H is rational and $q(H)$ nodal then $\pi_1(q(H))$ is free group with the rank equal to the number of nodes. Nevertheless the following is still open:

Problem 2.4. (*M.Nori*) *Let D be an effective divisor of a surface X and $D^2 > 0$. Let N be a normal subgroup of $\pi_1(X)$ generated by the images of the fundamental groups of the normalizations of all irreducible components of D . Is the index of N in $\pi_1(X)$ finite? In particular, can a surface with infinite fundamental group contain a rational curve with positive self-intersection?*

One of the main consequence of theorem 2.3 is the following:

Corollary 2.5. *Let D and E be curves on smooth projective surface intersecting transversally and such that D has nodes as the only singularities. Assume that for each irreducible component C of D one has $C^2 > 2r(C)$. Then $N = \text{Ker}(\pi_1(X \setminus (D \cup E)) \rightarrow \pi_1(X \setminus E))$ is a finitely generated abelian group and the centralizer of N has a finite index in $\pi_1(X \setminus (D \cup E))$.*

This immediately implies that the fundamental group of a nodal curve in \mathbb{P}^2 is abelian (indeed, for irreducible curve of degree d the maximal number of nodes $r(C) = \frac{(d-1)(d-2)}{2}$ satisfies $2r(C) < d^2$). Moreover, for an irreducible plane curve with $r(C)$ nodes and $\kappa(C)$ cusps (with local equation $u^2 = v^3$) one obtains that $\pi(\mathbb{P}^2 \setminus C)$ is abelian if $C^2 > 6\kappa(C) + 2r(C)$ (apply 2.5 to resolution of cusps only). On non-simply connected surfaces, the kernel $\pi_1(X \setminus C) \rightarrow \pi_1(X)$ belongs to the center if $C^2 > 4r(C)$ (though for $4r(C) \geq C^2 > 2r(C)$ the centralizer of this kernel still has a finite index, see [174] p. 324).

A result pointing out toward a positive answer to the Problem 2.4 appears in [126] and can be stated as follows. Let X be a smooth projective variety and Y be a subvariety such that $\pi_1(Y) \rightarrow \pi_1(X)$ is surjective. Let $f : Z \rightarrow Y$ be dominant morphism where all irreducible components of Z are normal. Let N be the normal subgroup of $\pi_1(X)$ generated by the images of irreducible components of Z . Then for any n , $\pi_1(X)/N$ has only finitely many n -dimensional complex representations, all of which are semi-simple (an obvious attribute a finite group).

Applications of Nori's results include [197], [198]. Paper [199] studies further exact sequence of the fiber spaces. Papers [211], [212] give conditions in opposite direction than the one considered by Nori, guaranteeing that the fundamental group of the complement is NON abelian. An important outcome of Nori's Weak Lefschetz theorem is that it provides an instance for the finiteness of the index of the image of the fundamental groups for compositions $H \rightarrow C \rightarrow X$ where as above H, X are smooth and $H \rightarrow C$ is dominant. This more general context was considered in [110] in the framework of the study of the representations of the fundamental groups of varieties dominating divisors in the moduli spaces of (pointed) curves (with level structure), under the heading of "non-abelian strictness theorems".

3. ALEXANDER INVARIANTS

3.1. Alexander polynomials. Alexander polynomial of knots and links was introduced by James W. Alexander in 1928 (cf. [3]). In response to a question by D.Mumford (cf. [168]), who noticed its relation to a construction used by O.Zariski, the Alexander polynomials were put in [129] in the context of complements to plane algebraic curves. This extension blends the algebraic geometry and the methods introduced by Fox (cf. [90]) and Milnor (cf. [161]) for the study of knots. Various generalizations, in which (a zero set of) polynomial was replaced by a subvariety of a torus and involving germs of singularities (cf. [140]), extensions to higher dimensions (cf. [134]) and to curves in complex surfaces (cf. [62]), were considered as well. A twisted versions (cf. [44], [158], [143]) were studied more recently. Below we shall describe the Alexander polynomials in the context of divisors on simply connected surfaces and refer to [144] for the history of the subject and further references.

Let X be a smooth simply connected projective surface and let D be a divisor on X with irreducible components D_i . Let $\{[D_i]\} = H^2(D, \mathbb{Z}) = \oplus_i H^2(D_i, \mathbb{Z})$ denote a free abelian group generated by the cohomology classes corresponding to the irreducible components of D . For $\alpha \in H_2(X, \mathbb{Z})$, we put $D_\alpha = \sum_i (\alpha, [D_i])[D_i] \in \{[D_i]\}$, where $[D_i] \in H_2(X, \mathbb{Z})$ is the fundamental class of the component D_i and denote by $\{D_\alpha\}$ the subgroup of $\{[D_i]\}$ generated by the classes $D_\alpha, \alpha \in H_2(X, \mathbb{Z})$. $\{D_\alpha\}$ is the image of the homomorphism $H_2(X, \mathbb{Z}) \rightarrow H^2(D, \mathbb{Z})$ obtained using the excision and duality isomorphisms giving $H_2(X, X - D, \mathbb{Z}) = H_2(T(D), \partial T(D), \mathbb{Z}) = H^2(D, \mathbb{Z})$ where $T(D)$ is a tubular neighborhood of D in X and $\partial T(D)$ is its boundary. From the exact sequence:

$$(5) \quad H_2(X, \mathbb{Z}) \rightarrow H^2(D, \mathbb{Z}) \rightarrow H_1(X \setminus D, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z}) = 0$$

we deduce that

$$(6) \quad \{[D_i]\}/\{D_\alpha\} = H_1(X \setminus D, \mathbb{Z}).$$

For example for an irreducible projective (resp. affine) plane curve D of degree d we obtain $H_1(\mathbb{P}^2 \setminus D, \mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$ (resp. $H_1(\mathbb{C}^2 \setminus D) = \mathbb{Z}$).

Alexander polynomial is an invariant of the complement to a reduced divisor D and a surjection $\phi : \pi_1(X \setminus D) \rightarrow C$ where C is a cyclic group. We state the definition for a finite CW complex Y endowed with a surjection $\phi : \pi_1(Y) \rightarrow C$ such that $H_1(Y_\phi, \mathbb{Q})$ is finite dimensional where Y_ϕ is the covering space corresponding to the subgroup $\text{Ker}(\phi) \subset \pi_1(Y)$ (cf.[115] sect.1.3)⁵.

If C is finite then the finiteness of the dimension of $H_1(Y_\phi, \mathbb{Q})$ is automatic ⁶. If $H_1(Y, \mathbb{Z})$ is infinite cyclic then $H_1(Y_\phi, \mathbb{Q})$ also is finite-dimensional as follows for example from (8) below.

For the covering map $Y_\phi \rightarrow Y$, we have the exact compactly supported homology sequence corresponding to the sequence of chain complexes

$$(7) \quad 0 \rightarrow C_*(Y_\phi, \mathbb{Q}) \xrightarrow{t^{-1}} C_*(Y_\phi, \mathbb{Q}) \rightarrow C_*(Y, \mathbb{Q}) \rightarrow 0$$

Here the first two terms are viewed as the modules over the group ring $\mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$, where t denotes preferred generator of $C = \mathbb{Z}$ in multiplicative notations, and the left map being multiplication by $t - 1$. Hence

$$(8) \quad \begin{aligned} H_2(Y, \mathbb{Q}) &\rightarrow H_1(Y_\phi, \mathbb{Q}) \xrightarrow{t^{-1}} H_1(Y_\phi, \mathbb{Q}) \rightarrow H_1(Y, \mathbb{Q}) \\ &\rightarrow H_0(Y_\phi, \mathbb{Q}) \xrightarrow{(t-1)} H_0(Y_\phi, \mathbb{Q}) \end{aligned}$$

Consider the cyclic decomposition of $H_1(Y_\phi, \mathbb{Q})$, viewed as a module over $\mathbb{Q}[t, t^{-1}]$,

$$(9) \quad H_1(Y_\phi, \mathbb{Q}) = \bigoplus \mathbb{Q}[t, t^{-1}]^{a_0} \oplus_p \mathbb{Q}[t, t^{-1}]/(p(t))$$

where the summation is over a finite number of monic polynomials p .

One of immediate consequences is that if $\text{rk} H_1(Y, \mathbb{Q}) = 1$, then the multiplication by $t - 1$ in the top row in (8) is surjective (since clearly the multiplication by $t - 1$ is trivial on H_0) and hence in (9) $a_0 = 0$ ⁷ Moreover, $(t - 1)^\alpha$, $\alpha \in \mathbb{N}$ is not among the polynomials $p(t)$.

Definition 3.1. Let Y be a CW-complex as above.

If $a_0 = 0$ in the decomposition (9) one defines the Alexander polynomial $\Delta(t)$ of (Y, ϕ) as the order of the $\mathbb{Q}[t, t^{-1}]$ -module $H_1(Y_\phi, \mathbb{Q})$ i.e. as the product:

$$(10) \quad \Delta(t) = \prod p(t)$$

In the case when X is a smooth projective surface and D is a reduced divisor, we call $\Delta(t)$, the *global* Alexander polynomial of $X \setminus D$ (and the surjection ϕ of its fundamental group).

⁵In this case we call Y_ϕ 1-finite

⁶An example of infinite cyclic covers which is infinite in dimension 1 is given by the complement to a set [3] containing 3 points in \mathbb{P}^1 . Let (a, b) be generators of the free group $\pi_1(\mathbb{P}^1 \setminus [3])$ and ϕ is the quotient of the normal subgroup generated by b . Then $\mathbb{P}^1 \setminus [3]$ is homotopy equivalent to a wedge of two circles and $(\mathbb{P}^1 \setminus [3])_\phi$ can be viewed as a real line with the circle attached at each integer point of this line with the covering group \mathbb{Z} acting via translations. In particular $H_1(\mathbb{P}^1 \setminus [3])_\phi, \mathbb{Z}$ is a free abelian group with countably many generators.

⁷the condition $a_0 = 0$ is equivalent to finite dimensionality of $H_1(Y_\phi, \mathbb{Q})$ over \mathbb{Q}

$\Delta(t)$ has integer coefficients, is well defined up to $\pm t^i, i \in \mathbb{Z}$ and, it follows from (8) that, $rkH_1(Y, \mathbb{Q}) = 1$ implies $\Delta(1) \neq 0$. If the target of ϕ is a finite cyclic group then, since $\mathbb{Q}[\mathbb{Z}_n] = \mathbb{Q}[t, t^{-1}]/(t^n - 1)$, instead of (9) one has

$$(11) \quad H_1(Y_\phi) = \oplus[\mathbb{Q}[t, t^{-1}]/(t^{\text{ord}C} - 1)]^{\alpha_o} \oplus \mathbb{Q}[t, t^{-1}]/p(t)$$

and the Alexander polynomial defined to be the order (10) of this $\mathbb{Q}[t, t^{-1}]$ -module.

This construction, when applied to the intersection of D with a small sphere about a singular point P of D and when ϕ is given by evaluation of the linking number in this sphere with D , yields the *local Alexander polynomial*. It is not hard to show the following (cf. [128]):

Proposition 3.2. *The local Alexander polynomial coincides with the characteristic polynomial of the local monodromy of the singularity of D at P*

3.2. A Divisibility Theorem. This is the central result on the Alexander polynomials allowing to obtain information about $\Delta(t)$ in terms of geometry of D . In many cases it leads to its determination or makes possibilities for $\Delta(t)$ rather limited. The case of curves in \mathbb{P}^2 appears in [129].

Theorem 3.3. *Let $D = D_1 \cup D_2$ be a divisor on X such that D_1 is ample. Let $\phi_{X \setminus D} : \pi_1(X \setminus D) \rightarrow H_1(X \setminus D, \mathbb{Z}) \rightarrow C$ be a surjection onto a cyclic group C (either infinite or finite) and $T(D_1)$ denotes a small regular neighborhood of the divisor D_1 . Assume also that ϕ maps the meridian ⁸ of each irreducible component of D_1 to the generator of C corresponding to the variable t of the Alexander polynomial. Then*

1. *The cyclic cover $(X \setminus D)_\phi$ is 1-finite and so is $(T(D_1) \setminus D \cap T(D_1))_{\phi_T}$ where ϕ_T is the composition $\pi_1((T(D_1) \setminus D \cap T(D_1))) \rightarrow H_1((T(D_1) \setminus D \cap T(D_1))) \rightarrow H_1(X \setminus D) \rightarrow C$ of the map induced by embedding and the surjection $\phi_{X \setminus D}$.*

2. *Let $\Delta_{\phi_{X \setminus D}}, \Delta_{\phi_T}$ be the Alexander polynomials of $X \setminus D$ and $T(D_1) \setminus D \cap T(D_1)$ corresponding to surjections ϕ and ϕ_T respectively. One has the following divisibility:*

$$(12) \quad \Delta_{\phi_{X \setminus D}}(t) | \Delta_{\phi_T}$$

3. *Let $\{p_i\}$ be the set consisting of singular points of D_1 and the points $D_1 \cap D_2$. For each p_i let B_{p_i} denotes a small ball in X centered at this point. Let Δ_{p_i} denotes the Alexander polynomial of $B_{p_i} \setminus D \cap B_{p_i}$ relative to the map $\phi_i : H_1(B_{p_i} \setminus D \cap B_{p_i}) \rightarrow C$ induced by embedding $B_{p_i} \setminus D \cap B_{p_i} \rightarrow X \setminus D$. Then*

$$(13) \quad \Delta_{\phi_T} = (t - 1)^\alpha \prod \Delta_{p_i} \quad \alpha \in \mathbb{Z}.$$

In particular, the roots of the Alexander polynomials $\Delta_{\phi_{X \setminus D}}$ and Δ_{ϕ_T} are roots of unity.

Proof. Ampleness of D_1 implies that for $n \gg 0$ there exist a smooth curve \tilde{D}_1 on X linearly equivalent to nD_1 and belonging to $T(D_1)$. Moreover, we can assume that \tilde{D}_1 is transversal to all components of D .

⁸i.e. a loop consisting of a path connecting the base point with a point in vicinity of the irreducible component of D , the oriented boundary of a small disk in X transversal to this component of D at its smooth point and not intersecting the other components of D , with the same path used to return back to the base point; orientation of the small disk must be positive i.e. such that its orientation will be compatible with the complex orientations of smooth locus of divisor and the ambient manifold. As an element of the fundamental group, only the conjugacy class of a meridian is well defined.

Weak Lefschetz theorem (cf. [114],[174]) implies that the composition in the middle row of the following diagram is a surjection:

$$(14) \quad \begin{array}{ccccc} & & Ker\phi_T & \rightarrow & Ker\phi_{X \setminus D} \\ & & \downarrow & & \downarrow \\ \pi_1(\tilde{D}_1 \setminus \tilde{D}_1 \cap D) & \rightarrow & \pi_1(T(D_1) \setminus D \cup T(D_1)) & \rightarrow & \pi_1(X \setminus D) \\ & & \downarrow & & \downarrow \\ & & H_1(T(D_1) \setminus D \cap T(D_1), \mathbb{Z}) & \rightarrow & H_1(X \setminus D, \mathbb{Z}) \\ & & & \searrow & \downarrow \\ & & & & C \end{array}$$

Therefore the right map in that row and hence also $Ker\phi_T \rightarrow Ker\phi_{X \setminus D}$ both are surjective. The condition that meridians are taken by ϕ_T to non-zero element of C implies that the covering space $(T(D_1) \setminus D \cap T(D_1))_{\phi_T}$ is 1-finite (cf. [99]) and surjectivity of the maps of the kernels implies that so is $(X \setminus D)_{\phi}$. Since the map in the top row in (14) is surjective, $\mathbb{Q}[C]$ -module $H_1((X \setminus D)_{\phi}, \mathbb{Q})$ is a quotient of $H_1((T(D_1) \setminus D \cap T(D_1))_{\phi_T}, \mathbb{Q})$ hence the divisibility relation (12) follows.

Finally, taking $T(D_1)$ sufficiently thin, $(T(D_1) \setminus D_1) \setminus \bigcup_i (B_{p_i} \setminus D \cap B_{p_i})$ can be assumed isotopic to the trivial C^∞ -fibration $(T(D_1) \setminus D_1) \setminus \bigcup_i (B_{p_i} \setminus D \cap B_{p_i}) \rightarrow D_1 \setminus \{p_i\}$, with the fiber being isotopic to a punctured 2-disk. Due to assumption that meridians of all components are mapped to generator corresponding to t , the Alexander polynomial of $(T(D_1) \setminus D_1) \setminus \bigcup_i (B_{p_i} \setminus D \cap B_{p_i})$ is a power of $t - 1$. The decomposition

$$(15) \quad T(D_1) \setminus D \cap T(D_1) = \left[(T(D_1) \setminus D_1) \setminus \bigcup_i (B_{p_i} \setminus D \cap B_{p_i}) \right] \cup \bigcup_i (B_{p_i} \setminus D \cap B_{p_i})$$

induces decomposition of the cover $(T(D_1) \setminus D)_{\phi_T}$ of $T(D_1) \setminus D$ corresponding to subgroup $Ker\phi_T$ of $\pi_1(T(D_1) \setminus D \cap T(D_1))$ into a union of preimages of each subspace on the right in (15). Now the Mayer-Vietoris sequence implies the part 3 of the Theorem (also the 1-finiteness of $T(D_1) \setminus T(D_1) \cap D$). \square

Corollary 3.4 ([129]). *Let C be an irreducible curve in \mathbb{P}^2 and L be the line at infinity. Then $H_1(\mathbb{P}^2 \setminus C \cup L, \mathbb{Z}) = \mathbb{Z}$ and the Alexander polynomial of $\mathbb{P}^2 \setminus C \cup L$ with respect to the abelianization, divides the product of the Alexander polynomials of links of all singularities of $C \cup L$. It also divides the Alexander polynomial of the link at infinity i.e. the Alexander polynomial of the complement $S_\infty \setminus C \cap S_\infty$ where S_∞ is the boundary of a small (in the metric on \mathbb{P}^2) regular neighborhood of $L \subset \mathbb{P}^2$.*

Proof. It follows from (6) and Theorem 3.3 applied to C and L separately. More precisely, the part 2 (resp. part 2 and 3) of Theorem 3.3 show that the global Alexander polynomial divides the Alexander polynomial at infinity (resp. of the product of local Alexander polynomials). \square

Corollary 3.5 ([62]). *Let D be a divisor of a simply connected surface X . Let S be a subset of the set of singular points of D belonging to an irreducible component D' of D such that on log-resolution \tilde{X} of singularities of D' outside of S , for proper preimage \tilde{D}' one has $(\tilde{D}')^2 > 0$. Then one has divisibility:*

$$(16) \quad \Delta_{\phi_{X \setminus D}} \mid \prod_{p_i \in S} \Delta_{p_i}$$

Proof. Condition on self-intersection implies that \tilde{D}' is ample. Now the claim follows immediately from the Theorem 3.3 applied to the proper preimage of D on \tilde{X} and its

component \tilde{D}' since $X \setminus D = \tilde{X} \setminus \tilde{D}$ because only points on deleted divisor D are blown up. \square

Example 3.6. *Milnor fibers of homogeneous polynomials and arrangements of lines* Let $\mathcal{A} \subset \mathbb{P}^2$ be an arrangement of lines given by equations $L_i(x, y, z) = 0, i = 1, \dots, N$. Milnor fiber $\prod L_i(x, y, z) = 1$ of the cone $\prod L_i = 0$ over this arrangement (denoted below $M_{\mathcal{A}_L}$) can be identified with the $\mathbb{Z}/N\mathbb{Z}$ -cyclic cover of the complement $\mathbb{P}^2 \setminus \mathcal{A}$. Theorem 3.3 gives restrictions on the degree of the characteristic polynomial of the monodromy operator acting on $H_1(M_{\mathcal{A}}, \mathbb{Q})$ (which can be identified with the Alexander polynomial of $\mathbb{P}^2 \setminus \mathcal{A}$) in terms of multiplicities of point of \mathcal{A} along one of the lines (cf.[139]). For example, if \mathcal{A} has only triple points along one of the lines, it follows that the characteristic polynomial of the monodromy of Milnor fiber has form $(t-1)^{N-1}(t^2+t+1)^\kappa, \kappa \geq 0$. See [73], [74], [175] for other numerous applications.

3.3. Branched covers. A branched cover of a complex space Y is a finite dominant morphism $f : X \rightarrow Y$. We will consider only the case when X is normal and Y is smooth. Ramification locus $R_f \subset X$ is the support of the quasi-coherent sheaf $\Omega_{X/Y}$ and the branch locus is $f(R_f) \subset Y$. It has codimension 1 (Nagata-Zariski purity of the branch locus cf. [222]).

Given an irreducible divisor $D \subset Y$ on a complex manifold Y one associates to (Y, D) a discrete valuation $v_D : \mathbb{C}(Y) \rightarrow \mathcal{N}_D$ of the field of meromorphic functions on Y given by $v_D(\phi) = \text{ord}_D(\phi), \phi \in \mathbb{C}(Y)$ (cf.[113], p.130). Here \mathcal{N}_D is the subgroup of \mathbb{Z} generated by the values of $v_D(\phi), \phi \in \mathbb{C}(Y)$. For a branched cover $X \rightarrow Y$ and a pair of irreducible divisors $D \subset Y, \Delta' \subset X$ where Δ' is a component of $f^*(D)^{\text{red}}$ the map $f^* : \mathbb{C}(Y) \rightarrow \mathbb{C}(X)$ induces the map $f^*_N : \mathcal{N}_D \rightarrow \mathcal{N}_{\Delta'}$. The index $[N_{\Delta'} : f^*_N \mathcal{N}_D]$ (cf.[223] cf. Chapter 6, §12) is the ramification index e_D of f along the component Δ' . One has $e_{D'} = 1$, unless D' is a component of R_f . Restriction of a branched cover $X \rightarrow Y$ onto the complement to the ramification divisor induces étale map $X \setminus R_f \rightarrow Y \setminus D$ where $D = f(R_f)$ is the branch locus. In particular given a branched cover $f : X \rightarrow Y$, selection of base point $p \in Y \setminus D$ allows to construct monodromy:

$$(17) \quad \pi_1(Y \setminus D) \rightarrow \text{Sym}(f^{-1}(p))$$

into permutation group of points in the preimage of p assigning to each loop and a point $a \in f^{-1}(p)$ the end of the lift of the loop starting at a .

The set of equivalence classes of unramified covers $f_Z : Z \rightarrow Y \setminus D$, where f_{Z_1}, f_{Z_2} are considered to be equivalent iff there exists biholomorphic isomorphism $h : Z_1 \rightarrow Z_2$ such that $f_{Z_1} = f_{Z_2} \circ h$, is in one to one correspondence with the subgroups of $\pi_1(Y \setminus D, p)$ where $p \in Y \setminus D$ is a base point. The correspondence is given by assigning to f_Z the subgroup $(f_Z)_* \pi_1(Z, p') \subset \pi_1(Y \setminus D, p)$. This correspondence depends on a choice of a base point $p' \in f^{-1}(p) \subset X \setminus R_f$, but the subgroups corresponding to $p', p'', p' \neq p''$ are conjugate. A cover $Y \rightarrow X$ is called Galois if the corresponding subgroup is normal. The quotient of the fundamental group by this subgroup is the Galois group of the cover. This group is the image of the monodromy (17).

A branched cover is Galois if and only if the extension of the fields of meromorphic functions $\mathbb{C}(X)/\mathbb{C}(Y)$ is Galois and the Galois group of the cover is the Galois group of this field extension.

It follows from above discussion that the Galois group G acts freely on $X \setminus R_f$ with the quotient $Y \setminus D$ and one has the exact sequence $0 \rightarrow \pi_1(X \setminus R_f, p') \rightarrow \pi_1(Y \setminus D, p) \rightarrow G \rightarrow 0$. Vice versa, given an unramified cover $f : X \setminus R_f \rightarrow Y \setminus D$, it follows from Riemann Extension

Theorem for normal spaces (cf. [101] Ch.7, §4, sect.2) that this action on $X \setminus R_f$ extends to the G -action on X via biregular automorphisms.

For an irreducible component $\Delta \subset R_f$ of the ramification divisor, the subgroup $I(\Delta)$ of G of automorphisms which fixes all $x \in \Delta$ is called the decomposition group of Δ or inertia group of Δ (cf. [104], Expose V, sect 2)⁹. Action of inertia group on the tangent space at a smooth point $x \in \Delta$ which it fixes, induces the action on the normal space of Δ . The character ψ_I of this 1-dimensional representation of the cyclic group $I(\Delta)$ generates the group of characters $Char(I(\Delta))$. In particular one has a well defined map $Char I(\Delta) \rightarrow \mathbb{Z} : \chi \rightarrow i_\chi$ where $\chi = \psi^{i_\chi}, 0 \leq i_\chi < \text{ord} I(x)$.

The extension is called abelian (resp. cyclic) if it is Galois and the Galois group is abelian (resp. cyclic). For a branched Galois cover $X \rightarrow Y$, the ramification index $e_{\Delta'}$ is the same for all irreducible components $\Delta' \subset X$, having the same image $D \subset Y$. Moreover, the order of the inertia group $|I(\Delta')| = e_{\Delta'}$. If r is the number of f -preimages of a generic point $D \subset Y$ then one has $|G| = r e_{\Delta'}$.

The above correspondence between subgroups of the fundamental group and covers in Galois case becomes the correspondence (for fixed pair (Y, D)) between surjections $P : \pi_1(Y \setminus D) \rightarrow G$ and covers with Galois group G . Given a surjection, one can construct the corresponding cover, i.e. unique, up to homeomorphism over $Y \setminus D$, topological space Y' and the map $f : Y' \rightarrow Y \setminus D$ making Y' into unramified covering space with group G , as follows. This space Y' can be viewed as the quotient of the space of paths in $Y - D$ with a fixed initial points $p \in Y \setminus D$ with two paths γ_1, γ_2 being equivalent iff they have the same end point and the homotopy class of $\gamma_1^{-1} \circ \gamma_2 \in \pi_1(Y \setminus D, p)$ has trivial image in G . Y' , being an unramified cover, inherits from $Y \setminus D$ the structure of complex manifold so that f' is étale. We have the following theorem:

Theorem 3.7. (*Grauert-Remmert, cf. [101] Chapter 7, Grothendieck, [104], SGA1, Ch XII, sect. 5*) *Let $f^* : X^* \rightarrow Y \setminus D$ be a finite unramified map of complex spaces where Y is a smooth complex manifold and D a divisor on Y . Then there is a unique normal space X containing X^* as a dense subset and morphism $f : X \rightarrow Y$ such that $f^* = f|_{X^*}$.*

The inertia group of a component $\Delta' \subset X$ of ramification divisor in a Galois cover with a group G is the cyclic subgroup of G generated by the image in G of a representative g in the conjugacy class $\gamma \in \pi_1(Y \setminus D)$ of a meridian of D . If the order of this image is s , then g^s can be lifted to $X \setminus \Delta'$ as a closed path (i.e. g^s belongs to the kernel of surjection $\pi_1(Y \setminus D, p) \rightarrow G$) and is homotopic in the complement to the ramification divisor in X to a meridian of Δ' .

3.4. Abelian Covers. We discuss two ways to enumerate branched covers with abelian Galois group over a manifold with given branch locus. One is topological, which follows immediately from the discussion of previous section (cf. [137]) and another is algebro-geometric (cf. [178], [4]). A different important perspective, from a view point of root-stacks is discussed in [196].

Since by Hurewicz theorem $H_1(Y \setminus D, \mathbb{Z})$ is the abelianization of $\pi_1(Y \setminus D)$, any surjection of the fundamental group onto an abelian group G factors as $\pi_1(Y \setminus D) \rightarrow H_1(Y \setminus D, \mathbb{Z}) \rightarrow G$. Hence we have the following:

⁹Recall that the ground field here is \mathbb{C} . For varieties over non-algebraically closed fields, the inertia group $I(x)$ of $x \in \Delta$ (which is the subgroup of the decomposition group consisting of automorphisms inducing trivial automorphism of the extension of the residue fields of $f(x)$ by the residue field of x (cf. [104] Expose V, sect. 2) is a proper subgroup of the decomposition group.

Corollary 3.8. *Equivalence class of a branched cover of a complex manifold Y and having a divisor $D = \bigcup D_i$ as its branch locus is determined by the surjection $H_1(Y \setminus D) \rightarrow G$ taking neither of meridians of D to identity. Vice versa, an abelian branched cover $f : X \rightarrow Y$ determines the branch divisor $D \subset Y$ and the above surjection of the homology group. Moreover, this correspondence induces the map from the set of irreducible components of the branch locus to the set of cyclic subgroups of the Galois group (inertia subgroups of the irreducible components of ramification divisor).*

An algebro-geometric description of abelian covers (cf. [178]) is given in terms of the collections of line bundles labeled by the characters of G . Given such a cover $f : X \rightarrow Y$, one obtains the decomposition into eigen-sheaves:

$$(18) \quad f_*(\mathcal{O}_X) = \bigoplus_{\chi \in \text{Char}G} \mathcal{L}_\chi^{-1}$$

The left hand side has the structure of a sheaf of algebras and the work [178] describes the data specifying such structure on the right in (18). This is done in terms of classes of components of branch locus in $\text{Pic}(Y)$, the Galois group G , the collection of cyclic subgroups H and generators ψ_H of each group $\text{Char}H$ satisfying the following compatibility conditions. Once the \mathcal{O}_Y -algebra structure, say \mathcal{A} , on the right in (18) is specified, the branched cover is just $\text{Spec}\mathcal{A} \rightarrow Y$.

For a pair (H, ψ) , where $H \in \text{Cyc}(G)$ and ψ is a generator of $\text{Char}H$, let $D_{H,\psi}$ be the union of irreducible components D of the branch locus which have H as its inertia group and ψ as the character of representation of H on the normal space to a component of the ramification locus over D fixed by H . To a character $\chi \in \text{Char}G$ corresponds $r_{H,\psi}^\chi \in \mathbb{N}$ such that $\chi|_H = \psi^{r_{H,\psi}^\chi}$, $0 \leq r_{H,\psi}^\chi < \text{ord}H$. For a pair $\chi_1, \chi_2 \in \text{Char}G$, let us set $\epsilon_{\chi_1, \chi_2}^{H,\psi}$ to be 0 (resp.1) if $\chi_1|_H = \psi^{i_{\chi_1}}$, $\chi_2|_H = \psi^{i_{\chi_2}}$, $0 \leq i_{\chi_1}, i_{\chi_2} < \text{Card}H$ and $i_{\chi_1} + i_{\chi_2} < \text{Card}H$ (resp. $i_{\chi_1} + i_{\chi_2} \geq \text{Card}H$).

Then the bundles \mathcal{L}_χ in (18) satisfy the relations (cf.[178]):

$$(19) \quad \mathcal{L}_{\chi_1\chi_2} = \mathcal{L}_{\chi_1} \otimes \mathcal{L}_{\chi_2} \bigotimes_{H \in \text{Cyc}(G), \psi \in \text{Char}H} \mathcal{O}(D_{H,\psi})^{\epsilon_{\chi_1, \chi_2}^{H,\psi}}$$

In fact, if χ_1, \dots, χ_s are generators of a decomposition of $\text{Char}(G)$ into a direct sum of cyclic subgroups and d_j is the order of χ_j , $j = 1, \dots, s$ then:

$$(20) \quad d_\chi \mathcal{L}_\chi = \sum_{H,\psi} \frac{d_\chi r_{H,\psi}^\chi}{|H|} D_{H,\psi}$$

Vice versa (cf. [178]), given

- (a) a finite abelian group G ,
 - (b) a smooth compact complex manifold Y ,
 - (c) a divisor D on Y with assignment to each irreducible component a cyclic subgroup H of G and a generator ψ_H of $\text{Char}(H)$
 - (d) collection of line bundles \mathcal{L}_χ , $\chi \in \text{Char}(G)$ labeled by the characters of G
- with (a),(b),(c),(d) satisfying the relations (19), there is abelian branched cover X of Y satisfying (18) (cf. [178]). The data (a),(b),(c),(d) subject to (19) called *the building data*.

We will show how to recover from Y, D and surjection $H_1(Y \setminus D, \mathbb{Z}) \rightarrow G$ the parts (c),(d) of the building data and vice versa, the building data determines the surjection onto the covering group.

Proposition 3.9. *Let Y be a smooth projective manifold and let $D = \bigcup_{i=1}^r D_i$ be a divisor with irreducible components D_i . The surjection $\pi : H_1(Y \setminus D) \rightarrow G$ onto an abelian group G determines for each character $\chi \in \text{Char}G$ the bundle \mathcal{L}_χ so that the bundles $\mathcal{L}_\chi, \chi \in \text{Char}G$ satisfy the relations (19). Moreover, the bundles \mathcal{L}_χ^{-1} are the eigenbundles of decomposition (18) for the covering corresponding to π (cf. Cor. 3.8). Vice versa, a building data determines the surjection $H_1(Y \setminus D, \mathbb{Z}) \rightarrow G$.*

Proof. To a unitary character $\chi \in H^1(Y \setminus D, U(1))$ one associates the element in $\text{Pic}(X)$ as follows. One has the following high dimensional version of the exact sequence (5):

$$(21) \quad H_2(Y, \mathbb{Z}) \rightarrow H^{2 \dim D}(D, \mathbb{Z}) \rightarrow H_1(Y \setminus D, \mathbb{Z}) \rightarrow H_1(Y, \mathbb{Z}) = 0$$

Applying $\text{Hom}(\cdot, \mathbb{K}), \mathbb{K} = \mathbb{Z}, \mathbb{R}, U(1)$ to the terms of (21) we obtain:

$$(22) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}(H_1(Y \setminus D, \mathbb{Z}), \mathbb{Z}) & \rightarrow & H_{2 \dim D}(D, \mathbb{Z}) & \xrightarrow{\iota_{\mathbb{Z}}} & \text{Pic}(Y) \subset H^2(Y, \mathbb{Z}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & H^1(Y \setminus D, \mathbb{R}) & \rightarrow & H_{2 \dim D}(D, \mathbb{R}) & \xrightarrow{\iota_{\mathbb{R}}} & \text{Pic}(Y) \otimes \mathbb{R} \subset H^2(Y, \mathbb{R}) \\ & & \downarrow & & \downarrow \text{exp} & & \downarrow \text{exp} \\ 0 & \rightarrow & H^1(Y \setminus D, U(1)) & \xrightarrow{\varrho} & H_{2 \dim D}(D, U(1)) & \xrightarrow{\iota_{U(1)}} & H^2(Y, \mathbb{Z}) \otimes U(1) \end{array}$$

Here ϱ is evaluation of a character on the meridian of the irreducible component¹⁰, the vertical arrows are induced by the exponentiation $\text{exp} : \mathbb{R} \rightarrow U(1), a \rightarrow e^{2\pi i a}$ and the map ι (for each choice of coefficients) assigns to a homology class, the class in $H^{2 \dim D}(Y)$ which corresponds to the linear function on $H_{2 \dim D}(X)$ given by the intersection index with this class. A lift $\text{exp}^{-1}(\varrho)$ of $\chi \in H^1(Y \setminus D, U(1))$ determines uniquely the element $\tilde{\chi}$ in the unit cube in $H_{2 \dim D}(D, \mathbb{R})$, which is a fundamental domain for the action of the group $H_{2 \dim D}(D, \mathbb{Z})$ on the latter and which has $H_{2 \dim D}(D, U(1))$ as the quotient. Since $\iota_{U(1)}$ takes $\text{exp}(\tilde{\chi})$ to the trivial class in $H^2(Y, \mathbb{Z}) \otimes U(1)$ we obtain that $\iota_{\mathbb{R}}(\tilde{\chi}) \in H^2(Y, \mathbb{R})$ is an integral class. Since it has the Hodge type (1, 1), this class defines a line bundle. We shall denote it as \mathcal{L}_χ .

Let $\chi(\gamma_{D_i}) = \text{exp}(2\pi i \alpha_i), \alpha_i \in \mathbb{Q}, 0 \leq \alpha_i < 1$. If $\text{ord} \chi = d$ then $\alpha_i = \frac{v_i}{d}, 0 \leq v_i < d, i \in \mathbb{N}$. It follows that $\iota_{\mathbb{R}}(\text{exp}^{-1}(\varrho(\chi))) = \sum \frac{v_i}{d} [D_i]$ defines an integral class in $H^2(Y, \mathbb{Z})$ and \mathcal{L}_χ is the bundle with the first Chern class corresponding to this integral class¹¹. More directly, integrality can be seen as follows: since $\chi(D_\gamma) = 1, \forall \gamma \in H_2(Y, \mathbb{Z})$, it follows from (6) that one has $\prod_i \text{exp}(2\pi i \alpha_i)^{\langle \gamma, D_i \rangle} = 1$. Hence $(\gamma, \sum_i \frac{v_i}{d} [D_i]) \in \mathbb{Z}$ for all $\gamma \in H_2(Y, \mathbb{Z})$ i.e. $\sum_i \frac{v_i}{d} D_i$ is an integral class. Let \mathcal{L}_χ be the bundle with the first Chern class corresponding to this class. The bundle $\mathcal{L}_\chi^d = \sum v_i \mathcal{O}(D_i)$ has a section and it follows from the calculation in [84] Section 3.6 (cf. also [127] Remark 4.1.7 and Remark 3.10 below) that for the cyclic cover $\pi_\chi : Y_\chi \rightarrow Y$ corresponding to $\text{Ker}(\chi)$ one has $\pi_\chi(\mathcal{O}) = \sum_0^{d-1} \mathcal{L}_\chi^{-k}$. Moreover, \mathcal{L}_χ^{-k} is the eigensheaf with corresponding character χ^k . Considering the full G -cover and factoring it through Y_χ , one sees that this is also the eigenbundle in the G -cover corresponding to π . Now the relations (19) follows from [178], Theorem 2.1.

Vice versa, the map $\text{Char}G = \text{Hom}(G, U(1)) \rightarrow H_{2 \dim D}(D, U(1))$ sending a component with inertia group H to $\text{exp}(2\pi i \frac{r_{H, \psi}^X}{\text{ord} H})$ due to relations (19) lifts to the map to $H^1(Y \setminus D, U(1))$ and hence by duality induces the surjection $H_1(Y \setminus D, \mathbb{Z}) \rightarrow G$. \square

¹⁰recall that this follows from identification $H_{2 \dim D}(D, U(1)) = H^{2 \dim D}(Y, Y \setminus D, U(1))$ obtained by excision and Lefschetz duality

¹¹recall that Y is simply connected and hence \mathcal{L}_χ is well defined.

Example 3.10. (cf. [127], Sect. 4.1.B or [84]) Let Y be a smooth projective variety, $D \subset Y$ be a very ample divisor. Let \mathcal{L} be a very ample line bundle such that $\mathcal{L}^d = \mathcal{O}(D)$. Clearly, the bundles \mathcal{L}^i form a part of a building data for $G = \mathbb{Z}_d$. The corresponding cover can be obtained as follows. Let $s \in H^0(Y, \mathcal{O}(D))$ be a section with zero-scheme D and $\nu_d : [\mathcal{L}] \rightarrow [\mathcal{O}(D)]$ the map of the total spaces of the line bundles given by $\nu \in \mathcal{L} \rightarrow \nu^{\otimes d} \in \mathcal{L}^d$. Then $\nu_d^{-1}(s(Y))$ is a smooth subvariety Y_d of the total space of the line bundle \mathcal{L} and its projection π onto the base endows Y_d with the structure of the branched cover over Y with branch locus D . The divisibility of the fundamental class of D by d , implies that if $H_1(Y, \mathbb{Z}) = 0$, then there is well defined surjection $H_1(Y \setminus D, \mathbb{Z}) \rightarrow \mathbb{Z}_d$. It assigns to a 1-cycle δ representing a class in $H_1(Y \setminus D, \mathbb{Z})$, the modulo d intersection index of a 2-chain in Y having δ as its boundary. So Y_d is the cyclic cover of Y branched over D with Galois group \mathbb{Z}_d and corresponding to this surjection of $H_1(Y \setminus D, \mathbb{Z})$. The inertia group of any point of D is \mathbb{Z}_d . On the other hand $\pi_*(\mathcal{O}_{Y_d}) = \bigoplus_{i=0}^{d-1} \mathcal{L}^{-i}$ and \mathcal{L}^{-i} is the eigen-bundle corresponding to the character of \mathbb{Z}_d given by $\chi_i : j \rightarrow \exp(\frac{2\pi\sqrt{-1}ij}{d})$. The relation (19) is immediate.

Vice versa, given the surjection $H_1(Y \setminus D, \mathbb{Z}) \rightarrow \mathbb{Z}_d$, the diagram (22) shows that the character χ_i , taking value $\exp(\frac{2\pi\sqrt{-1}i}{d})$ on generator of $\mathbb{Z}/d\mathbb{Z}$, has as the lift $i_{\mathbb{R}}(\exp^{-1}(\varrho(\chi_i))) = c_1(\mathcal{L}^i)$ and in this way producing a building data.

In the case $Y = \mathbb{P}^2$, D is an irreducible curve of degree d with equation $f(x_0, x_1, x_2) = 0$, one has $H_1(\mathbb{P}^2 \setminus D, \mathbb{Z}) = \mathbb{Z}_d$ and the cover corresponding to this isomorphism is biholomorphic to a hypersurface $V_f : u^d = f(x_0, x_1, x_2)$ in \mathbb{P}^3 . Moreover the decomposition (18) becomes $f_*(\mathcal{O}_{V_f}) = \bigoplus_{i=0}^{d-1} \mathcal{O}_{\mathbb{P}^2}(-i)$.

Example 3.11. (cf. [119], [118]) Let \mathcal{A} be an arrangement of r lines in \mathbb{P}^2 . Then $H_1(\mathbb{P}^2 \setminus \mathcal{A}, \mathbb{Z}) = \mathbb{Z}^r / \{(1, \dots, 1)\} = \mathbb{Z}^{r-1}$. Let $H_1(\mathbb{P}^2 \setminus \mathcal{A}, \mathbb{Z}) \rightarrow G = \mathbb{Z}_n^r / \{(1, \dots, 1)\} = \mathbb{Z}_n^{r-1}$ sending the meridian the i -th line to $(0, \dots, 0, 1, \dots, 0) \bmod n$. A character of G can be identify with a vector $(\frac{a_1}{n}, \dots, \frac{a_r}{n}), 0 \leq a_i < n, \sum_1^r \frac{a_i}{n} \in \mathbb{Z}$. Let us denote this character χ_{a_1, \dots, a_r} . The inertia group H_i of the i -th line is the subgroup of G isomorphic to \mathbb{Z}_n and generated by $(0, \dots, 0, 1, \dots, 0) \bmod n$ (all components except the i -th are zero) and the character ψ of H_i takes the value $\exp \frac{2\pi i}{n}$ on the corresponding generator. It follows from discussion of Proposition 3.9 that

$$\mathcal{L}_{\chi_{a_1, \dots, a_r}}^{-1} = \mathcal{O}_{\mathbb{P}^2} \left(- \left(\frac{\sum_1^r a_i}{n} \right) \right)$$

See [119] for a direct calculation of the direct image of the structure sheaf using that this abelian cover is the restriction of the Kummer cover: $\mathbb{P}^{r-1} \rightarrow \mathbb{P}^{r-1}$ given by $(x_1, \dots, x_r) \rightarrow (x_1^n, \dots, x_r^n)$.

Example 3.12. Let D be the hypersurface in \mathbb{C}^n given by $f_1(x_1, \dots, x_n) \cdots f_r(x_1, \dots, x_n) = 0$ where $f_i \in \mathbb{C}[x_1, \dots, x_n]$ are irreducible. Using a non-compact version of the calculation (6) one obtains $H_1(\mathbb{C}^n \setminus D) = \mathbb{Z}^r$. Let $p : H_1(\mathbb{C}^n \setminus D) \rightarrow G$ be a surjection onto an abelian group. Then to p corresponds the cover $P : V_{p, \bar{D}} \rightarrow \mathbb{P}^n$ branched over the projective closure \bar{D} of D and possibly over the hyperplane at infinity with the following properties. The order r_i of $p(\gamma_i) \in G$ coincides with the ramification index of the branched cover P at $P^{-1}(s)$ where s is a generic point in D_i . At a generic point $s \in \mathbb{P}^{n-1}$ the ramification index at $P^{-1}(s)$ is the order in G of the class $p(\sum(\deg f_i)\gamma_i) \in H_1(\mathbb{C}^n \setminus D, \mathbb{Z})$. An explicit model of such covering can be obtained as the normalization of the projective closure of affine complete intersection in \mathbb{C}^{n+r} given by equations:

$$(23) \quad z_i^{r_i} = f_i(x_1, \dots, x_n) \quad i = 1, \dots, r.$$

3.5. Characteristic varieties.

3.5.1. *Alexander invariants and jumping loci of local systems.* A multivariable generalization of Alexander polynomials was proposed in [137] as follows. Let Y be a finite CW complex and $\phi : \pi_1(Y) \rightarrow A$ be a surjection onto a finitely generated abelian group. The unbranched abelian cover $\pi_\phi : Y_\phi \rightarrow Y$ corresponding to ϕ comes with a free action of A via cellular maps. Hence the compact supported homology $H_k(Y_\phi, \mathbb{C})$ and also its exterior powers

$$(24) \quad \Lambda^i H_k(Y_\phi, \mathbb{C})$$

can be considered as the modules over the group algebra $\mathbb{C}[A]$ of A .

Definition 3.13. (cf. [132]) The affine subvariety $Char_i^k(Y, \phi)$ of the torus $Spec\mathbb{C}[A]$ defined as support of the module $\Lambda^i H_k(Y_\phi, \mathbb{C})$ (cf.[81]) is called *the depth i characteristic variety of Y in dimension k (corresponding to surjection ϕ)*¹²

Standard results from commutative algebra (cf. [35] or [81] Ch. 20) show that $Char_i^k(Y)$ is the zero set of the i -th Fitting ideal of $\mathbb{C}[A]$ -module $H_k(Y_\phi, \mathbb{C})$ i.e. the ideal generated by $(n - i + 1) \times (n - i + 1)$ minors of the matrix of a presentation of this module via n generators and m relations. Moreover, for $k = 1$, which unless otherwise stated will be our focus for the rest of this section, presentation of the module $H_1(Y_\phi, \mathbb{C})$ can be studied using the matrix of Fox derivatives giving presentation of $\mathbb{C}[A]$ -module $H_1(Y_\phi, \tilde{p}, \mathbb{C})$ where $\tilde{p} = \pi_\phi^{-1}(p)$ is the preimage of $p \in Y$ (cf. [132]). As a consequence, this implies that for a CW complex having as its fundamental group a group with deficiency 1 and the homomorphism ϕ being the abelianization, the characteristic variety $Char_1^1(Y)$ of depth 1, has codimension 1 in $Spec\mathbb{C}[A]$. For example this is the case for $Y = S^3 \setminus L$ where L is a link. In fact, $Char_1^1(S^3 \setminus L)$ is the zero set of the multivariable Alexander polynomial of L ([122]). In the case of algebraic curves in \mathbb{C}^2 , the codimension of $Char_1^1$ is typically larger than 1 except for the case $A = \mathbb{Z}$ in which case it is the zero set of the 1-variable Alexander polynomial discussed in section 3.1¹³

There is a different interpretation of these subvarieties of the complex tori $Spec\mathbb{C}[A]$ ¹⁴. Recall (cf. for example [49], Ch.5) that a rank l local system on a CW complex Y is a l -dimensional linear representation of the fundamental group $\rho : \pi_1(Y) \rightarrow GL(l, \mathbb{C})$. (Co)homology of a local system are obtained as the cohomology of the chain complex:

$$(25) \quad \dots \rightarrow C_i(\tilde{Y}, \mathbb{C}) \otimes_{\pi_1(Y)} \mathbb{C}^l \rightarrow \dots$$

where $C_i(Y, \mathbb{C})$ are the chains with compact support on the universal cover considered as a module over the group ring of the fundamental group.

An important feature of local systems is the following: in the case when Y is a smooth quasi-projective variety, the cohomology of local systems admit Hodge-deRham description given by Deligne (cf. [67]). First of all representations of fundamental groups can in interpreted as locally trivial vector bundles with constant transition functions (cf. [67], Cor.1.4), which in turn, in the case when Y is a smooth manifold, can be interpreted as flat (integrable) connections $\nabla : \mathcal{V} \rightarrow \Omega_Y^1 \otimes \mathcal{V}$ on a holomorphic vector bundle \mathcal{V} (i.e. a

¹²the support is assumed to be a reduced variety

¹³However $Char_1(\pi_1(\mathbb{P}^1 \setminus [3]), ab) = (\mathbb{C}^*)^2$ where $[3]$ is a subset containing 3 points and ab is the abelianization of the free group

¹⁴ $Spec\mathbb{C}[A]$ is algebraic group with $CardTor A$ connected components with $(\mathbb{C}^*)^{r \cdot k \cdot A}$ being the component of identity

\mathbb{C} -linear map satisfying Leibnitz rule). This differential operator can be extended to higher degree forms and lead to a twisted deRham complex:

$$(26) \quad \dots \rightarrow \Omega^p(Y) \otimes \mathcal{V} \xrightarrow{\nabla} \Omega^{p+1} \otimes \mathcal{V} \rightarrow \dots$$

The (co)homology of the latter are identified with the (co)homology $H^*(Y, \rho)$ of the complex (25) since both are derived functors of the functor sending a representation to the subspace of invariants (cf. [67], Prop. 2.27, 2.28).

Using (co)homology with twisted coefficients of rank one local systems, one can define *jumping loci*:

$$(27) \quad \mathcal{V}_i^k(Y) = \{\rho \in \text{Hom}(\pi_1(Y), \mathbb{C}) \mid H_k(Y, \rho) \geq i\}$$

There is the canonical identification of $\text{Hom}(H_1(Y, \mathbb{Z}), \mathbb{C}^*)$ and $\text{Spec} \mathbb{C}[H_1(Y, \mathbb{Z})]$ making two collections of subvarieties of both tori correspond to each other ¹⁵:

$$(28) \quad \text{Char}_i^1(Y) \setminus \{1\} = \mathcal{V}_i^1(Y) \setminus \{1\}$$

This was shown to be the case for any finite CW complex Y in [117] for $k = 1$ and arbitrary i (cf. also [137], [75]) i.e. when one is interested in invariants of $\pi_1(Y)$ and for $i = 1$ but with arbitrary k (cf. [177]) when one considers invariants of the homotopy type.

3.5.2. *Homology of abelian covers.* Characteristic varieties determine the homology of covering spaces as follows.

Proposition 3.14. (cf. [132], [116], [70]) *Let $\phi : \pi_1(Y) \rightarrow A$ be a surjection onto a finite abelian group A and let Y_ϕ be the corresponding unbranched cover of Y with the Galois group A . Let $\nu(A)$ be the image of embedding $\phi^* : \text{Spec} \mathbb{C}[A] \rightarrow \text{Spec} \mathbb{C}[H_1(Y, \mathbb{Z})]$. Then:*

$$(29) \quad \text{rk} H_k(Y_\phi, \mathbb{C}) = \sum_i \text{Card}(\nu(A) \cap \mathcal{V}_i^k(Y))$$

This follows from the definition of jumping loci (27), the extension of classical Shapiro lemma (cf. [70]) from group cohomology to arbitrary spaces (cf. [49]) i.e. in our notations the identification $H_k(Y_\phi, \mathbb{C}) = H_k(Y, \mathbb{C}[A])$, and the decomposition $\mathbb{C}[A] = \bigoplus_{\chi \in \text{Spec} \mathbb{C}[A]} \mathbb{C}_\chi$ where \mathbb{C}_χ is the 1-dimensional representation of A given by the character χ .

Now consider the case of abelian branched covers which according to Cor. 3.8 are specified by the branching locus and the abelian quotient of the fundamental group of the complement to the latter. The proof below is a version of the argument due to M.Sakuma (cf. [188]).

Proposition 3.15. *Let X be a smooth simply-connected projective surface and $D = \bigcup D_i$ a reduced divisor. Let $\phi : H_1(X \setminus D, \mathbb{Z}) \rightarrow A$ be a surjection onto a finite abelian group. For a character $\chi \in A^*$ where $A^* = \text{Hom}(A, \mathbb{C}^*)$, let D^χ be the union of irreducible components D_i of D such that for the meridian $\delta_i \in H_1(X \setminus D, \mathbb{Z})$ of D_i one has $\chi(\delta_i) \neq 1$. Denote by $d(D^\chi, \chi)$ ¹⁶ the maximum of the integers i such that $\chi \in \text{Char}_i^1(X \setminus D^\chi)$, where $\text{Char}_i^1(X \setminus D^\chi)$ is the depth i characteristic variety of the curve D^χ as defined in 3.13.*

¹⁵the order of vanishing of the Fitting ideal of $H_1(Y_\phi, \mathbb{C})$ at $(1, \dots, 1)$ in general is different than $\text{rk} H_1(Y, \mathbb{C})$ which is the first Betti number of trivial local system

¹⁶The integer $d(D^\chi, \chi)$ is called *the depth* of the character χ of the curve D^χ

Let \widetilde{X}_ϕ be a resolution of singularities of the branched cover \bar{X}_ϕ of X ramified along D corresponding to above surjection ϕ . Then

$$(30) \quad rkH_1(\widetilde{X}_\phi, \mathbb{C}) = \sum_{\chi \in A^*} d(D^\chi, \chi)$$

Proof. Denote by $Sing\bar{X}_\phi$ the set of singularities of \bar{X}_ϕ . This is a finite set mapped by the covering map into the set $Sing(D)$ of singularities of D . We will start by showing that

$$(31) \quad rkH_1(\widetilde{X}_\phi, \mathbb{C}) = rkH_1(\bar{X}_\phi \setminus Sing(\bar{X}_\phi), \mathbb{C})$$

Indeed, if E is the exceptional set of a resolution $\widetilde{X}_\phi \rightarrow \bar{X}_\phi$, then

$$(32) \quad \widetilde{X}_\phi \setminus E = \bar{X}_\phi \setminus Sing(\bar{X}_\phi)$$

On the other hand, the exact sequence of the pair $(\widetilde{X}_\phi, \widetilde{X}_\phi \setminus E)$ and the identification $H^i(\widetilde{X}_\phi, \widetilde{X}_\phi \setminus E) = H_{4-i}(E)$ yield:

$$(33) \quad 0 \rightarrow H^1(\widetilde{X}_\phi) \rightarrow H^1(\widetilde{X}_\phi \setminus E) \rightarrow H_2(E) \rightarrow H^2(\widetilde{X}_\phi)$$

Together with injectivity of the right map in (33), which is a consequence of Mumford theorem on non-degeneracy of the intersection form on a resolution of a surface singularity (cf. [167]), we obtain (31).

By universal coefficients theorem, allowing to switch to cohomology, the identity (30) will follow from the following calculation of dimensions of χ -eigenspaces

$$(34) \quad dimH^1(\bar{X}_\phi \setminus Sing(\bar{X}_\phi))_\chi = d(D^\chi, \chi)$$

for all characters $\chi \in Char(A)$.

To show (34), note that the group A acts on $\bar{X}_\phi \setminus Sing(\bar{X}_\phi)$ with the quotient $X \setminus Sing(D)$. For any character χ of group A we consider the cyclic branched cover $(X \setminus Sing(D))_\chi$ of $X \setminus Sing(D)$ corresponding to composition $\pi_1(X \setminus D) \rightarrow A \rightarrow Im(\chi)$. One has the biregular isomorphism:

$$(35) \quad \bar{X}_\phi \setminus Sing(\bar{X}_\phi)/Ker\chi = (X \setminus Sing(D))_\chi$$

The group $A/Ker(\chi) = Im(\chi)$ acts on the left side of (35) and the identification (35) is $Im(\chi)$ -equivariant. The transfer $H^*(\bar{X}_\phi \setminus Sing(\bar{X}_\phi)) \rightarrow H^*(\bar{X}_\phi \setminus Sing(\bar{X}_\phi)/Ker)$ (cf. [33], p.118) provides $Im(\chi)$ -equivariant isomorphism $H^*(\bar{X}_\phi \setminus Sing(\bar{X}_\phi))^{Ker(\chi)} = H^*(\bar{X}_\phi \setminus Sing(\bar{X}_\phi)/Ker(\chi))$ which implies that

$$(36) \quad H^*(\bar{X}_\phi \setminus Sing(\bar{X}_\phi))_\chi = H^*(\bar{X}_\phi \setminus Sing(\bar{X}_\phi))_\chi^{Ker(\chi)} = H^*((X \setminus Sing(D))_\chi)_\chi$$

(34) is obvious for trivial character and the isomorphism (36) shows that (30) follows from the cyclic case of the Proposition for non-trivial χ ¹⁷.

Finally, the cyclic cover $(X \setminus Sing(D))_\chi$ is a totally ramified cover of $X \setminus SingD$ branched over D^χ and the cyclic case with non-trivial χ follows from the calculation of homology of unbranched covers in Proposition 3.14 since the action of $Im(\chi)$ on kernel and cokernel of the map induced by the embedding of the cyclic unbranched cover $(X \setminus D^\chi)_\chi$ with Galois group $Im(\chi)$

$$(37) \quad H^1((X \setminus SingD)_\chi) \rightarrow H^1((X \setminus D^\chi)_\chi)$$

is trivial. □

¹⁷we also use that removal 0-dimensional set $SingD$ from a 4 -dimensional manifold does not change the first Betti number

3.5.3. *Structure of characteristic varieties.* The central result on the structure of characteristic varieties of quasi-projective manifolds is obtained from their interpretation (cf. [117]) as the jumping loci of the cohomology of local systems, which allows to apply deep Hodge theoretical methods [9]. It asserts that the irreducible components of characteristic varieties are a finite order cosets of subtori of $\text{Spec}\mathbb{C}[H_1(X \setminus D, \mathbb{Z})]$ and that these components are the pull backs of the characteristic varieties of fundamental groups of curves via holomorphic maps. The origins of such correspondence are going back to deFranchis ([221]), Beauville ([30]), Green-Lazarsfeld ([103]), Simpson ([195]) in projective case with quasi-projective case being addressed by Arapura ([9]). In a very special case when the quasi-projective manifold is a complement to an irreducible plane singular curve the assertion of the finiteness of the order of cosets becomes the cyclotomic property of the roots of Alexander polynomials ([129], cf. Theorem 3.3) and does not require Hodge theory (unlike the Theorem 3.16). We shall quote an orbifold version (cf. [19]) of this correspondence between the holomorphic maps and the components of characteristic varieties

Theorem 3.16. (cf. [9], [19]). *Let $\mathcal{V}_i^1(X \setminus D)^{irr}$ be an irreducible component of jumping locus of 1-dimensional cohomology of a smooth quasi-projective variety $X \setminus D$. Then $\mathcal{V}_i^1(X \setminus D)^{irr}$ is a coset of finite order of a subtorus of the commutative algebraic group $\text{Spec}\mathbb{C}[H_1(X \setminus D, \mathbb{Z})]$. Moreover, there exist an orbifold curve C^{orb} , an irreducible component $\mathcal{V}_i(\pi_1^{orb}(C^{orb}))^{irr}$ and holomorphic orbifold map $f : X \setminus D \rightarrow C^{orb}$ such $\mathcal{V}_i^1(X \setminus D)^{irr} = f^*(\mathcal{V}_i(\pi_1^{orb}(C^{orb}))^{irr})$.*

Corollary 3.17. *The array of Betti numbers of finite abelian covers of X branched over D determines the characteristic varieties of the fundamental group of the complement.*

Proof. Translated subgroups are specified by the points of finite order on the torus which they contain. A point χ of the finite order belongs to the i -th characteristic variety if and only if the multiplicity of χ in the cyclic cover corresponding to the group $\text{Im}(\chi)$ is at least i . The claim follows. \square

A very important application of translated subgroup property is that it provides a necessary conditions on a group be quasi-projective i.e. to be a fundamental group of a smooth quasi-projective variety. For application of these and ideas from different ideas, not discussed here, to the problem of characterisation of quias-projective and quasi-Kahler groups (in particular the comparison with the fundamental groups of 3-manifolds) see: [133], [76], [77], [19], [10], [124], [91], [32]

3.6. **Isolated non-normal crossings.** A generalization of results on Alexander invariants from sections 3.1- 3.5 providing invariants of the homotopy type beyond fundamental groups was proposed in [141]. The starting point is the following:

Theorem 3.18. (cf. [141], Th. 2.1) *Let $X, \dim X > 2$ be a smooth simply connected projective variety and let D be a divisor such that all its irreducible components are smooth and ample. Then $\pi_1(X \setminus D)$ is abelian and $\pi_i(X \setminus D) = 0$ for $2 \leq i \leq \dim X - 1$.*

This theorem and Lefschetz hyperplane sections theorem have the following as an immediate corollary:

Corollary 3.19. ([141]) *Let X be as in Theorem 3.18, let $D = \bigcup_1^r D_i$ be a divisor with ample irreducible components and let $\text{NNC}(D)$ be the subvariety of X consisting of points $x \in X$ at which D fails to be a normal crossings divisor¹⁸. Let $s = \dim \text{NNC}(D)$ ¹⁹.*

¹⁸We call D a divisor with isolated non-normal crossings, if $\dim \text{NNC}(D) = 0$.

¹⁹the convention is that if $x \notin D$ then D does have normal crossing at x and the dimension of empty set is -1

Then $\pi_i(X \setminus D) = 0$ for $i \leq d - s - 2$. Moreover, if $H = \bigcap_1^s H_i$ is a sufficiently general intersection of very ample divisors on X then $D \cap H$ is a divisor on $X \cap H$ with isolated non normal crossings and

$$(38) \quad \pi_i(X \setminus D) = \pi_i((X \setminus D) \cap H) \quad i \leq d - s - 1$$

In particular the first non-trivial homotopy group of the complement to a divisor with ample components can be calculated using the divisor $D \cap H$ with isolated non-normal crossings on H .

The high dimensional analogs of the results on the Alexander invariants of the complement to curves described in [141] give a similar description of the homotopy group $\pi_{d-1}(X \setminus D)$ where X as in Theorem 3.18, D is a divisor with ample components but now D is allowed to have *isolated* non normal crossings (INNC) i.e. $\dim NNC(D) = 0$. The role of the first homology of the infinite abelian cover in the case of complements to curves is played by the first non-vanishing homotopy group $\pi_i(X \setminus D), i > 1$. In fact one has the identification:

$$(39) \quad H_{\dim X - 1}(\widehat{X \setminus D}, \mathbb{Z}) = \pi_{\dim X - 1}(X \setminus D)$$

where $\widehat{X \setminus D}$ is the universal (hence also universal abelian, cf. theorem 3.18) cover of $X \setminus D$. The $\mathbb{Z}[\pi_1(X \setminus D)]$ -module structure equivalently can be obtained using the Whitehead product (cf. [141]) and the characteristic variety of $X \setminus D$ in dimension $\dim X - 1$ is the support of the module $\pi_{\dim X - 1}(X \setminus D) \otimes \mathbb{C}$.

In the case when a point in $NNC(D)$ belongs to only one component, failure to be normal crossing means that the point is just an isolated singularity of D . In this case the local information about the Alexander invariants is contained in the Milnor fiber of the singularities of D and includes the characteristic polynomial of the monodromy as well as some the Hodge theoretical invariants (cf. [141]). Reducible analog of isolated singularities are isolated non-normal crossings i.e. the intersections of hypersurfaces which are smooth and transversal everywhere except for a single point. It turns out that many features of isolated singularities described by J.Milnor in [163] have counterparts in INNC case. They include the analogs of high connectivity of Milnor fibers, analogs of monodromy action on the cohomology, its cyclotomic properties and others.

To be specific, recall that if $f(x_0, \dots, x_n) = 0$ is a germ of an isolated singularity, V_f is its zero set, B_ϵ is a small ball about the singular point of f , ∂B_ϵ is the boundary sphere then we have the following: ²⁰

Theorem 3.20. (cf.[163]) (i) *The complement $B_\epsilon \setminus (V_f \cap B_\epsilon)$ is homotopy equivalent to $\partial B_\epsilon \setminus (V_f \cap \partial B_\epsilon)$.*

(ii) *There is a locally trivial fibration $\partial B_\epsilon \cap (V_f \cap \partial B_\epsilon) \rightarrow S^1$, with the fiber (i.e. the Milnor fiber) being homotopy equivalent to a wedge of spheres of dimension n .*

Corollary 3.21. *If $f(x_0, \dots, x_n)$ is a germ of isolated singularity, then the universal cyclic cover of $B_\epsilon \setminus (B_\epsilon \cap V_f)$ is homotopy equivalent to a finite wedge of spheres of dimension n . Moreover, the action of the deck transformation on the homology of the universal cyclic cover coincides with the action of the monodromy operator on the homology of the Milnor fiber. In particular, the characteristic polynomial of the monodromy coincides with the Alexander polynomial of $\partial B_\epsilon \setminus (V_f \cap \partial B_\epsilon)$.*

A generalization of these results to multi-component germs is as follows (which is the local counterpart of Corollary 3.19):

²⁰only the last claim in (ii) requires singularity of f to be isolated.

Theorem 3.22. (cf. [140]) Let X_r be a germ $f_1(x_0, \dots, x_n) \cdot \dots \cdot f_r(x_0, \dots, x_n) = 0$ of a hypersurface which is product of r irreducible germs. Then for $n > 1$

$$(40) \quad \pi_1(\partial B_\epsilon \setminus (X_r \cap \partial B_\epsilon)) = \mathbb{Z}^r \quad \pi_i(\partial B_\epsilon \setminus (X_r \cap \partial B_\epsilon)) = 0, 2 \leq i \leq n-1$$

The role of the Milnor fiber is now played by the universal abelian ²¹ cover of the complement $\overline{\partial B_\epsilon \setminus (X_r \cap \partial B_\epsilon)}$ and the monodromy action is replaced by the action of fundamental group of the complement to INNC germ on the universal abelian cover via deck transformations. The homology $H_n(\partial B_\epsilon \setminus (X_r \cap \partial B_\epsilon), \mathbb{C})$ is equipped with the action of the group algebra $\mathbb{C}[H_1(\partial B_\epsilon \setminus (X_r \cap \partial B_\epsilon))]$. The $H_1(\partial B_\epsilon \setminus (X_r \cap \partial B_\epsilon), \mathbb{Z})$ -action can be, as in global case, identified the Whitehead product of the elements of $\pi_n(\partial B_\epsilon \setminus (X_r \cap \partial B_\epsilon))$ with the elements of $\pi_1(\partial B_\epsilon \setminus (X_r \cap \partial B_\epsilon)) = H_1(\partial B_\epsilon \setminus (X_r \cap \partial B_\epsilon))$. The group ring of the latter is the ring of Laurent polynomials and the role of the characteristic polynomial of the monodromy is played by the subvariety of the torus $\text{Spec} \mathbb{C}[H_1(\partial B_\epsilon \setminus (X_r \cap \partial B_\epsilon))]$ which is the support of this module ([81]).

Example 3.23. (cf. [140]) Let $f_{d_i}(x_0, \dots, x_n) = 0, i = 1, \dots, r$ be the equations of smooth sufficiently general hypersurfaces in \mathbb{P}^n . Then the union in \mathbb{C}^{n+1} of cones over these r hypersurfaces is isolated non-normal crossing. The support of this module is the zero set of $t_1^{d_1} \cdot \dots \cdot t_r^{d_r} - 1 = 0$.

Example 3.24. Let $f_1(x, y) \dots f_r(x, y)$ be a germ of reducible curve in \mathbb{C}^2 . The support of the universal abelian cover of the complement to the link coincides the zero set of the multivariable Alexander polynomial of the link. Further properties of this support and its Hodge theoretical properties are discussed in [39], [40].

We refer to the work [141] Sections 5 and 6 for description of the structure of characteristic varieties of *global* INNC in terms of the homotopy groups of local INNC (of germs), as well as the relationship between the cohomology of local systems, the homology of branched cover and the translated subgroups of $\text{Spec} \mathbb{C}[\pi_1(X \setminus D)]$ which are the irreducible components of $\text{Supp} \pi_{\dim X-1}(X \setminus D) \otimes \mathbb{C}$. The divisibility relations extending the divisibility theorem from section 3.2 to the case of hypersurfaces in \mathbb{P}^n with isolated singularities, the Thom Sebastiani theorems for the orders of the homotopy groups and other results on the topology of the complements to are discussed in [134]. For some results in non-isolated case, cf. Section 3.8.

3.7. Twisted Alexander Invariants. A generalization of Alexander polynomials was proposed in the context of knot theory which uses as an additional input a linear representation of the fundamental group (cf. [123] for a discussion of this generalization). A twisted version of Alexander polynomials of algebraic plane curves was considered in [44]. In [143] a multivariable extension of this construction called a *characteristic varieties of a CW complex twisted by a unitary representation* was defined as follows.

Let $\pi : \pi_1(X) \rightarrow U(V)$ be a unitary representation of the fundamental group of a CW complex such that $H_1(X, \mathbb{Z})$ is a free abelian group of a positive rank ²². Here V is a complex vector space endowed with a Hermitian bilinear form and viewed as a left $\mathbb{C}[\pi_1(X)]$ -module. Let \tilde{X} be the universal cover of X . For a (left or right) module M over the algebra $\mathbb{C}[\pi_1(X)]$ (which is associative but possibly non-commutative), we denote by M^b the module obtained by restriction of the coefficients to the group algebra $\mathbb{C}[\pi'_1(X)] \subset \mathbb{C}[\pi_1(X)]$ of the commutator subgroup $\pi'_1(X)$ of $\pi_1(X)$. Let $C_*(\tilde{X})$ denotes

²¹which is also the universal cover for $n \geq 2$ since the fundamental group is abelian in this case

²²torsion freeness condition of $H_1(X, \mathbb{Z})$ is introduced to simplify the exposition.

chain complex of \tilde{X} endowed with the natural structure of (a right) $\mathbb{C}[\pi_1(X)]$ -module. Consider the following complex of tensor products of $\mathbb{C}[\pi_1'(X)]$ -modules.

$$(41) \quad C_*(\tilde{X})^b \otimes_{\mathbb{C}[\pi_1'(X)]} V^b : g(c \otimes v) = cg^{-1} \otimes gv$$

The group $\pi_1(X)$ acts on the module (41) and the restriction of this action to the commutator $\pi_1'(X)$ is trivial. Hence (41) obtains the structure of $\mathbb{C}[H_1(X, \mathbb{Z})]$ -module. It passes to the homology of the complex (41). We denote the resulting homology modules as $H_i(X_{ab}, V_{ab})$

Definition 3.25. The support of $\mathbb{C}[H_1(X, \mathbb{Z})]$ -module $\Lambda^l H_i(\tilde{X}_{ab}, V_{ab})$ we call *the ρ -twisted degree i , l -th characteristic variety of X* . This is a subvariety of the torus $\text{Spec} \mathbb{C}[H_1(X, \mathbb{Z})]$ which we denote as $Ch_i^l(X, \rho)$.

In the case when $rk H_1(X, \mathbb{Z}) = 1$ the support is a finite subset of \mathbb{C}^* and hence the zero set of a unique monic polynomial of degree $\text{Card}(Ch_i^l(X, \rho))$. More generally, a surjection $\epsilon : H_1(X, \mathbb{Z}) \rightarrow \mathbb{Z}$ defines the surjection of the group algebras and hence the embedding $\epsilon^* : \mathbb{C}^* = \text{Spec} \mathbb{C}[\mathbb{Z}] \rightarrow \text{Spec} \mathbb{C}[H_1(X, \mathbb{Z})]$. The Alexander polynomial $\Delta_n^l(X, \rho, \epsilon)$ is the unique monic polynomial of minimal degree having in \mathbb{C}^* the roots $(\text{Im} \epsilon^*) \cap Ch_n^l(X, \rho)$.

We refer to [143] Sect. 5 for other results on the relation between the twisted Alexander polynomials and characteristic varieties in the context of complements to plane curves. For example the cyclotomic property of the roots of Alexander polynomials becomes the following: the roots of a ρ -twisted Alexander polynomial belong to a cyclotomic extension of the extension of \mathbb{Q} generated by the eigenvalues of ρ (cf. [143], Th. 5.4).

3.8. Alexander Invariants of the complements without isolatedness properties. Investigations of the Alexander invariants of the complements to hypersurfaces with isolated singularities which were discussed in sections 3.1, 3.2, 3.5 and 3.6 were extended to the case of hypersurfaces with non-isolated singularities and further to smooth quasi-projective varieties by L.Maxim and his collaborators (cf. [158], [151],[148]). The D -modules (cf. [186],[192]), the category of complexes of constructible sheaves, the perverse sheaves and peripheral complex (going back to [41] and first studied in this context in [158]) are the key technical tools used by these authors. We will review two most important outcomes of this approach: propagation of characteristic varieties and extension of divisibility theorem for Alexander polynomials. The propagation property of characteristic varieties was first noticed in the context of arrangements of hyperplanes (cf. [82], [71]) and extended further in [149],[150]. The key observation is pure topological and concerns the spaces satisfying a version of cohomological duality.

Definition 3.26. (cf. [31], [71], [149]) Let X be a finite CW complex and $G = \pi_1(X, x_0)$, $x_0 \in X$. A topological space is called a duality space of dimension n if $H^p(X, \mathbb{Z}[G]) = 0$, $p \neq n$ and $H^n(X, \mathbb{Z}[G])$ is non-zero and torsion free. A spaces X is called *an abelian duality space* if for $A = H_1(X, \mathbb{Z})$ one has $H^p(X, A) = 0$, $p \neq n$ and $H^n(X, A) \neq 0$ is torsion free.

Theorem 3.27. (cf. [150], Theorem 3.16) *Let X be an abelian duality space of dimension n . Then the cohomology jumping loci of the characters of the fundamental group $\pi_1(X)$ satisfy the following properties (the so call propagation package ([149],[150]):*

(i) *Propagation: Subvarieties $\mathcal{V}^i(X)$ form descending chain:*

$$(42) \quad \mathcal{V}^n(X) \supseteq \mathcal{V}^{n-1}(X) \supseteq \dots \supseteq \mathcal{V}^0(X)$$

(ii) *Codimension bound:*

$$\text{codim} \mathcal{V}^{n-i} \geq i$$

(iii) *Irreducible components:* if V is an irreducible component of codimension d in $\mathcal{V}^n(X)$ then $V \subset \mathcal{V}^{n-d}$

(iv) *Generic vanishing:* for characters ρ of the fundamental group $\pi_1(X)$ in a Zariski open set in $\text{Hom}(\pi_1(X), \mathbb{C}^*)$ one has $H^i(X, L_\rho) = 0$ for $i \neq \dim X$.

(v) *Signed Euler characteristic property:* $(-1)^{\dim X} \chi(X) > 0$

(vi) *Betti numbers inequality:* $b_i(X) > 0$ for $0 \leq i \leq n$ and $b_1(X) \geq n$.

Use of jump loci of constructible complexes on semi-abelian varieties (which are Albanese varieties of X in appropriate sense) is the key step in the proof of this result.

This theorem suggests the problem of identifying the abelian duality spaces. In this direction one has the following:

Theorem 3.28. (cf. [149]) (i) *Let X a compact Kahler manifold which is abelian duality space. Then X is biholomorphic to an abelian variety.*

(ii) *Let X be quasi-projective manifold, such the albanese map $X \rightarrow \text{Alb}(X)$ is proper. Then X is an abelian duality space. In particular a complement to a union of hypersurfaces in \mathbb{C}^n satisfies the propagation package.*

Next we will describe the identity having as a special case the divisibility relation between product of local Alexander polynomials of singularities and the global Alexander polynomial from section 3.2 in the case of curves and in [134] in higher dimensions. Let $f(z_0, \dots, z_{n+1}) = 0$ be a homogeneous polynomial of degree d having as its zero set the hypersurface $V_f \subset \mathbb{P}^{n+1}$ and let $f_d(z_1, \dots, z_n) = f(0, z_1, \dots, z_n, 1)$ be the equation of the intersection of V_f with the hyperplane H_∞ at infinity. The map

$$(43) \quad f : \mathbb{P}^{n+1} \setminus (H_\infty \cup V_f) \rightarrow \mathbb{C}^*$$

which is the restriction of $f : \mathbb{C}^{n+1} = \mathbb{P}^{n+1} \setminus \mathbb{P}^n \rightarrow \mathbb{C}$ given by $(1, z_1, \dots, z_{n+1}) \rightarrow f(1, z_1, \dots, z_{n+1})$ allows to define the infinite cyclic cover $\mathbb{P}^{n+1} \setminus (H_\infty \cup V_f)$ corresponding to the kernel of the map $\pi_1(\mathbb{P}^{n+1} \setminus (H_\infty \cup V_f)) \rightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z}$ induced by (43). For each $1 \leq i \leq n$ one has a well defined (up to a unit of $\mathbb{C}[t, t^{-1}]$) polynomial $\Delta_i(t)$ which is the order of the $\mathbb{C}[t, t^{-1}]$ -module $H_i(\mathbb{P}^{n+1} \setminus (H_\infty \cup V_f))$ (cf. [158]). Let $\psi_f \mathbb{Q}_{\mathbb{C}^{n+1}}$ denotes Deligne's nearby cycles complex associated to f and let $\psi_i(t)$ be the order of $H^{2n+1-i}(V_f \cap \mathbb{C}^{n+1}, \psi_f \mathbb{Q}_{\mathbb{C}^{n+1}})$. Note that $\Delta_i(t) = \psi_i(t)$ for $i < n$ and $\Delta_n(t)$ divides ψ_n . Let $h(t)$ be the characteristic polynomial of the Milnor fiber $f_d(z_1, \dots, z_n) = 1$. The final ingredient is the determinant $\det \phi$ of the bilinear form:

$$(44) \quad H_{n+1}(\mathbb{P}^{n+1} \setminus (H_\infty \cup V_f), \mathbb{Q}(t)) \otimes H_{n+1}(\mathbb{P}^{n+1} \setminus (H_\infty \cup V_f), \mathbb{Q}(t)) \xrightarrow{\phi} \mathbb{Q}(t)$$

given by $(\alpha, \beta) \rightarrow \alpha \cdot i(\beta)$ where $i : \partial(\mathbb{P}^{n+1} \setminus (H_\infty \cup V_f)) \rightarrow \mathbb{P}^{n+1} \setminus (H_\infty \cup V_f)$ is the embedding of the boundary $\partial(\mathbb{P}^{n+1} \setminus (H_\infty \cup V_f))$ of a small regular neighborhood of $V_f \cup H_\infty$, and " \cdot " is the Poincare pairing in the homology of the infinite cyclic cover (cf. [162]) of the pair $(\mathbb{P}^{n+1} \setminus (H_\infty \cup V_f), (\partial \mathbb{P}^{n+1} \setminus (H_\infty \cup V_f)))$. With these definitions one can describe the relation between the global Alexander invariants and the data of the starta of a singular hypersurface as follows.

Theorem 3.29. (cf. [151]) *Let $f \in \mathbb{C}[x_1, \dots, x_{n+1}]$ be defining polynomial of an affine hypersurface $F \subset \mathbb{C}^{n+1}$, f_d be the top degree form of homogenization of f with corresponding Milnor fiber F_h and $h_i(t)$ be the Alexander polynomial associated to $H_i(F_h)$. Let $\psi_n(t)$ denotes the Alexander module associated to $H_c^{2n-i}(V_0, \psi_f \mathbb{Q}_{\mathbb{C}^{n+1}})$. Finally, let ϕ be the intersection form on the infinite cover associated with f . Then*

$$(45) \quad h_n(t) \psi_n(t) = \delta_n(t)^2 \det(\phi)$$

In the case when V_f is transversal to H_∞ and has only isolated singularities, this is translated into the following relation (cf. [151]):

$$(46) \quad (t-1)^{|e(\mathbb{P}^{n+1} \setminus (H_\infty \cup V_f))| + (-1)^{n+1}} (t^d - 1)^{\frac{1}{d}((d-1)^{n+1} + (-1)^n)} \prod_{p \in \text{Sing}(V_f)} \Delta_p = \Delta_n^2 \det(\phi)$$

i.e. one obtains topological interpretation of terms converting divisibility into equality. In the case of plane curves one recovers the result in [44].

4. IDEALS OF QUASIADJUNCTION AND MULTIPLIER IDEALS

Now we will turn to calculation of components of characteristic varieties in terms of dimensions of linear systems determined by the singular points of the curve. For earlier expositions of this material in the case of plane curves cf. [142] or [18].

4.1. Ideals and polytopes of quasi-adjunction.

Definition 4.1. Let X be a complex n -dimensional manifold, $P \in X$ be a point and let B_P be a small ball centered at P . Let D be a reduced divisor on X containing P , $f \in \mathcal{O}_P$ be a reduced germ of a holomorphic function having D as its divisor and let $f(x_1, \dots, x_n) = f_1(x_1, \dots, x_n) \cdot \dots \cdot f_r(x_1, \dots, x_n)$ be its prime factorization. Let

$$(47) \quad (j_1, \dots, j_r), (m_1, \dots, m_r), 0 \leq j_i < m_i$$

be two arrays of integers. Consider abelian branched cover V_{m_1, \dots, m_r} of B ramified over D and corresponding to the component-wise reduction $H_1(B_P \setminus D \cap B_P, \mathbb{Z}) = \mathbb{Z}^r \rightarrow \bigoplus_1^r \mathbb{Z}/m_i \mathbb{Z}$ (cf. Section 3.3). After selecting local coordinates near P , this cover can be viewed as a germ at the origin in \mathbb{C}^{n+r} with coordinates $(z_1, \dots, z_r, x_1, \dots, x_n)$ given by the local equations:

$$(48) \quad z_1^{m_1} = f_1(x_1, \dots, x_n), \dots, z_r^{m_r} = f_r(x_1, \dots, x_n),$$

The covering map of subvariety (48) onto B_P is given by projection

$$(z_1, \dots, z_r, x_1, \dots, x_n) \rightarrow (x_1, \dots, x_n).$$

The ideal of quasi-adjunction $\mathcal{A}(f|j_1, \dots, j_r|m_1, \dots, m_r)$ of $f(x_1, \dots, x_n)$, corresponding to the array (47), is the ideal of germs $\phi \in \mathcal{O}_P$ in the local ring of P , such that the n -form

$$(49) \quad \omega_\phi = \frac{\phi(x_1, \dots, x_n) dx_1 \wedge \dots \wedge dx_n}{z_1^{m_1 - j_1 - 1} \dots z_r^{m_r - j_r - 1}}$$

defined on the smooth locus of (48), can be extended over a log resolution of (48).

One shows that the ideal $\mathcal{A}(f|j_1, \dots, j_r|m_1, \dots, m_r)$ is independent of a resolution of (48) and that it depends only on the vector:

$$(50) \quad \left(\frac{j_1 + 1}{m_1}, \dots, \frac{j_r + 1}{m_r} \right) \in [0, 1]^r \subset \mathbb{R}^r$$

rather than on specific values of j_i, m_i . To see that dependence is only on (50), let us consider the following resolution of the germ (48): select a log-resolution

$$(51) \quad \mu : (\tilde{X}, \tilde{D}) \rightarrow (B_P, B_P \cap D)$$

with the exceptional set $E = \bigcup_1^K E_k$ i.e. assume that $E \cup \mu^*(D)$ is a normal crossing divisor. Consider a resolution \mathcal{X}_m of the *normalization* $\tilde{X}_{m_1, \dots, m_r}$ of the fiber product $\tilde{X} \times_X V_{m_1, \dots, m+r}$

$$(52) \quad \begin{array}{ccc} \mathcal{X}_{m_1, \dots, m_r} & & \\ \downarrow & & \\ \tilde{X}_{m_1, \dots, m_r} & & \\ \downarrow & & \\ \tilde{X} \times_X V_{m_1, \dots, m_r} & \rightarrow & V_{m_1, \dots, m_r} \\ \tilde{\pi}_{m_1, \dots, m_r} \downarrow & & \pi_{m_1, \dots, m_r} \downarrow \\ \tilde{X} & \xrightarrow{\mu} & \mathbb{C}^n \end{array}$$

The normalization $\tilde{X}_{m_1, \dots, m_r}$ has abelian quotient singularities and hence is \mathbb{Q} -Gorenstein Kawamata log-terminal (cf. [27]). In particular an n -form on the smooth locus of $\tilde{X}_{m_1, \dots, m_r}$ extends over exceptional locus of $\mathcal{X}_m \rightarrow X_{m_1, \dots, m_r}$ iff it is holomorphic on the smooth locus of $\tilde{X}_{m_1, \dots, m_r}$ (cf. [102]). Let $a_{k,i} = \text{ord}_{E_k} \mu^*(f_i)$, $c_k = \text{ord}_{E_k} \mu^*(dx_1 \wedge \dots \wedge dx_n)$, $e_k(\phi) = \text{ord}_{E_k} \mu^*(\phi)$. We obtain that $\tilde{\pi}_{m_1, \dots, m_r}(\omega_\phi)$ extends over the smooth locus of normalization $\tilde{X}_{m_1, \dots, m_r}$ iff for all irreducible components $E_k \subset X$ one has:

$$(53) \quad \sum_i a_{k,i} \frac{j_i + 1}{m_i} > \sum_i a_{k,i} - e_k - c_k - 1 \quad e_k = e_k(\phi)$$

Definition 4.2. For a choice of collection $\mathcal{E} = \{e_k\}$, $k = 1, \dots, K$ of non-negative integers labeled by irreducible components of resolution (51) and such that there exist a germ $\phi \in \mathcal{O}_P$ such that $e_k = e_k(\phi)$, the closure $\mathcal{P}(\mathcal{E})$ in the r -cube $[0, 1]^r$ of the set of solutions to inequalities (53) is called a *polytope of quasi-adjunction*²³ of the germ $f = f_1 \cdot \dots \cdot f_r$.

Using polytopes $\mathcal{P}(\mathcal{E})$ or inequalities (53), one can describe necessary and sufficient conditions on (47) assuring extendability of the form ω_ϕ (cf. (49)) on resolution of singularities of abelian covers in the tower (48). While the polytopes $\mathcal{P}(\mathcal{E})$ form a set which is partially ordered by inclusion and closely related to Alexander invariants, sometimes it is convenient also to use a partition of $[0, 1]^r$ into a union of (locally closed) *non-intersecting and possibly non-convex* polytopes \mathcal{Q} compatible with polytopes $\mathcal{P}(\mathcal{E})$.

To describe these polytopes consider the hyperplanes in \mathbb{R}^r

$$(54) \quad \sum_i a_{k,i}(\gamma_i - 1) + c_k = \epsilon_k, \epsilon_k \in \mathbb{Z}_{<0}$$

labeled by the exceptional components E_k , $k = 1, \dots, K$ of resolution μ (cf. (51)) where collections of integers ϵ_k are such that there exist $\phi \in \mathcal{O}_P$ satisfying $\epsilon_k = e_k(\phi)$ and the remaining coefficients in (54) are defined just before (53). The hyperplanes (54) partition the unit cube $[0, 1]^r$ into a union of locally closed polytopes \mathcal{Q} of various dimensions, such that $v = (\dots, \gamma_i, \dots) = (\dots, \frac{j_i+1}{m_i}, \dots)$ and $v' = (\dots, \gamma'_i, \dots) = (\dots, \frac{j'_i+1}{m'_i}, \dots)$ belong to the same polytope iff v, v' satisfy the same sets of inequalities (53). Equivalently, each polytope \mathcal{Q} coincides with the polytope formed by an equivalence class of points in $[0, 1]^r$ when one considers points equivalent if they have the same set of polytopes $\mathcal{P}(\mathcal{E})$ containing each. We will call the polytopes \mathcal{Q} , defining the decomposition of the unit cube into disjoint union,

²³By a polytope we mean a set of solutions to a finite collection of inequalities. All polytopes considered here are bounded (subsets of a unit cube) and hence are the convex hulls of the sets of their vertices. Faces are subsets of the boundary of a polytope which are the convex hulls of a subset of the set of vertices of the polytope. The dimension of a polytope (including a face) is the maximal dimension of the balls in its interior (with the dimension of a vertex being zero).

the strict polytopes of quasi-adjunction to distinguish them from the ordinary polytopes of quasi-adjunction $\mathcal{P}(\mathcal{E})$.

For a fixed vector v with coordinates (50) the germs ϕ which satisfy the inequalities (53) form the ideal $\mathcal{A}_Q \subset \mathcal{O}_P$ depending only on the polytope Q and not on a choice of $v \in Q$. In particular, the ideal of quasi-adjunction $\mathcal{A}(f|j_1, \dots, j_r|m_1, \dots, m_r)$ is the ideal \mathcal{A}_Q such that $(\dots, \frac{j_i+1}{m_i}, \dots) \in Q$.

Definition 4.3. 1. We define an ideal of quasi-adjunction as an ideal coinciding with an ideal \mathcal{A}_Q for some strict polytope of quasi-adjunction $Q \subset [0, 1]^r$.

2. For a polytope in $[0, 1]^r \subset \mathbb{R}^r$ as above, a *face of quasi-adjunction* \mathcal{F} of dimension p is the p -dimensional intersection of the polytope of quasi-adjunction with an affine half-space in \mathbb{R}^r , transversal to all coordinated hyperplanes in \mathbb{R}^r , such that each point of this intersection is the boundary point of the polytope and the half-space. ²⁴

Corollary 4.4. *The polytopes, ideals and faces of quasi-adjunction depend only on the germ $f(x_1, \dots, x_n)$ and not on a resolution. The inequalities (53) show that there exist firstly, the collection of polytopes \mathcal{P} , which are the unions of the polytopes Q , and secondly to each of these polytopes correspond the ideal \mathcal{A}_Q such that for any $v \in Q$ and a germ $\phi \in \mathcal{O}_P$, the corresponding form ω_ϕ can be extended to a holomorphic form on X_{m_1, \dots, m_r} iff $\phi \in \mathcal{A}_Q$. The boundary of such a polytope Q is a union of faces each being also a face the boundary of a polytope \mathcal{P} and each such face being a close polytope in the intersection of hyperplanes given by the equations (54) with*

$$(55) \quad \gamma_i = 1 - \frac{j_i + 1}{m_i} \quad i = 1, \dots, r.$$

4.2. Ideals of quasi-adjunction and multiplier ideals. For an exposition of the theory of multiplier ideals we refer to [127] Part III. Here we recall the key definitions and relate them to the ideals of quasi-adjunction.

Definition 4.5. (cf. [127] Def.9.2.1). Let X be a smooth complex variety and $D \in \text{Div}(X) \otimes \mathbb{Q}$ an effective \mathbb{Q} -divisor. Let $\mu : X' \rightarrow X$ be a log resolution, $K_{X'/X} = K_{X'} - \mu^*(K_X)$ is a relative canonical class. The multiplier ideal sheaf $\mathcal{J}(X, D)$ of D is the direct image

$$(56) \quad \mu_*(K_{X'/X} - \lfloor \mu^*(D) \rfloor)$$

where for a \mathbb{Q} -divisor $D = \sum \gamma_i D_i$, $a_i \in \mathbb{Q}$, $D_i \in \text{Div}(X)$, $\lfloor D \rfloor = \sum \lfloor \gamma_i \rfloor D_i$ and $\lfloor \gamma \rfloor \in \mathbb{Z}$ denotes the integral part of $\gamma \in \mathbb{Q}$.

For a collection of \mathbb{Q} -divisors F_1, \dots, F_r (resp. the ideals \wp_1, \dots, \wp_r), the mixed multiplier ideal $\mathcal{J}(c_1 F_1, \dots, c_r F_r)$ (resp. $\mathcal{J}(\wp_1^{c_1} \cdot \dots \cdot \wp_r^{c_r})$) (cf. [127] 9.2.8, [2]) is defined as the multiplier ideal

$$(57) \quad \mu_*(K_{X'/X} - \lfloor c_1 F_1 + \dots + c_r F_r \rfloor)$$

(in the case of mixed multiplier ideals attached to \wp_1, \dots, \wp_r , the divisors F_i are determined from $\wp_i \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$).

Proposition 4.6. *The ideal of quasi-adjunction $\mathcal{A}(f|j_1, \dots, j_r|m_1, \dots, m_r)$ coincides with the multiplier ideal $\mathcal{J}(\sum((1 - \frac{j_i+1}{m_i})D_i)$*

Proof. A germ ϕ is a section of the sheaf given by (56) where $D = \sum \gamma_i D_i$ if and only if it satisfies the inequality

$$(58) \quad e_k(\phi) + c_k - \sum \gamma_i a_{k,i} \geq 0$$

²⁴i.e. the set of solutions to a linear inequality

This is equivalent to (53). \square

Following [145] one can define the LCT polytope, which is a multi-divisor analog of the log canonical threshold (cf. [127]):

Definition 4.7.

$$LCT(D_1, \dots, D_r) = \{(\lambda_1, \dots, \lambda_r) \in \mathbb{R}_+^r \mid (X, \sum_1^r \lambda_i D_i) \text{ is log canonical}\}$$

Proposition 4.8. (*log-canonical polytopes and polytopes of quasi-adjunction*). *Let $I : [0, 1]^r \rightarrow [0, 1]^r$ be the involution of the unite cube given by $(\gamma_1, \dots, \gamma_r) \rightarrow (1-\gamma_1, \dots, 1-\gamma_r)$. Then $LCT(D_1, \dots, D_r)$ is the I -image of the part in the interior of the unite cube of the boundary of the polytope of quasiadjunction containing the origin.*

Proof. Clearly for the vectors (50) sufficiently close to zero, the n -form (49) is extendable for any ϕ in the local ring of P i.e. the “ideal” of quasi-adjunction is not proper. The ideal of quasi-adjunction is constant for all vectors (50) in the polytope bounded by the faces of quasi-adjunction closest to the origin. Hence the claim follows from the characterization of log-canonical thresholds in terms of multiplier ideals (cf. [127] Section 9.3.B) and Proposition 4.6). \square

Remark 4.9. 1. We often use the following correspondence between vectors (50) and the characters of local fundamental group of the complement to the germ of $D = \bigcup D_i$. Consider the embedding of $[0, 1]^r$ into $H^1(B_P \setminus B_P \cap \bigcup D_i, \mathbb{R})$ using the basis dual to the meridians δ_i of the divisors D_i i.e. assigning to a vector $(\gamma_1, \dots, \gamma_r)$ the cohomology class h such that $h(\delta_i) = \gamma_i$.

This embedding induces the identification of the cube $[0, 1]^r$ with the group of characters of the local fundamental group. Indeed, any point in $H^1(B_P \setminus B_P \cap \bigcup D_i, \mathbb{R})$ via exponential map $t \rightarrow \exp(2\pi i t)$ determines an element in $H^1(B_P \setminus B_P \cap \bigcup D_i, U(1))$ i.e. a unitary character of the local fundamental group²⁵. Vice versa, to a character in $H^1(B_P \setminus B_P \cap \bigcup D_i, U(1))$ we can assign its unique preimage belonging to the fundamental domain of the action of $H^1(B_P \setminus B_P \cap \bigcup D_i, \mathbb{Z})$ on $H^1(B_P \setminus B_P \cap \bigcup D_i, \mathbb{R})$ which is the unit cube in the coordinates of the above basis.

2. Exact sequence of a pair $(B_P, B_P \setminus B_P \cap D)$ gives the identification:

$$(59) \quad H^1(B_P \setminus B_P \cap D, \mathbb{R}) = H^2(B_P, B_P \setminus D, \mathbb{R}) = H_2(B_P \cap D, \partial(B_P \cap D), \mathbb{R})$$

and similarly for the coefficients \mathbb{Z} and $U(1)$. $B_P \cap D$ is homeomorphic to a bouquet of disks and $\partial(B_P \cap D)$ is a disjoint union of circles, both having the cardinality coinciding with the number of branches of D at P . In particular (59) is a vector space with canonical direct sum decomposition with summands corresponding to the branches of D at P .

4.3. Local polytopes of quasi-adjunction and spectrum of singularities.

4.3.1. *Cyclic Theory.* The relation between the constants of quasi-adjunction (i.e. the faces of quasi-adjunction in cyclic case) and the Hodge theory was first discussed in [152] in the case of isolated singularities. This was continued in [36], [39].

Let $f(x_0, x_1, \dots, x_n)$ be a germ of an isolated singularity at the origin. Recall that the cohomology of the Milnor fiber F_f support a mixed Hodge structure defined by Steenbrink and Varchenko (cf. [207], [180], [216], [11]). This mixed Hodge structure provides the cohomology $H^n(F_f, \mathbb{C})$ with a decreasing filtration $F^p \cap H_\lambda \subseteq F^{p-1} \cap H_\lambda$ on each summand

²⁵this is a local version of the global construction used in Prop. 3.9. Here the rank of the fundamental group coincides with the number of irreducible components of the divisor.

of the direct sum decomposition of $H^n(F_f, \mathbb{C})$ into the eigenspaces of the monodromy operator. A rational number α is an element of the spectrum of f of multiplicity k if there is an eigenvalue of the monodromy λ such that

$$(60) \quad n - p - 1 < \alpha \leq n - p \quad \exp(2\pi i \alpha) = \lambda \quad \dim F^p \cap H_\lambda / F^{p+1} \cap H_\lambda = k$$

(cf. [152])

Theorem 4.10. *A rational number α belongs to the spectrum of $f(x_0, \dots, x_n) = 0$ and satisfies $0 < \alpha < 1$ if and only iff $-\alpha$ is a face of quasi-adjunction of f (i.e. is a constant of quasi-adjunction in terms of [130]; the definition given there is in terms of the adjoint ideals and is a special case of Def. 4.3 corresponding to cyclic covers of \mathbb{C}^2 .)*

There are many cases when spectrum can be easily calculated explicitly. In the case of quasi-homogeneous singularities (with weights w_0, \dots, w_n i.e. when defining polynomial is a sum of monomials $a x_0^{m_0} \dots x_n^{m_n}$ such that $\sum_{i=0}^n w_i m_i = 1$) the generating function i.e. $\sum_\alpha t^\alpha$ where α runs through the spectrum of the singularity is given by

$$(61) \quad \frac{1}{t} \prod_{i=0}^n \frac{t^{w_i} - t}{1 - t^{w_i}}$$

cf. [187]; we included the factor $\frac{1}{t}$ since we use the same normalization of the spectrum as in [152] where the left end of the spectrum is -1 , cf. (60). In particular for an irreducible germ with one characteristic pair $x^p = y^q$ we obtain $(t^{-\frac{1}{p}} + \dots + t^{-\frac{p-1}{p}})(t^{\frac{1}{q}} + \dots + t^{\frac{q-1}{q}})$ which for the cusp $x^2 = y^3$ gives $t^{-\frac{1}{6}} + t^{\frac{1}{6}}$. Explicite calculations of the spectrum and hence the constants of quasi-adjunction of irreducible plane curve singularities in terms of Puiseux pairs was made in [187] For related calculations see [36], [96].

4.3.2. *Local abelian theory.* Calculations of the polytopes of quasi-adjunction of singularities of plane curves were made in [138],[39],[40]. Curves on surfaces with rational singularities were considered in [2].

Several examples of calculation of polytopes and ideals of quasi-adjunction for isolated non-normal crossings were considered in [140] [143] which lead to expressions for characteristic varieties mentioned in Example 3.23. Related results are presented in [145].

4.4. **Ideals of quasi-adjunction and homology of branched covers.** One of the first applications of ideals of quasi-adjunction was a procedure that allowed to express the Hodge numbers of abelian covers in terms of dimensions of linear systems determined by the branch locus and the data of singularities. The relation between the constants and ideals of quasi-adjunction and Hodge numbers of the cyclic covers in [130] followed by numerous works (cf. for example [152], [215], [170], [171], [37]), many using the terminology of multiplier ideals. ²⁶

In the case of curves on surfaces one has the following result. For a high dimensional extension leading to calculations of the dimensions of the space of holomorphic forms on the resolutions of branched covers (the only Hodge numbers which are independent of resolutions) see [83], [134], [37] etc.

Proposition 4.11. *Let $f : \bar{X}_\phi \rightarrow X$ be an abelian branched cover of a smooth projective surface X ramified over a divisor D with r irreducible components and let $f_*(\mathcal{O}_{\bar{X}_\phi}) = \oplus L_\chi^{-1}$ be the decomposition (18). For a character χ , let $\mathcal{A}_\chi \subset \mathcal{O}_X$, $\chi = \exp(\dots 2\pi i \alpha(\chi), \dots)$, $\alpha \in$*

²⁶the term introduced by A.Nadel in 1990, cf. [169]

$[0, 1]^r$ be the ideal sheaf having as stalk at $p \in \text{Sing}(D)$ the ideal of quasi-adjunction of singularity at p corresponding to the polytope of quasi-adjunction containing $\alpha(\chi)$

Then the dimension of the χ -eigenspace of the covering group acting on the space of holomorphic 1-forms on a resolution \widetilde{X}_ϕ of singularities of X_ϕ is given by

$$(62) \quad \dim H^0(\widetilde{X}_\phi, \Omega_{\widetilde{X}_\phi}^1)_\chi = H^1(X, K_X \otimes L_\chi \otimes \mathcal{A}_{\text{Sing}D}(\chi))$$

4.5. Hodge decomposition of characteristic varieties.

4.5.1. *Calculation of Characteristic varieties: Deligne extensions.* In this section we sketch a method for calculation of the variety of unitary characters with corresponding local systems having positive first Betti number and which is based on the Hodge theoretical description of the cohomology of local systems due to Deligne and Timmersheidt ([210],[67],[68],[69], related works include works of S.Zucker, M.Saito, El Zein and Illusie cf. discussion in H.Esnault review of [210] in Math. Reviews). For details we refer to [143], [37],[84], [19].

The starting point is deRham type description of the cohomology of local systems on smooth quasi-projective varieties using logarithmic forms already mentioned in section 3.5.1. With notations used in this section, we assume now that Y is smooth and quasi-projective and that \bar{Y} is a smooth projective compactification, such that $\bar{Y} \setminus Y$ is a normal crossing divisor $Y^\infty = \bigcup_{i \geq 1} Y_i$. The connection ∇ in (26) can be selected to be holomorphic on Y and meromorphic on \bar{Y} i.e. in having a matrix given in local coordinates by meromorphic functions with poles along $\bar{Y} \setminus Y$. Moreover, this selection can be made so that the matrix of connection has as its entries the 1-forms having logarithmic poles along Y^∞ (i.e. linear combinations: $\omega = \sum \alpha_i \frac{dz_i}{z_i}$ where z_i are local equations of irreducible components of Y^∞ and α_i are holomorphic in a chart in \bar{Y} cf. [67], Prop.3.2). Globally, logarithmic connection can be viewed as a \mathbb{C} -linear map $E \rightarrow \Omega^1(\log Y^\infty) \otimes E$ where E is a vector bundle on \bar{Y} satisfying Leibnitz rule. Note that the matrix of connection, in the rank one case, is just a logarithmic 1-form. One has a well defined Poincare residue map $\Omega^1(\log Y^\infty) \otimes E \rightarrow \mathcal{O}_{Y_i} \otimes E$ along each irreducible component Y_i . Locally, residue depends on trivialization and globally on the bundle E .

Definition 4.12. *Deligne's extension of a flat connection ∇ on Y is a logarithmic connection on a bundle E on \bar{Y} such that its residues satisfy the inequality*

$$(63) \quad 0 \leq \text{Res}_{Y_i}(\nabla) < 1$$

for any i .

The description of the jumping loci of local systems in terms of Deligne's extensions, is based on the degeneration of the Hodge-DeRham spectral sequence

$$(64) \quad E_1^{p,q} = H^p(\Omega^q(\log Y^\infty) \otimes V_\rho) \rightarrow H^{p+q}(V_\rho).$$

in term E_1 . Here V_ρ is a Deligne's extensions of a unitary connection corresponding to the local system \mathcal{V}_ρ where $\rho : \pi_1(Y) \rightarrow U(n)$ is a unitary representation (cf. [210]). For a rank one local system \mathcal{V}_χ , corresponding to a character $\chi = \exp(2\pi i u)$ and the Deligne extension L_χ of the corresponding connection, this degeneration implies $\text{rk} H^1(Y, \mathcal{V}_\chi) = \text{rk} H^0(\Omega^1(\log Y^\infty) \otimes L_\chi) + \text{rk} H^1(L_\chi)$ and the following:

Proposition 4.13. *A subset of $[0, 1]^{b_1(X \setminus D)} \subset H^1(X \setminus D, \mathbb{R})$ such that the Deligne extension of the connection corresponding to a character $\chi = \exp(2\pi i u)$ coincide with a fixed line bundle L is a polytope Δ_L in $H^1(X \setminus D, \mathbb{R})$. The rank $\text{rk} H^1(Y, \mathcal{V}_\chi)$ is constant when u*

varies within Δ_L . The subset of the torus $\mathbb{C}^{*b_1(X \setminus D)} = \text{Char}H_1(X \setminus D, \mathbb{Z})$ consisting of the characters $\exp(2\pi i u)$, $u \in \Delta_L$ is a jumping set of the Hodge numbers of unitary local systems. The Zariski closure in $(\mathbb{C}^*)^{b_1(X \setminus D)}$ of this subset is translated by a finite order character a connected subgroup of $\mathbb{C}^{*b_1(X \setminus D)}$ which is an irreducible component of the characteristic variety of the fundamental group. Vice versa, any irreducible component of characteristic variety of $\pi_1(X \setminus D)$, is Zariski closure of a set $\exp(\Delta_L)$.

Indeed, after selecting a basis in $H^1(X \setminus D, \mathbb{R})$, one readily sees that inequality (63) for each component translates into a linear inequality on components of logarithm of χ in coordinates in this basis. It is not hard to see that there are only finitely many bundles on X which are the Deligne extensions of a connection corresponding to a character in $[0, 1)^{b_1(X \setminus D)}$ can occur (cf. [143]). This provides an explicit description of the unitary part of the components of characteristic varieties. Since by [9] all irreducible components of the characteristic variety are translated subtori of $\mathbb{C}^{*b_1(X \setminus D)}$, it follows that in this way we obtain all the components as the Zariski closures of the exponential images of the polytopes Δ_L .

4.5.2. Calculation of Characteristic Varieties: quasi-adjunction. We will focus on the case of characteristic varieties of curves on surfaces. Similar results are expected for characteristic varieties associated with higher homotopy groups. We refer to [137], [141] for some of the results in this direction.

Calculation in terms of ideal of quasi-adjunction is based on comparison of topological and algebro-geometric calculation of the dimensions of eigenspaces of the action of the Galois group on the homology of abelian covers given respectively by Propositions 3.15 and 4.11. However, the Proposition 3.15 considers only the characters of $\pi_1(X \setminus D)$ which values on the meridians of all irreducible components of D are non-trivial. This motivates the following definition.

Definition 4.14. Let $D = D_1 \cup D_2$ be a decomposition of a reduced divisor on a smooth projective simply connected surface X . Let $s_{D/D_1} : \pi_1(X \setminus D) \rightarrow \pi_1(X \setminus D_1)$ be the surjection of the fundamental groups induced by inclusion $X \setminus D \subset X \setminus D_1$ and let $s_{D/D_1}^{H_1}, s_{D/D_1}^{\pi_1}, s_{D/D_1}^{\pi_1/\pi_1}$ be the corresponding surjections respectively on the homology, commutator of the fundamental group and the abelianization of the commutator. Let $s_{D, D_1}^{Char_i} : \text{Char}_i^1(X \setminus D_1) \rightarrow \text{Char}_i^1(X \setminus D)$ be induced map of supports of the i -th exterior powers of the homology modules (over $\mathbb{C}[H_1(X \setminus D_1)]$ and $\mathbb{C}[H_1(X \setminus D)]$ respectively, cf. Definition 3.13). An irreducible component C_D of the characteristic variety of $\pi_1(X \setminus D)$ is called *non-essential* if there is a decomposition $D = D_1 \cup D_2$ and a component C_{D_1} of the characteristic variety of $\pi_1(X \setminus D_1)$ such that $s_{D, D_1}^{Char_i}(C_{D_1}) = C_D$. An *essential component* is an irreducible component of $\text{Char}_i(X \setminus D)$ which fails to be non-essential.

Below we describe a calculation of only essential components, for simplicity assuming that $H_1(X \setminus D, \mathbb{Z})$ has no torsion. The results certainly can be described without this assumption. In fact, the first examples of calculations of characteristic varieties (the roots of the Alexander polynomials) in terms of ideals of quasi-adjunction, were made in [130] in the cases when D is an irreducible plane curve i.e. when $H_1(\mathbb{P}^2 \setminus D, \mathbb{Z})$ is a cyclic group of order $\deg D$. See also example (4.21) where the complement is torsion as well. Non essential components may exhibit a subtle behavior: the depth of a component may increase considered as component of C_D instead of C_{D_1} . We refer to discussion of this phenomenon to [17],[19], [21].

Recall (cf. section 4.1 and Remark 4.9) that with each singular point P of the reduced divisor $D = \bigcup_1^N D_i$ on a surface X we associated a collection of polytopes of quasi-adjunction $Q_j(P)$, $j = 1, \dots, n(P)$ in $U_{s(P)} = [0, 1]^{s(P)} \subset H^1(B_P \setminus D \cap B_P, \mathbb{R}) = H_2(B_P \cap D, \partial B_P \cap D, \mathbb{R})$ (recall that B_P is a small ball in X centered at P). Note that the latter locally homology groups can be endowed with the maps to the corresponding global groups for each group of coefficients $\mathbb{K} = \mathbb{Z}, \mathbb{R}, U(1)$ leading to the diagram:

$$(65) \quad \begin{array}{ccc} H^1(B_P \setminus B_P \cap D, \mathbb{K}) & \xrightarrow{\delta_P} & H_2(B_P \cap D, \partial(B_P \cap D), \mathbb{K}) \\ \uparrow i_P & & \uparrow \epsilon_P \\ H^1(X \setminus D, \mathbb{K}) & \rightarrow & H_2(D, \mathbb{K}) \end{array}$$

Here the top horizontal map δ_P is the isomorphisms (59), the left vertical map i_P is induced by embedding and the right vertical map ϵ_P is the homology boundary map: $H_2(D, \mathbb{K}) \rightarrow H_2(D, D \setminus B_P \cap D, \mathbb{K}) = H_2(B_P \cap D, \partial(B_P \cap D), \mathbb{K})$.

For each polytope $Q_j(P) \subset H^1(B_P \setminus D \cap B_P, \mathbb{R})$ we consider preimages

$$Q_j^X(P) = (i_P^*)^{-1}(Q_j(P)) \subset H^1(X \setminus D, \mathbb{R}) \text{ and } Q_j^D(P) = \epsilon_P^{-1}(\delta_P(Q_j(P))) \subset H_2(D, \mathbb{R})$$

In (65), each group in the top row for $\mathbb{K} = \mathbb{R}$ and the group $H_2(D, \mathbb{R})$ contains the canonical fundamental domain for the action of respective lattice which one obtains taking for each group $\mathbb{K} = \mathbb{Z}$. These fundamental domains are the unit cubes in the bases corresponding to the fundamental classes of appropriate irreducible component of D . The image of each such fundamental domain in $H^1(B_P, B \setminus B_P \cap D, \mathbb{R})$, induced by embedding $H^1(B_P, B \setminus B_P \cap D, \mathbb{R}) \rightarrow \bigoplus_{P \in \text{Sing}(D)} H_2(B_P \cap D, \partial(B_P \cap D), \mathbb{R})$, is a face of the unit cube in the latter. The intersection of the image of $H_2(D, \mathbb{R})$ in $\bigoplus_P H_2(B_P \cap D, \partial(B_P \cap D), \mathbb{R})$ is either such a face, if the branches of D at P belong to different irreducible components of D , or a diagonal in such a face, if different branches at P belong to the same irreducible component of D . We denote by $U_{X,D}$ the unit cube in $H_2(D, \mathbb{R})$. From now on we will use the same notation $Q_j^X(P)$, $Q_j^D(P)$ for their intersections with the respective unite cubes: these and only parts of respective polytopes contain the information we need below.

The unit cubes in $H_2(D, \mathbb{R})$ and $H_2(D \cap B_P, \partial(D \cap B_P), \mathbb{R})$, have canonical involution corresponding to the lift of the conjugation of characters via inverse of the map induced on cohomology by $\exp : \mathbb{R} \rightarrow U(1)$. For example on each unit cube in $H^1(B_P \setminus B_P \cap D, \mathbb{R})$ this involution is given by $(u_1, \dots, u_r) \rightarrow (1 - u_1, \dots, 1 - u_r)$ and similarly in other cases. For a subset \mathcal{U} in such a cube we denote the image of this involution as $-\mathcal{U}$.

Definition 4.15. Let $\text{Sing}(D)$ be the set of singularities of D and let Q be the set of collections $Q = \{Q_j(P_k) | P_k \in \text{Sing}(D), j = 1, \dots, J(P_k)\}$ (here $J(P_k)$ is the cardinality of the set of local polytopes of quasi-adjunction of singularity P_k) of (strict²⁷) local polytopes of quasi-adjunction $Q_j(P_k)$, one for each singularity $P_k \in \text{Sing}(D)$.

a. *The divisorial global polytope of quasi-adjunction* is the intersection

$$(66) \quad \mathcal{G}_Q = \bigcap_{P_k \in \text{Sing}(D), Q_j(P_k) \in Q} pr_{P_k}^{-1} Q_j(P_k) \subset U_{X,D} \subset H_2(D, \mathbb{R})$$

of preimages of polytopes of quasiadjunction, one for each singular point of D .²⁸

b. A *global divisorial face of quasi-adjunction* is a face \mathcal{F} of a polytope \mathcal{G}_Q (cf. (66)) corresponding to a collection Q of local polytopes of quasi-adjunction. We say that a face \mathcal{F} of a global polytope of quasi-adjunction *correspond to a subset* $S \subset \text{Sing}(D)$ if \mathcal{F} is a

²⁷strict polytopes of quasiadjunction were described just before Def. 4.3

²⁸the number of global polytopes of quasi-adjunction is at most $\prod_{k \in \text{Sing}(D)} n(P_k)$ where $n(P_k)$ is the number of local polytopes of quasi-adjunction at singular point P_k

face of a polytope determined already by the local polytopes of singularities only from \mathcal{S} : $\bigcup_{P_k \in \mathcal{S}, Q_j(P_k) \in \mathcal{Q}} PR_{P_k}^{-1} Q_j(P_k)$.

c. *The sheaf of quasi-adjunction \mathcal{A}_Q (or $\mathcal{A}_{\mathcal{G}(Q)}$) corresponding to a choice collection Q of local polytopes of quasi-adjunction, is the ideal sheaf in \mathcal{O}_X having as the stalk at $P \notin \mathcal{S}$ the local ring of $P \in X$ and the ideal of quasi-adjunction $\mathcal{A}_{Q_j(P)}$ corresponding to selected local polytope of quasi-adjunction for singularity $P \in \mathcal{S}$*

d. *The homological global polytope (resp. face) of quasi-adjunction is a polytope in $H^1(X \setminus D, \mathbb{R})$, viewed as the trivial coset of $H_2(D, \mathbb{R})/H^1(X \setminus D, \mathbb{R})$, which is the translation to this trivial coset²⁹ of the intersection of divisorial global polytope (resp. face) of quasiadjunction described in a. (resp. b.) of this definition with a coset in $H_2(D, \mathbb{R})/H^1(X \setminus D, \mathbb{R})$ which image in $H^2(X, \mathbb{R})$ (i.e. the image via the map $i_{\mathbb{R}}$ in (22)) is an integral cohomology class (i.e. the first Chern class of a line bundle).*

The following Proposition shows that in the case when irreducible components of D are big and nef, only characters of the fundamental group, which after lift to $H^1(X \setminus D, \mathbb{R})$ give classes belonging to the faces of quasi-adjunction, can have non-trivial eigenspaces for the action of Galois groups on the abelian covers of X ramified along D .

Proposition 4.16. *Assume that irreducible components of D are big and nef. Let $u \in U_{X,D}$ be such that u is in interior of all global polytopes of quasi-adjunction \mathcal{G}_Q of divisor D or their images $\tilde{\mathcal{P}}_Q$ for the involution $u \rightarrow \bar{u}$ sending $(x_1, \dots, x_k) \in U_{D,X}$ to $(1-x_1, \dots, 1-x_k)$. If \tilde{X}_G is a resolution of singularities of a cover X_G of X with abelian Galois group G and $\chi = \exp(2\pi i u)$, then the eigenspace $H^1(\tilde{X}_G, \mathbb{C})_{\chi} = \{v \in H^1(\tilde{X}_G, \mathbb{C}) | g \cdot v = \chi(g)v, \forall g \in G\}$ is trivial.*

Proof. Since the characters of $H_1(X \setminus D, \mathbb{Z})$ having a finite order are the characters $\chi = \exp(2\pi i u)$ with $u \in \mathbb{Q}$, the density of those in $U_{X,D}$ implies that we can assume that χ is a character of a finite abelian group G . Let \mathcal{L}_{χ}^{-1} be the corresponding line bundle (cf. Prop. 3.9). Since the action of G is holomorphic and hence preserves the Hodge decomposition of $H^1(\tilde{X}_G, \mathbb{C})$, after possibly replacing the character χ by the conjugate $\bar{\chi}$, we can assume that χ has non-trivial eigenspace for G acting on the holomorphic forms of the cover. Then one has (cf. Prop. 4.11)

$$(67) \quad \dim H^0(\Omega_{\tilde{X}_G}^1)_{\chi} = \dim H^1(X, \Omega_X^2 \otimes \mathcal{L}_{\chi} \otimes \mathcal{A}_{\mathcal{G}(Q)})$$

where $\mathcal{P}(Q)$ is the global polytope of quasi-adjunction containing u , $\chi = \exp(2\pi i u)$.

Since u is an interior point of $\mathcal{G}(Q)$ one can take a small perturbation of it along the intersection with $U_{X,D}$ with the coset of $H^1(X \setminus D, \mathbb{R})$ corresponding to \mathcal{L}_{χ} (i.e. an affine subspace) so that it remains inside $\mathcal{G}(Q)$. The corresponding line bundle \mathcal{L}_{χ} is unchanged in this deformation of u . Using multiplier ideal interpretation of ideals of quasi-adjunction and Kawamata-Viehweg-Nadel vanishing (cf. [127] sect. 9.4B) we obtain that the terms in (67) are zeros. \square

Definition 4.17. A global divisorial face of quasi-adjunction $\mathcal{F} \subset \mathcal{G}(\mathcal{F})$ is called *contributing* if for $u \in \mathcal{F}$ and the resolution of singularities of the cyclic cover $\pi_* X_{\chi} \rightarrow X$ corresponding to the surjection $\chi : H_1(X \setminus D, \mathbb{Z}) \rightarrow \text{Im}(\chi) \subset \mathbb{C}^*$ one has $H^1(X, \Omega_X^2 \otimes \mathcal{L}_{\chi} \otimes \mathcal{A}_{\mathcal{G}(\mathcal{F})}) \neq 0$ (here \mathcal{L}_{χ} is the dual χ -eigenbundle of $\pi_*(\mathcal{O}_{\tilde{X}_{\chi}})$). A homological face of quasi-adjunction is called contributing if its translation to a coset $H_2(D, \mathbb{R})/H^1(X \setminus D, \mathbb{R})$ is a contributing divisorial face.

²⁹cf. construction described in Proposition 3.9

Theorem 4.18. *Let X be a simply connected smooth projective surface and let $D = \sum D_i$ be a reduced divisor with irreducible components D_i which are big and nef. Assuming as above that $H_1(X \setminus D, \mathbb{Z})$ is torsion free, let $r = \text{rk} \text{Coker} H_2(X, \mathbb{Z}) \rightarrow H^2(D, \mathbb{Z})$ denote its rank (cf. (6)). For any essential component of characteristic variety \mathcal{V}_i having positive dimension i.e. a coset of the r -dimensional torus $H^1(X \setminus D, U(1))$ such that $\dim \mathcal{V}_i \geq 1$ there is:*

a) a collection of singularities \mathcal{S} of D

b) a contributing face \mathcal{F} of a global polytope of quasi-adjunction $\mathcal{G}(\mathcal{S})$ which is determined by the collection \mathcal{S} and a collection of the polytopes of quasi-adjunction $\mathcal{Q}(P)$, one at each of singularities $P \in \mathcal{S}$

c) a line bundle $\mathcal{L}_{\mathcal{G}(\mathcal{S})}$

such that \mathcal{V}_i is the Zariski closure of $\exp(\pm 2\pi i \mathcal{F})$ in the maximal compact subgroup of the r -dimensional torus $\text{Char} H_1(X \setminus D, \mathbb{Z})$ and

$$\dim \mathcal{V}_i = \dim \mathcal{F} = \dim H^1(X, \Omega_X^2 \otimes \mathcal{L}_{\pm \mathcal{Q}(\mathcal{S})}^{-1} \otimes \mathcal{A}_{\mathcal{G}(\mathcal{S})}) + 1$$

Moreover, $\mathcal{L}_{\pm \mathcal{G}(\mathcal{S})}$ is the line bundle which is part of the building data of the cyclic cover corresponding to surjection $\chi : \pi_1(X \setminus D) \rightarrow \mathbb{Z}_{\text{ord}(\chi)}$ for a character χ which is generic in the component \mathcal{V}_i .

Vice versa, given a maximal ³⁰ contributing face $\mathcal{F} \subset \mathcal{G}(\mathcal{S})$ of a global polytope of quasiadjunction, with the ideal of quasiadjunction $\mathcal{A}_{\mathcal{G}(\mathcal{S})}$, such that the line bundle corresponding to the characters $\exp(2\pi i u)$, $u \in \mathcal{F}$ is a \mathcal{L} (satisfying $H^1(X, \Omega_X^2 \otimes \mathcal{L} \otimes \mathcal{A}_{\mathcal{G}(\mathcal{S})}) \neq 0$) then the Zariski closure of the set of characters $\exp(2\pi i u)$, $u \in \mathcal{F}$ is a component of characteristic variety of $\pi_1(X \setminus D)$.

Proof. Let Q be a maximal contributing face of quasi-adjunction. The Zariski closure in $H^1(X \setminus D, \mathbb{C}^*)$ of the set $\exp(2\pi i u)$, $u \in Q$, belongs to a component of characteristic variety, as follows from the assumptions. If this Zariski closure is a proper subset of a component, then preimage of the unitary part (i.e. the intersection with $H^1(X, U(1)) \subset H^1(X, \mathbb{C}^*)$) of the full component must belong to the same $H^1(X \setminus D, \mathbb{R})$ coset in $H_2(D, \mathbb{R})$ as Q and, as follows from Prop. 4.16, its preimage in $H_2(D, \mathbb{R})$ must be a face of the same polytope as \mathcal{F} i.e. coincide with \mathcal{F} due to maximality assumption.

Now, let \mathcal{V}_i be an irreducible component of characteristic variety. A theorem of D.Arapura implies that \mathcal{V}_i is a translated subtorus of the torus $H^1(X \setminus D, \mathbb{C}^*)$. The subset $\exp^{-1}(\mathcal{V}_i \cap H^1(X \setminus D, U(1))) \subset H^1(X \setminus D, \mathbb{R})$ consist of a set of $H^1(X \setminus D, \mathbb{Z})$ -translates of a linear subspace of $H^1(X \setminus D, \mathbb{R})$. The eigenbundles of the characters in \mathcal{V}_i , for the push forward of the structure sheaf of a cyclic cover of X corresponding to characters from \mathcal{V}_i , define a collection of translates of $H^1(X \setminus D, \mathbb{R}) \subset H_2(D, \mathbb{R})$ (cf. Prop. 3.9) which intersect the fundamental domain (i.e. the unit cube) $U_{X,D}$ for the action of $H_2(D, \mathbb{Z})$ on $H_2(D, \mathbb{R})$. Due to identification in Prop. 4.11 of cohomology of the local systems and the cohomology of sheaves of quasi-adjunction, one obtains that at least one of translates belongs to a contributing face of quasi-adjunction. It is maximal since otherwise \mathcal{V}_i will be a proper subset of a component of larger dimension. \square

Corollary 4.19. *Let X, D be as in theorem 4.18 and let C be a smooth big and nef curve intersecting all irreducible components of D at smooth points transversally. Then $H_2(D, \mathbb{R}) \subset H_2(D + C, \mathbb{R})$ has codimension one and divisorial contributing faces of quasi-adjunction of D coincide with those of $D + C$.*

³⁰i.e. not contained properly in a contributing face of the same strict global polytope of quasi-adjunction.

Proof. Since polytope quasiadjunction of ordinary node coincides with the unit square (node does not impose conditions of quasi-adjunction) it follows that the global polytopes in $H_2(D + C, \mathbb{R})$ are the cylinders over the global polytopes of $H_2(D, C)$ (preimages of projection of $H_2(D+C, \mathbb{R})$ onto the later). Kawamata-Viehweg-Nadel vanishing implies that the characters in a contributing faces of the eigenbundles \mathcal{L}_χ must have trivial ramification along C i.e. belong to $H_2(D, \mathbb{R})$ (triviality of ramification also follows from Divisibility Theorem 3.3). \square

Remark 4.20. The removal a line at infinity, transversal to a curve, was used extensively in [129], [137]. The main theorem in [137] follows immediately from Theorem 4.18 and Corollary 4.19.

Numerous examples to the Theorem 4.18 can be found in the paper [137] in the case of line arrangements is a plane and in ([130]) in the case of irreducible curves. The local counterpart of the Theorem 4.18 and many examples of calculations of multivariable Alexander polynomials of the links (i.e. the characteristic varieties, cf. discussion after Def. 3.13) of singularities in terms of polytopes and ideals of quasi-adjunction are given in [39]. For results on zero dimensional components of characteristic varieties we refer to [19], [20]. We will finish this section with an example of calculation on a large class of surfaces generalizing 6-cuspidal sextic of Zariski.

Example 4.21. Let X be a smooth projective simply connected surface and let L be a very ample line bundle on X . Let $s_2 \in H^0(X, L^2)$, $s_3 \in H^0(X, L^3)$ be generic sections of the corresponding tensor powers of L . Let D be the zero set of $s = s_2^3 + s_3^2 \in H^0(X, L^6)$. Then the Alexander polynomial of this curve with $6L^2$ cusps, corresponding to the surjection $H_1(X \setminus D, \mathbb{Z}) \rightarrow \mathbb{Z}_6$, is $t^2 - t + 1$.

To see this, first note that the existence of the surjection follows from (6) since the class of D in $H_2(X, \mathbb{Z})$ is divisible by 6. Using (62), the eigenspace of the generator of \mathbb{Z}_6 acting on $H^{1,0}$ of the 6-fold cyclic can be identified with $H^1(X, K_X \otimes L^5 \otimes \mathcal{I}_{Sing})$ where \mathcal{I}_{Sing} is the ideal sheaf such that $\mathcal{O}_X/\mathcal{I}_{Sing}$ is the reduced 0-dimensional subscheme of X with support at the set of cusps of D . One has the following Koszul resolution of \mathcal{I}_{Sing} :

$$0 \rightarrow L^{-5} \rightarrow L^{-2} \oplus L^{-3} \rightarrow \mathcal{I}_{Sing} \rightarrow 0$$

After taking the tensor product of this sequence with $K \otimes L^5$ and considering the corresponding cohomology sequence:

$$(68) \quad H^1(X, K_X \otimes L^2) \oplus H^1(X, K_X \otimes L^3) \rightarrow H^1(X, K_X \otimes L^5 \otimes \mathcal{I}_{Sing}) \rightarrow H^2(X, K_X) \rightarrow 0$$

we see that Kodaira vanishing implies that $\dim H^1(X, K_X \otimes L^5 \otimes \mathcal{I}_{Sing}) = 1$. This shows that $\frac{1}{6} \in [0, 1]$ is the contributing face of quasi-adjunction and now the claim about the Alexander polynomial follows from the Theorem 4.18. Note that this example also can be analyzed using methods of orbifold pencils discussed in [19],[20],[21].

4.6. Bernstein-Sato ideals and polytopes of quasi-adjunction. Let f_1, \dots, f_r be germs of holomorphic functions in n variables. The Bernstein-Sato ideal $\mathcal{B}(f_1, \dots, f_r)$ is the ideal generated by polynomials $b(s_1, \dots, s_r)$ such that there exist a differential operator $P \in \mathbb{C}[x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, s_1, \dots, s_r]$ satisfying the identity:

$$(69) \quad b(s_1, \dots, s_r) f^{s_1} \dots f^{s_r} = P f_1^{s_1+1} \dots f_r^{s_r+1}$$

(cf. [185] [25], [160], [109], in 1-dimensional case cf. [155], [121]). In the case of plane curves singularities one has the following:

Theorem 4.22. *Let f_1, \dots, f_r be the germs of holomorphic functions in two variables. Let P be the product of the linear forms $L_i(s_1 + 1, \dots, s_r + 1)$ where L_i runs through linear forms vanishing on $r - 1$ -dimensional faces of polytopes of quasi-adjunction corresponding to a germ with r irreducible components f_1, \dots, f_r . Then any $b \in \mathcal{B}(f_1, \dots, f_r)$ is divisible by P .*

The same argument as used in [39], provides extension to isolated non-normal crossings (cf. [140]). For a general conjecture of the structure of Bernstein ideals we refer to [38] and for a discussion of the case of arrangements, other related problems and references cf. [217]

5. ASYMPTOTIC OF INVARIANTS OF FUNDAMENTAL GROUPS

The problem of characterization of fundamental groups of smooth quasi-projective varieties is intractable at the moment. Nevertheless some questions about distribution of Alexander invariants can be addressed. We will see below that one can make some conclusions about distribution of dimensions of characteristic varieties of such fundamental groups. A different type of asymptotics, is suggested by the relation between the degrees of Alexander polynomials and Mordell-Weil ranks of isotrivial families of abelian varieties (cf. [46],[146]) since it allows to restate the problem of asymptotic behavior of such degrees in terms of the conjectures on distribution of Mordell Weil ranks of curves over the function fields. In this section we shall survey the results in [47] concerning distribution of the dimensions of characteristic varieties ³¹.

Let X be a smooth simply connected projective variety, D a reduced divisor and let Δ be a subset of the effective cone $Eff(X) \subset NS(X)$ in the Neron Severi group of X . We shall call the set Δ *saturated* if $d_1 \in \Delta$ and $d_2 \in Eff(X)$ are such that $d_1 - d_2 \in Eff(X)$ implies that $d_2 \in \Delta$ and $d_1 - d_2 \in \Delta$. We are interested in distribution of invariants of $\pi_1(X \setminus D)$ when the class of $D \subset Eff(X)$ is a linear combination of classes in Δ with non-negative coefficients. We are specifically interested in curves D with large dimension of a component of characteristic variety of $\pi_1(X \setminus D)$ and D being a curve with all its irreducible components having classes in Δ . It follows from [9], that existence of a component of dimension r implies existence of surjection $\pi_1(X \setminus D) \rightarrow F_r$. Vice versa, existence of the latter implies that the characteristic variety of $\pi_1(X \setminus D)$ contains a component of dimension not smaller than r . Note right away that for the purpose of enumeration of reduced divisors D for which one has a surjection $\pi_1(X \setminus D) \rightarrow F_r$ we must impose some conditions on such surjections. For example, given any D with such property and any reduced divisor D' one has

$$(70) \quad \pi_1(X \setminus D \cup D') \rightarrow \pi_1(X \setminus D) \rightarrow F_r$$

and hence, given a curve admitting a surjection of its fundamental group onto F_r , there are enlargements of this curve with the same property parametrized by all the curves on the surface. This motivates the following:

Definition 5.1. (cf. [137]) Let \mathcal{D} be a reduced divisor on a smooth projective surface X . A surjection $\pi_1(X \setminus \mathcal{D}) \rightarrow F_r$ is called *essential* if \mathcal{D} does not admit split $\mathcal{D} = D \cup D'$ for which one has factorization (70).

³¹Such circle of problems is inspired by conjectural asymptotic of number fields extensions having a given group as the Galois group or the group of its Galois closure, which are unramified outside an arbitrary subset of primes while the size of the norm of discriminant grows ([156]): Malle conjectures implies a positive answer to the inverse problem of the Galois with little hope for solution in near future (as is obtaining a characterization of quasi-projective group)

A surjection $\pi_1(X \setminus \mathcal{D}) \rightarrow F_r$ is called *reduced* if there exist a choice of ordered system of generators $\{x_1, \dots, x_{r+1} | x_1 \cdot \dots \cdot x_{r+1} = 1\}$ of F_r such that this surjection takes meridian of each irreducible component of \mathcal{D} to a conjugate of a generator.

We also will say that singularities of \mathcal{D} satisfy condition (*) if all singular points belonging to more than one irreducible component are ordinary singularities i.e. are intersections of smooth transversal branches ³².

A rather detailed information about such curves was obtained in [136] in the case $X = \mathbb{P}^2, \Delta = \{[1]\} \in \mathbb{Z} = \text{Pic}(\mathbb{P}^2)$ i.e. the fundamental groups of the complements to arrangements of lines in a plane (see [86],[157] for related results).

Theorem 5.2. [136] [179] *Let \mathcal{A} be an arrangement of lines in \mathbb{P}^2 . If there exist an essential surjection $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}) \rightarrow F_r, r \geq 4$ then \mathcal{A} is a union of concurrent lines, in which case the last surjection is an isomorphism.*

Moreover, there is only one known example of essential surjections of the complements to an arrangement line which admits surjection onto F_3 ³³ and for any d there exist an arrangement of non-concurrent lines admitting essential surjection onto F_2 (e.g. $3d$ lines forming the zero set of $(x^d - y^d)(y^d - z^d)(x^d - z^d) = 0$).

Work [47] contains an extension of this theorem to reduced divisors on arbitrary simply connected surfaces. Before stating the main result, let us describe the analog of the case of concurrent lines in Theorem 5.2, which is a family of the curves with irreducible components in Δ and for which the fundamental group of the complements may have a free quotient of arbitrary large rank. For this family of curves, the fundamental groups of the complement form a finite set of groups, having cardinality depending on Δ and, moreover, a presentation of each group in this set can be described in terms of geometric data we specify. However, unlike the case of Theorem 5.2, the problem of characterizing which specific data is realizable by curves in this class remains open in general. Enumeration of fundamental groups of such curves for a class $\delta \in \Delta \subset \text{Pic}(X)$ can be made as follows.

Proposition 5.3. *For any $r \geq 1$ and a movable divisor $\delta \in \text{Pic}(X)$, ³⁴ there is a divisor D with classes of components in the linear system of δ and such that $\pi_1(X \setminus D)$ admits essential surjection onto F_r . Vice versa, if D has all its irreducible components being members of a pencil of curves in complete linear system of δ (i.e. a line in $\mathbb{P}(H^0(X, \mathcal{O}_X(\delta)))$), and $\pi_1(X \setminus D)$ admits surjection onto $F_r, r \geq 2$ this group is an amalgamated product $\mathcal{G} *_{F_a} \mathcal{H}$ with \mathcal{G} belonging to a finite collection of groups depending on δ , obtained by a construction below and \mathcal{H} is an extension:*

$$(71) \quad 0 \rightarrow F_a \rightarrow \mathcal{H} \rightarrow F_{r'} \rightarrow 0 \quad r' \leq r$$

defined by a homomorphism $F_{r'} \rightarrow \text{Aut}(F_a)$ coming from a finite set cardinality depending only on δ . The number of isomorphism classes of such groups $\pi_1(X \setminus D)$ with a fixed class δ , stabilises for large r .

Proof. Indeed, for any pencil in the linear system containing δ , a union on its $r + 1$ members yields a divisor $D \in \mathbb{P}(H^0(X, (r + 1)\delta))$ with $\pi_1(X \setminus D)$ admitting a surjection onto F_r since such a pencil induces a dominant map onto the complement in \mathbb{P}^1 to $r + 1$ points.

³² The results in this section make this assumption. It should be possible to eliminate it with essential conclusions remaining intact.

³³i.e. the Hesse arrangement of 12 lines formed by lines containing triples of inflection points of plane smooth cubic cf. [137]

³⁴i.e. such that the codimension of the base locus of the linear system it defines is at least 2

To enumerate all possible fundamental groups of the complements to the curves with all irreducible components belonging to a pencil let us consider the discriminant $Disc(\mathbb{P}(H^0(X, \mathcal{O}_X(\delta))))$ of the complete linear system $\mathbb{P}(H^0(X, \mathcal{O}_X(\delta)))$ i.e. the subvariety consisting of the divisors having singularities worse than singularities of a generic element in $\mathbb{P}(H^0(X, \mathcal{O}_X(\delta)))$. Consider also the stratification of the discriminant into connected components of equisingularity strata, adding to this stratification the complement to the discriminant as a codimension zero stratum (cf. [5] on some information about geometry of these strata).

We will use finite sets of collections of such equisingularity strata $\mathcal{S}_1, \dots, \mathcal{S}_t$ for which there exists a pencil \mathcal{P} in $\mathbb{P}(H^0(X, \mathcal{O}_X(\delta)))$ with the following property: there exists a union D of members of \mathcal{P} such that the curve D satisfies condition (*) (cf. Def. 5.1). Let $N(\delta, t)$ be the number of isotopy classes of pencils in $\mathbb{P}(H^0(X, \mathcal{O}_X(\delta)))$ such that the number of the strata of this stratification intersected by the pencil is t and let T be the least upper bound for integers t for all pencils in δ . Finiteness of these numbers is a consequence of the finiteness of the number of strata of stratifications since those are an algebraic subsets of discriminant (cf. [100]).

Let D is a curve having $r + 1$ irreducible components belonging to a pencil \mathcal{P} in $\mathbb{P}(H^0(X, \mathcal{O}_X(\delta)))$ in which the members of \mathcal{P} have t (where $t \leq r + 1$) equisingularity types. We claim that $\pi_1(X \setminus D)$, can have at most 2^t isomorphism types. More precisely for each subset \mathcal{T} of the set of strata $\mathcal{S}_1, \dots, \mathcal{S}_t$ there is at most one isomorphism type of the fundamental groups $\pi_1(X \setminus D)$ where the set of equisingularity strata of components of $D \in \mathbb{P}(H^0(X, \mathcal{O}_X((r + 1)\delta)))$ having non-generic equisingularity type in the pencil coincides with \mathcal{T} . This is the case when D is a union of $|\mathcal{T}|$ curves from the strata $\mathcal{S}_1, \dots, \mathcal{S}_t$ and $r' = r + 1 - |\mathcal{T}|$ curves from codimension zero stratum and none of remaining $t - |\mathcal{T}|$ singular members of the pencil are not components of D . In particular, for $r > t$ there are at most $\sum_{t=0}^T 2^t N(\delta(t))$ isomorphism classes of the fundamental groups and for $r > T$ the number of isomorphism classes of fundamental groups of curves with components in the linear system of δ and admitting surjections onto F_r is bounded, with bound depending only on $\delta \in \Delta$.

To describe the structure of the fundamental groups of the complement to a union D of several members of a pencil \mathcal{P} of curves in δ , with the set equisingularity types of singular members of \mathcal{P} consisting of equisingularity strata $\mathcal{S}_1, \dots, \mathcal{S}_t$, such that non-generic types of components of D are exactly those in \mathcal{T} , and also to enumerate such fundamental groups, consider the blow up \tilde{X} of X at the base points of the pencil. We obtain a regular map $\pi : \tilde{X} \setminus \tilde{D} \rightarrow \mathbb{P}^1 \setminus S_{r+1}$ where S_{r+1} a finite subset of \mathbb{P}^1 with cardinality $r + 1$.

Let $\mathbb{P}^1 = B_1 \cup B_2$ be partition into union of two disks intersecting along their common boundary and having the following properties: B_1 contains all $t - |\mathcal{T}|$ fibers of π which do not have generic equisingularity type in $\mathbb{P}(H^0(X, \mathcal{O}_X(\delta)))$ and are not components of D , while $B_2 = \mathbb{P}^1 \setminus B_1$ contains $|\mathcal{T}|$ non-generic fibers of π which are components of D and remaining $r' = r + 1 - (t - |\mathcal{T}|)$ fibers of π which all are generic in the latter linear system. Over the complement in B_2 to the subset over which the fibers of π are the components of D , the map π is a locally trivial fibration which global type is determined by δ . Van Kampen theorem 2.2 implies the following: if Σ is generic fiber of π , $\mathcal{G} = \pi_1(\pi^{-1}(B_1))$, $\mathcal{H} = \pi_1(\pi^{-1}(B_2))$ then

$$(72) \quad \pi_1(X \setminus D) = \mathcal{G} *_{\pi_1(\Sigma)} \mathcal{H}, \quad \text{and} \quad 1 \rightarrow \pi_1(\Sigma) \rightarrow \mathcal{H} \rightarrow F_{r'} \rightarrow 1$$

Σ is complement in the generic fiber of the pencil to the set of base points of the pencil i.e. $\pi_1(\Sigma)$ is a free group F_a for some a . The group \mathcal{G} belongs to a collection having at most 2^t elements (i.e. the number of subsets in $\mathcal{S}_1, \dots, \mathcal{S}_t$). The claim follows. \square

Example 5.4. Let us enumerate the fundamental groups of the complements to conic-line arrangements which admit a surjection onto a free group of rank greater than 5. The starting point is that a conic-line arrangement (satisfying condition (*)) having such fundamental group is a union of $r + 1$ (possibly reducible) quadrics belonging to a pencil. This is content of improvement for conic-line arrangements of the general bound in Theorem 5.5 below (cf. Example 5.6 2.) Equisingular stratification of $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)))$ consists of 3 strata: smooth quadrics, reduced and reducible quadrics i.e. a union of two transversal lines and non-reduced quadrics i.e. the double lines. The degree of discriminant is 3. We denote these equisingular strata respectively as $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$.

Any pencil of quadrics containing as generic element a smooth quadric in \mathcal{S}_0 , has at most 3 singular fibers which are either 3 reducible quadrics or contains 2 singular fibers one reduced and one non reduced. In the latter case, the condition (*) on D fails. Moreover, there are pencils with generic element inside the stratum \mathcal{S}_1 . For such a pencil, the divisor $D \in \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(2(r + 1))))$ is a union of $2r + 2$ concurrent lines and hence $\pi_1(\mathbb{P}^2 \setminus D) = F_{2r+1}$.

There are 4 equisingular classes of divisors $D \in \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}((r + 1)2)))$ with components formed by curves in a pencil δ , corresponding to the cases when the number of quadrics which are the singular elements of the pencil and formed by components of D , is either 0 (i.e. all components of D are smooth quadrics), or is 1, 2 or 3. Respectively, there are 4 corresponding types of fundamental groups.

For example, let us take as D a union of $r + 1$ quadrics belonging to a pencil, one of which is reducible. Let B_1 be a disk containing remaining 2 reducible fibers of the pencil and let $B_2 = \mathbb{P}^1 \setminus B_1$. Then B_2 is a disk containing the points corresponding to the fibers containing the components of the pencils comprising D . Over B_1 , the map π is a fibration with generic fiber being a smooth quadrics and which has two special fibers which are the union of lines and therefore can be calculated using van Kampen theorem 2.2. Over the complement in B_2 to the points corresponding to the components of D one has a locally trivial fibration with the fiber being the complement in a smooth quadric to 4 base points of the pencil. Hence $\mathcal{H} = \pi_1(B_2)$ is an extension of free group F_3 by the free group F_r with only one type of extension possible since there is only one isotopy class of generic pencils of quadrics.

Now we turn to the main result of [47] which can be stated as follows:

Theorem 5.5. *Given a saturated set Δ of classes in $NS(V)$ consider the following trichotomy for the distribution of the curves \mathcal{D} with classes of irreducible components in Δ having a free essential reduced quotient of a fixed rank r and satisfying conditions (*)*

- 1) *There exist infinitely many isotopy classes of curves \mathcal{D} admitting surjections $\pi_1(V \setminus \mathcal{D}) \rightarrow F_r, r > 1$.*
- 2) *There are finitely many isotopy classes of curves \mathcal{D} admitting surjections $\pi_1(V \setminus \mathcal{D}) \rightarrow F_r, r > 1$.*
- 3) *\mathcal{D} admitting a surjection $\pi_1(V \setminus \mathcal{D}) \rightarrow F_r$, is composed of curves of a pencil. There are finitely many isotopy classes of such \mathcal{D} for given Δ .*

*All three cases are realizable at least for some (V, Δ) . Case (2) takes place for $r \geq 10$. There exists a constant $M(V, \Delta)$ such that for $r > M(V, \Delta)$ one has case (3). In the latter case, $\pi_1(V \setminus \mathcal{D})$ splits as an amalgamated product $H *_{\pi_1(\Sigma)} G$ where Σ is an open Riemann surface which is a smooth member of the pencil, H is coming from a finite set of groups associated with the linear system $H^0(V, \mathcal{O}(D))$, D is a divisor having class $\delta \in \Delta$ and G is*

an extension:

$$(73) \quad 0 \rightarrow \pi_1(\Sigma) \rightarrow G \rightarrow F_r \rightarrow 0$$

In specific cases of (X, Δ) information about the constants 10 and $M(X, \Delta)$ can be improved.

Example 5.6. (1) Above results for the arrangements of lines shows that in this case one can replace 10 by 2 and $M(\mathbb{P}^2, [1]) = 3$.

(2) Again in the case $X = \mathbb{P}^2$ but $\Delta = \{[1], [2]\}$, the curves \mathcal{D} for which there exist a surjection $\pi_1(\mathbb{P}^2 \setminus \mathcal{D}) \rightarrow F_r$ must have the type only as described in Example 5.4, provided $r > 5$. However, a generic pencil in the linear system:

$$(74) \quad \lambda_0 x_0(x_1^2 - x_2^2) + \lambda_1 x_1(x_2^2 - x_0^2) + \lambda_2 x_2(x_0^2 - x_1^2) = 0$$

has 6 members which are unions of lines and quadrics. This gives a curve \mathcal{D} of degree 18 for which $\pi_1(\mathbb{P}^2 \setminus \mathcal{D})$ admits a surjection onto F_5 and is not isotopic to a curve as in 5.4.

The Theorem 5.5 can be restated as follows: if $N(X, \Delta, r)$ denotes the number of equisingular isotopy classes of curves on X with irreducible components having numerical classes in Δ and fundamental groups admitting a surjection onto a free group F_r then for $r > M(X, \Delta)$, $N(X, \Delta, r)$ is finite and all curves have special type as in the case (C) of the trichotomy. For $10 < r \leq M(X, \Delta)$, $N(X, \Delta, r)$ is also finite but the type of the curves may vary. Finally, for $r < 10$ the number of isotopy classes $N(X, \Delta, r)$ may be infinite.

Some information on dependence of the constant $M(X, \Delta)$ on Δ and X is also available. For example if $X = \mathbb{P}^2$, $\Delta_d = \{[1], \dots, [d]\}$ then $M(\mathbb{P}^2, \Delta_d) \geq 3d$. Indeed, Ruppert (cf. [184]) found a pencil of curves of degree $d + 1$ with $3d$ fibers being a union of a line and a curve of degree d . In particular a union of these $3d$ fibers yields a curve of degree $3d(d + 1)$ with irreducible components in Δ_d and having surjection on the free group of rank $3d - 1$. In particular the sequence $M(X, \Delta_d)$ is unbounded. The Ruppert pencil is a generic pencil in 2-dimension linear system of curves given by equation (which for $d = 2$ it is given in Example 5.6):

$$(75) \quad \lambda_0 x_0(x_1^d - x_2^d) + \lambda_1 x_1(x_2^d - x_0^d) + \lambda_2 x_2(x_0^d - x_1^d) = 0$$

More precisely, the curve (75) is singular if and only if

$$(\lambda_0^d - \lambda_1^d)(\lambda_1^d - \lambda_2^d)(\lambda_2^d - \lambda_0^d) = 0$$

and all reducible fibers are unions of a line and a curve of degree d . Hence generic line in variables λ_i is a pencil with $3d$ reducible members as described.

We refer to [47] for examples of surjections onto free groups of the fundamental groups of the complements to curves on surfaces besides \mathbb{P}^2 .

This discussion suggests the following problems:

Problem 5.7. (1) Determine the rate of growth of $N(X, \Delta, r)$ for various X and Δ when $r \rightarrow \infty$, i.e. how many types of reducible curves admitting surjections onto F_r , which r large (i.e. $r > M(X, \Delta)$) exist?

(2) Find a bound on $M(X, \Delta)$ in terms of invariants of X, Δ i.e. how large should be r such that there exist curves admitting surjection onto F_r and which are not the unions of the fibers of a pencil.

(3) For $n \in \mathbb{N}$ let $\Delta_n = \{\sum n_i \delta_i \mid \delta_i \in \Delta, n_i \leq n\} \subset NS(X)$. Determine the asymptotic of the number of curves admitting surjection onto $F_r, r < 10$ with the classes in Δ_n when $n \rightarrow \infty$.

- (4) *Determine algebraic properties of the fundamental groups described in Proposition 5.3.*

Some partial results, mainly in the case of plane, are discussed above and in [47]: for example the curves $(x^n - y^n)(y^n - z^n)(x^n - z^n) = 0$ formed by $3n$ lines show that the growth in Problem 3 for \mathbb{P}^2 , [1] for $r = 2$ is at least linear. The growth of $N(X, \Delta)$ appears to be related to the asymptotic of the number of strata (cf. the proof of Prop. 5.3) and possibly is exponential.

6. SPECIAL CURVES

This section surveys examples of calculations of the fundamental groups and other topological information about the complements, the properties of fundamental groups and applications. An important step in each such inquiry is finding a class of curves with interesting topology of the complements. Most examples in this section are plane curves.

6.1. Arrangements of lines, hyperplanes and plane curves. .

There are many calculations of the fundamental groups of the complements to arrangements of lines. The braid monodromy can be calculated algorithmically. In the case of real arrangements finding the braid monodromy and the presentation are particularly simple: see [191], [116]. In some instances this leads to presentations allowing a more intrinsic characterization: for example in [87] conditions on arrangement were found for the fundamental groups to be products of free groups.

The fundamental groups and more subtle questions on the topology of the complements to arrangements formed by hyperplanes fixed by the groups generated by reflections were very actively studied in many case. In case of real reflection groups, the fundamental groups of the complements to corresponding complexified real arrangements were found [29] with presentations closely related to the Dynkin diagrams of the corresponding Coxeter groups. The topology of the complements to hyperplanes corresponding to the complex reflection groups also were actively studied with many deep results. The number of striking results is too large to survey here and we refer for example to [34] and [28] for some particularly important ones and for further references.

Several calculations were made for the fundamental groups of the complements to unions of lines and quadrics. Work [8] includes the arrangements formed by unions a quadric and lines with various tangency conditions. Few example of such type of arrangements, more specifically those real arrangements of quadrics and lines which admit projections to a line with all critical points being real, were considered in [172]. Here the standard methods of calculation of the braid monodromy are almost as simple as in the case of real arrangements of lines and lead quickly to presentations in terms generators and relators.

Cardinality of the set of connected components of the equisingular families of reducible curves with fixed combinatorial type (cf. Definition 6.2) was investigated in several cases of plane curves of small degree. In particular the classification for curves of degree 5 was carried out in [64]. The case of arrangements of small cardinality and irreducibility of equisingular component was studied for arrangements up to 9 lines as well as arrangements of 10 and 11 lines with many different types of combinatorics with some results in the case of arrangements of 12 lines (cf. [6], [89], [173],[14], [107] the latter are in connection with Rybnikov's example of combinatorially equivalent arrangements with distinct homotopy types). Specific types of presentations of the fundamental groups of arrangements were studied in [80].

6.2. Generic Projections. Study of the fundamental groups of the complements to the branching curves of generic projections ³⁵ was initiated by B.Moishezon in work [164] and continued jointly with M.Teicher and later by M.Teicher and her collaborators. Given a smooth surface $X \subset \mathbb{P}^N$, a projection from a generic \mathbb{P}^{N-3} gives a generic branched cover ramified along a curve $R \subset X$. The image of R is the branching curve $B \subset \mathbb{P}^2$ of this projection. If the center of projection \mathbb{P}^{n-3} is sufficiently generic, then B has nodes and cusps as the only singularities. The number of cusps and nodes can be found in terms of intersection indices of Chern classes of X and the class of hyperplane section (cf. [131]). Work [164] considers the case when X is a smooth surface in \mathbb{P}^3 . Then the branching curve B has degree $n(n-1)$, $n(n-1)(n-2)$ cusps and $\frac{1}{2}n(n-1)(n-2)(n-3)$ nodes (for $n=3$ one obtains sextic with six cusps). The fundamental group of the complement is isomorphic to the quotient of the braid group on n strings by its center (cf. [164]).

Works [166] consider generic projections of quadrics $X = \mathbb{P}^1 \times \mathbb{P}^1$ using a family of embeddings $i_{a,b}$, $a, b \in \mathbb{Z}$ corresponding to various ample divisors in $NS(X)$. Interest in this class stems from the fact that Galois covers of \mathbb{P}^2 with branching curve of generic projections of these surfaces provide examples of simply connected surface of general type for which $c_1^2 > 2c_2$. The key step in the showing the simply connectedness is the calculation of the fundamental group of the complement to the branching curve. The relation between the fundamental groups of the complements to the branching curves of generic projections and the fundamental groups of smooth models of Galois closures of these projections is discussed in [166], [147].

Since then, the class of surfaces which generic projections produces the curves for which one has a presentation of the fundamental groups of the complements was greatly increased. Calculations produced over the span of more than 30 years include complete intersections in projective spaces ([182]), very ample embeddings of Hirzebruch surfaces, embeddings of K3 surfaces, very ample embeddings of ruled surfaces which are the products of \mathbb{P}^1 and smooth curves of positive genus and others. In many instances a quite different than in the case of surfaces in \mathbb{P}^3 pattern emerged for the fundamental groups (cf. [208] for references to these calculations). One has to mention that the main technical tool in such calculation is appropriate degeneration of the surface resulting in degeneration of the branching curve. Steps of calculation include calculation of the braid monodromy of degenerate curve (which may be reducible) and then applying rules of regeneration i.e. relating the braid monodromy of degenerate curve to the braid monodromy of the curve prior to degeneration. We refer to a survey article [7] which has useful references to these numerous calculations.

An interesting property of branching curves of generic projections was discovered by Chisini: (with a small number of exceptions) the cover given by generic projection is determined by the curve alone, i.e. no subgroup of the fundamental group to specify the cover (cf. section 3.3) is needed. A proof of this result was found in [125] (cf. also, [42]).

6.3. Complements to discriminants of universal unfoldings. With a germ of isolated hypersurface singularity $f(x_1, \dots, x_n) = 0$ one associates the germ of the universal unfolding \mathbb{C}^N , $N = \dim \mathbb{C}[x_1, \dots, x_n]/(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ which comes with the germ of discriminantal hypersurface *Disc* (corresponding to the germs having a critical point (cf. [106])).

The fundamental groups of the complements to the germs *Disc* have appearance in a variety of questions spreading from singularity theory and topology to representation theory and beyond. An important feature of the fundamental groups of such complements

³⁵important results on geometry of such curves were obtained much earlier by italian school, notably B.Segre, Chisini and his school cf. [193]

(as well as complements to other discriminants) is that they come endowed with geometric monodromy i.e. the homomorphism to the mapping class group of the Milnor fiber, i.e. the group of diffeomorphisms of the Milnor fiber constant on its boundary modulo isotopy. This induces the homological monodromy via the action of the mapping class group on the homology of the Milnor fiber. For *ADE* singularities one obtains the corresponding Coxeter groups (cf. [79]). Moreover, these complements to germs often can be identified with the complement to the whole affine hypersurfaces in \mathbb{C}^N , so these local fundamental groups are quasi-projective. In the case of simple *ADE* surface singularities, the fundamental groups of the complement were identified by Brieskorn (cf. [29]) with the braid groups corresponding to the respective Coxeter systems.

Calculations for several more complicated classes of singularities were made also. An important case of Brieskorn-Pham polynomials $f(x_1, \dots, x_n) = x_1^d + \dots + x_n^d$ was considered by M.Lonne (cf. [153] and references there). Generators and relations of the fundamental group of the complement to discriminant are described in terms of combinatorial data given by the graph associated to singularity, analogous to Dynkin diagram or, equivalently, in terms of the corresponding bilinear form. Vertices correspond to the integer points in the interior of the cube $I_{d,n} = \{\mathbf{i} = (i_1, \dots, i_n) | 1 \leq i_k \leq d - 1\}$. Edges described in terms of bilinear form on the vector space with basis $v_i, \mathbf{i} \in I_{d,n}$ given by

$$(76) \quad \langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } |i_v - j_v| \geq 2 \text{ for some } v \\ 0 & \text{if } (i_v - j_v)(i_\mu - j_\mu) < 1 \text{ for some } \mu, v \\ -2 & \text{if } i = j \\ -1 & \text{otherwise} \end{cases}$$

The edges of the graph connect the pairs of vertices \mathbf{i}, \mathbf{j} such that $\langle v_i, v_j \rangle \neq 0$. In terms of this bilinear form or the graph the fundamental group of the complement to discriminant has generators t_i corresponding to the vertices and the relations as follows

$$(77) \quad \begin{aligned} t_i t_j &= t_j t_i && \text{if } \langle v_i, v_j \rangle = 0, \\ t_i t_j t_i &= t_j t_i t_j && \langle v_i, v_j \rangle \neq 0 \\ t_i t_j t_k t_i &= t_j t_i t_k t_j && \langle v_i, v_j \rangle < \langle v_j, v_k \rangle < \langle v_k, v_i \rangle \neq 0 \\ &&& i_v \leq j_v \leq k_v \text{ for all } v \end{aligned}$$

6.4. Complements to discriminants of complete linear systems. This class of singular curves comprised of the curves where the fundamental groups come endowed with the homomorphisms into non-abelian groups given by either geometric monodromy i.e. with values in a mapping class group or (co)homological monodromy (with values in the linear group of automorphisms of the homology). Homological monodromies often are surjective or are close to such (i.e. the fundamental group itself is non-abelian). The construction of these curve is as follows. Let X be a smooth projective variety and let \mathcal{L} be a line bundle. The linear system $\mathbb{P}(H^0(X, \mathcal{L}))$ contains the discriminant consisting of the elements having singularities worse than singularities of its generic element. With rare exceptions the discriminant has codimension 1 (identifying varieties with a small dual is an interesting problem). Its intersection with a generic plane ³⁶ in $\mathbb{P}(H^0(X, \mathcal{L}))$ produces a plane curve which fundamental group of the complement has monodromy map into the mapping class group of the generic fiber of the universal element of this linear system i.e. the group of diffeomorphisms modulo isotopy of the fiber of the incidence correspondence $I_{\mathcal{L}} \subset X \times$

³⁶generic choice assures that the fundamental group of the complement to the intersection with the plane inside this plane is isomorphic to the fundamental group of the complement to the discriminant of the complete linear system. Non-generic section were studied in very special cases. For a recent study cf. [85]

$\mathbb{P}(H^0(X, \mathcal{L}))$ set theoretically consisting of pairs $\{(x, C) | x \in X, C \in \mathbb{P}(H^0(X, \mathcal{L})), x \in C\}$. In [78] was considered the case $X = \mathbb{P}^2$ (resp. $X = V_2$ the quadric in \mathbb{P}^3) and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(3)$ (resp. $\mathcal{L} = \mathcal{O}_{V_2}(2)$) when one obtains as the fundamental group of the complement to discriminant the extension of $SL(2, \mathbb{Z})$ by the Heisenberg group over the field with 3 elements (resp. the ring \mathbb{Z}_4). The surjection onto $SL_2(\mathbb{Z})$ is the monodromy (the mapping class of 2-dimensional torus coincides with $SL_2(\mathbb{Z})$) and the kernel is the Heisenberg group. Recently, a progress was made in understanding the kernel of the monodromy in the case $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(4)$ cf. [112].

A much more difficult case $X = \mathbb{P}^n, \mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, including the case of discriminant of the family of cubic curves just described, was addressed by M.Lonne in [153]. It also includes apparently the only other known case of this construction i.e. $X = \mathbb{P}^1, \mathcal{L} = \mathcal{O}(d)$ considered by Zariski (and mentioned in [78]) when the corresponding fundamental group is the braid group of two dimensional sphere. The fundamental group $\pi_1(\mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \setminus Disc))$ is the quotient of the group with generators and relations (77) by the normal subgroup generated by additional relations which we now shall describe. They are defined in terms of enumeration functions: $\Upsilon_k, k = 0, \dots, n : \{1, \dots, (d-1)^n\} \rightarrow I_{n,d}$ or equivalently the orderings of the integral points of the cube $I_{n,d}$. Among them, Υ_0 considered as the ordering of the integral points in $I_{n,d}$ according to the reverse lexicographic order: $(i_1, \dots, i_n) < (i'_1, \dots, i'_n)$ iff the for the smallest subscript k for which $i_k \neq i'_k$ one has $i_k > i'_k$ (e.g. $(d-1, d-1, d-1) < (d-1, d-1, d-2) < (d-1, d-1, d-2) < \dots < (d-1, d-1, 1) < (d-1, d-2, d-1) < (d-1, d-2, d-2) < \dots$). The order $<_k$ obtained from this one as follows:

$$(78) \quad (i_1, \dots, i_n) <_k (j_1, \dots, j_n) \iff i_k < j_k \text{ or } i_k = j_k, (i_1, \dots, i_n) <_0 (j_1, \dots, j_n)$$

With this notations a presentation of $\pi_1(\mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \setminus Disc))$ is given by generators and relators (77) and

$$(79) \quad (t_i \delta_0)^{d-1} = (\delta_0 t_i^{-1})^{d-1}, \delta_0 \cdot \dots \cdot \delta_n = 1 \text{ where } \delta_k = \prod_{m=1}^{(d-1)^n} t_{\Upsilon_k(m)} \quad k = 0, \dots, n$$

It would be interesting to understand the algebraic structure of such groups and their relation to other geometrically defined group but see [153] for discussion of the relation of this presentation with those in cases known earlier. For results on the monodromy representations of the groups of the complements to discriminant using presentation (77), (79) we refer to [190] and for the case of monodromy of complements to discriminants on toric surfaces to [189] and [48].

6.5. Plane sextics and trigonal curves. In the last 10-20 years, many important results were obtained in the study of equisingular families of curves of degree 6 (and less; cf. [63], [52], [53], [55], [56], [59] and references below). The number of equisingular families of plane sextics measures in thousands and hence listing of possible cases is not a reasonable approach. Several classes of sextics were identified and we will describe some of them below. The methods include the use of Alexander invariants, connection with K3 surfaces and relation with the class trigonal curves on ruled rational surfaces. An interesting study of the moduli of sextics with six cusps i.e. the locus in the moduli space \mathcal{M}_g given by the curves in distinct equisingular families was done in [95]. Several good surveys of the subject are already available (cf. [57], Preface and section 7.2 in [54] and [1]).

A. Simple and non-simple sextics. A sextic is called *simple* if its only singularities are ADE singularities. Otherwise, a sextic is called *non-simple*. For irreducible non-simple

sextics the type of equisingular deformation type is determined by the combinatorial type i.e. the collection of the local types of all singularities (cf. [57] Theorem 3.2.1). The key to a classification of simple sextics is the relation with the theory of K3 surfaces. Consider a double cover X_C of \mathbb{P}^2 branched over a sextic C . Singularities of this surface, correspond to the singularities of the branching curve and are simple of the same ADE type as the singularity of the curve. Moreover, the minimal resolution \tilde{X}_C comes with the following data associated with the intersection form on $H_2(\tilde{X}_C, \mathbb{Z})$. Recall that as a lattice with bilinear form the latter is isomorphic to $\mathbb{L} = 2E_8 \oplus U^3$ where U is the intersection form of quadric surface. The data associated with the minimal resolution \tilde{X}_C consists of the sublattice of $H_2(\tilde{X}_C, \mathbb{Z})$ spanned by the classes of exceptional curves of the resolution. These curves form a root system σ in this sublattice. Let \tilde{S}_C be the primitive hull in $H_2(\tilde{X}_C, \mathbb{Z})$ of these sublattices and the pull back to \tilde{X}_C of the class of a line in \mathbb{P}^2 . An abstract oriented homological type of a K3 surface is a sublattice in $H_2(\tilde{X}_C, \mathbb{Z})$, which in the case of a double cover over a ADE sextics is the image of \tilde{S}_C , plus the orientation of the positive definite plane in real subspace spanned by transcendental lattice given by the holomorphic 2-form on \tilde{X}_C (cf. [60] p.214).

Theorem 6.1. (cf.[50],[213],[218]) *There is one to one correspondence between oriented abstract homological types arising from sextics and the set of equisingular deformations of sextic curves with simple singularities. Moreover, the moduli space of sextics in each equisingular component (i.e. its quotient by the group of projective isomorphisms) is isomorphic to the moduli space of K3 surfaces with such abstract homological type.*

Particularly well understood class of such sextics is the class of maximizing ones i.e. for which the sum of Milnor numbers is 19 (i.e. the maximal possible). However, there is no classification of the fundamental groups for the curves of this type though very large number of cases was made explicite.

B. Sextics of torus type. Those are sextics given by an equation of the form $f_2^3 + f_3^2 = 0$ where f_i denotes a form of degree i .

If f_i generic than for such C , $\pi_1(\mathbb{P}^2 \setminus C)$ is the quotient of the braid group B_3 by its center ([219]). The fundamental group varies when $f_i = 0$ become singular or tangent to each other and there are many explicite calculations. For curves with simple singularities, having such type, the commutativity of the fundamental group of the complement is detected by the Alexander polynomial (Oka conjecture cf. [51], [46])

C. Sextics with triple points. Blow up of the plane at a triple point of a sextic results in Hirzebruch surface F_1 and a cover of degree 3 of projective line induced by projection from the triple point. Such curves and their braid monodromy were studied extensively by Degtyarev in his book [57] in a more general framework of trigonal curves on arbitrary Hirzebruch surfaces F_d . Relation between the braid monodromy and the graphs in 2-spheres leads to enumeration of extremal irreducible trigonal curves which shows that their number grows exponentially (as a function of appropriate parameter).

6.6. Zariski pairs. One of applications of the fundamental groups of the complements (as was envisioned and implemented in some cases by Zariski cf. [219],[221]) is detecting the existence of different connected components of the space of equisingular deformations of curves on the surface. Indeed, those deformations do not alter the fundamental group. In fact there are several natural topological equivalence relations of curves on surfaces interrelationship between which is a natural question.

Definition 6.2. Let X be an algebraic surface and let D_1, D_2 be divisors on X . Pairs (X, D_1) and (X, D_2) are equivalent if one of the following conditions is satisfied:

(A) There exist an irreducible variety T , a holomorphic map $\Phi : X \rightarrow T$ with a fiber biholomorphic to X , a divisor $\mathcal{D} \subset X$ such that Φ is a locally trivial fibration of the pair (X, \mathcal{D}) ³⁷ and such that there exist a pair of points $t_1, t_2 \in T$ the fibers of $\Phi|_{\mathcal{D}}$ over t_1, t_2 are D_1, D_2 respectively.

(A') There is a symplectic isotopy of pairs (X, D_i) i.e. (X, \mathcal{D}) in (A) is a pair of symplectic spaces with symplectic Φ with fiber being symplectomorphic to D_1, D_2 respectively.

(B) There exists a diffeomorphism (resp. PL equivalence, reps. homeomorphism, resp. a homotopy equivalence, resp. proper homotopy equivalence of the complements) of pairs $(X, D_1) = (X, D_2)$ i.e. one selects the corresponding type of a continuous map $X \rightarrow X$ taking subcomplex D_1 to D_2 .

(C) There exists an isomorphism of fundamental groups $\pi_1(X \setminus D_1) = \pi_1(X \setminus D_2)$ (or sometimes just equality of the Alexander polynomials).

(D) There exist the following:

(i) a one to one correspondence between irreducible components of D_i such that corresponding components are homeomorphic and

(ii) a one to one correspondence between singularities of D_i preserving the local type in X compatible with correspondence (i) between the components.

(E) There exist an automorphism of fields \mathbb{C}/\mathbb{Q} which takes (a deformation as in (a)) of the pair (X, D_1) to the pair (X, D_2) .

The names used in literature are respectively, equisingular deformation equivalence for (A), Zariski pairs³⁸ for (D)-equivalent but not (B)-equivalent pairs, π_1 -equivalent for (C), combinatorially equivalent for (D) and conjugation equivalent for (E).

The relation between these conditions is as follows: (A) implies (B) (Thom isotopy theorem), (B) implies (D) and also (C) by topological invariance of the fundamental groups. Relation between equivalences in (B) corresponding to different types of homeomorphisms of pairs are unknown in real dimension 4 and finally (E) implies (D).

Large and continuing to increase volume of papers deals with finding examples confirming that these implications cannot be reversed, though until 80s connected components of the strata were viewed as an aberration. The conditions found in [105] delineate the range of combinatorial data for which the strata are connected but numerous examples found up to date outside of this range, suggest that disconnectedness of equisingular families is a widespread occurrence. At the same time no systematic theory of Zariski pairs as to classification or distribution did emerge. A good survey of this vast subject is given in [15]. Further non-trivial results on the relations between above equivalences are as follows.

(C) or (D) does not imply (A): Shirane [205] showed that curves in equisingular families constructed earlier by Shimada (cf. [199]) cannot be transformed by a homeomorphism of \mathbb{P}^2 though fundamental groups are isomorphic. Work [65] contains examples of such type in the case of sextics.

(D) does not imply (C) for arrangements of lines defined over \mathbb{Q} : cf. [107] and references there for other numerous examples found by those authors giving arrangements of lines for which (D) does not imply (C). (D) does not imply (C) for reducible curves with components being a smooth curve and a union of certain 3 tangent lines cf. [204]. k -tuples

³⁷i.e. for any $t \in T$ there is a neighborhood $U \subset T$ such that $\Phi^{-1}(U)$ and $T \times \Phi^{-1}(t)$ are equivalent as stratified spaces

³⁸ the term was coined in [22] in reference to first example found by Zariski in 1930s

of pairwise distinct reducible curves with one component of degree 4 and several quadrics were considered in [26] (also, see there the references to the works of these two authors presenting many other examples of failure of this implication).

(E) does not imply (C): [14] gives examples of conjugate line arrangements with non-isomorphic fundamental groups. See also [13] where one has conjugacy over \mathbb{Q} and isomorphism of the fundamental groups and even homeomorphism of the complements but there is no homeomorphism of pairs. Examples are the appropriately chosen unions of sextics and lines. Moreover, (E) and (C) do not imply (B) (cf. [12])

Distinct connected components often even contain curves conjugate over \mathbb{Q} (arithmetic Zariski pairs cf. [201]).

The examples of Zariski pairs or multiplets ³⁹ fall in the following groups

A. Arrangements of lines and conics. [108]

B. Curves of degree 6 and trigonal curves (cf [61])

C. Other sporadic examples such as reducible curves with components of low degree (cf. [176]).

Methods employed in these works include study of the Alexander invariants, Hurwitz equivalence classes of braid monodromy and more ad hoc invariants of the fundamental groups (e.g. existence of dihedral cover of the complement to a curve is an invariant of the fundamental group and hence can be used to distinguish classes of equisingular deformations) and other sporadic methods (cf. [202], [203], [206]). Problems here include the question of combinatorial invariance of the Alexander polynomials and more generally the characteristic varieties or existence of Zariski pairs defined over \mathbb{Q} .

Many examples of fundamental groups of Zariski pairs were computed in [58]

An interesting problem about Zariski multiplets is understanding the asymptotic of the number of connected components of equisingular families when the number of classes of the curves grows. Consideration of families of trigonal curves on Hirzebruch surfaces, shows that the number of equisingular components grows exponentially. One can show exponential growth of the number of connected components of equisingular families of plane curves with nodes and cusps when degree grows by considering generic projections of surfaces of general type in a families which have exponentially large growth of the number of connected components of the moduli spaces (cf. [154]). This follows from the explicit formulas in terms of Chern numbers of the surfaces for the numbers of cusps, nodes and the degree of the branching curves of generic projection (cf. [131]).

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³⁹Zariski k -tuples are sets of k curves in distinct classes of equivalence (B) but in the same class (D); sometimes, in a more loose usage, the reference is to sets of k curves in distinct classes for some equivalences (A)-(E) but not another.

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