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Homotopy groups of complements to ample divisors

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Abstract.

Homotopy groups of the complements to divisors with ample components on non-singular projective varieties are considered as the modules over the fundamental group. We prove a vanishing theorem and consider the calculation of supports of these modules by relating them to the cohomology of local systems. We review previous work on the local study of isolated non-normal crossings. As an application, we obtain information about the support loci of homotopy groups of arrangements of hyperplanes.

§1. Introduction

An interesting problem in the study of the topology of algebraic varieties is to understand the fundamental group of the complement to a divisor on a non-singular algebraic variety in terms of the geometry of the divisor. Works of Abhyankar ([1]) and Nori ([31]) show that, if C is an irreducible curve on a non-singular algebraic surface X, then for some effective constant $F(C)$ depending on the local type of singularities of C, the inequality $C^2 > F(C)$ implies that the kernel of the map $\pi_1(X - C) \to \pi_1(X)$ belongs to the center of $\pi_1(X - C)$. For example, if X is simply connected, then $\pi_1(X - C)$ is abelian. Historically, such results were originated in the so-called Zariski problem and we refer to [16] for a survey. The case of non-abelian fundamental groups of complements, notably when $X = \mathbf{P}^2$, is also very interesting. The geometric information, such as the dimensions of the linear systems defined by singularities of the curve, becomes essential in descriptions of fundamental groups and their invariants (cf. [37], [21] [24]). Recently, analogous questions about fundamental groups of the complements in the case when X is symplectic began to attract attention as well (cf. [4]).

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In the present work, we shall show that, in appropriate settings, the relationship between the topology of the complement and the geometry of the divisor can be extended to some higher homotopy groups. Some work in this direction already was done. In [22], we show that if V is a hypersurface in $\mathbf{C}^{n+1}$ with isolated singularities whose compactification in $\mathbf{P}^{n+1}$ is transversal to the hyperplane at infinity, then the first homotopy groups of the complement are the following:

$$(1) \qquad \pi_1(\mathbf{C}^{n+1} - V) = \mathbf{Z}, \quad \pi_i(\mathbf{C}^{n+1} - V) \text{ for } 2 \le i \le n-1$$

Moreover, the next homotopy group $\pi_n(\mathbf{C}^{n+1} - V)$ depends on the local type of the singularities and also on the geometry of a collection of singularities as a finite subset in $\mathbf{C}^{n+1}$. It can also be described via a generalization of the van Kampen procedure in terms of pencils of hyperplane sections (cf. [22], [8]). Recently, homotopy groups of arrangements were considered in [13] and [32].

Below, we shall extend these results in two directions. On the one hand, we shall consider complements on arbitrary algebraic varieties rather than just in projective space. The latter case, however, appears to be the most important one due to a variety of interfaces with other areas – e.g. the study of arrangements of hyperplanes. On the other hand, we do not assume here that V has isolated singularities, but rather that the divisor D has normal crossings except for finitely many points. The effect of this is that the fundamental group, which plays the key role in the description of higher homotopy, may be abelian rather than cyclic, as is the case in (1), and the theory which we obtain is *abelian* rather than *cyclic*.

In the next section, we prove the triviality of the action of the fundamental group on higher homotopy groups in certain situations (cf. Theorem 2.1). This implies that all information about homotopy groups in these cases is homological (π_1 in these situations is automatically abelian). In some instances, as result of homological calculations, one obtains a *vanishing* of homotopy groups in certain range. In particular if D is a divisor in $\mathbf{P}^{n+1}$ having only isolated non normal crossings and the number of components greater than $n+1$ then $\pi_i(\mathbf{P}^{n+1} - D) = 0$ in the range $2 \le i \le n-1$. The results of section 2 also isolate first non-trivial homotopy group in the sense that it is a non-trivial π_1-module.

In Section 3 we define our main invariant of the homotopy group, i.e. a sub-variety of the spectrum of the group ring of the fundamental group which is the support of the first non trivial homotopy group considered as the module over π_1. We call these sub-varieties *characteristic* and show that they are related to the jumping loci for the cohomology of local systems. The latter have a very restricted structure (cf. [2]),– i.e.

they are unions of translated by points of finite order subgroups which also suggest the numerical data that describe these varieties completely.

The methods of obtaining numerical data specifying the characteristic varieties from the geometry of the divisor are discussed in the Section 6. This is done by using the Hodge theory of abelian covers, which is studied in section 5, and relies on our local study of the isolated non normal crossings in [26] and [14]. The results of these papers are discussed in the Section 4 where, among other things, we compare cyclic theory of isolated singularities with abelian theory of isolated-non normal crossings having more than one components. In Section 7, we review cases to which the results of Section 6 can be applied. In particular, the Kummer configuration yields an arrangement of planes in $\mathbf{P}^3$ with non-trivial π_2 of the complement which we calculate. In the final section, we show the relationship between the invariants of the homotopy groups and the motivic zeta function of Denef-Loeser.

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§2. Action of the fundamental group on homotopy groups.

In this section we discuss homotopy groups, in a certain range of dimensions, for a class of quasi-projective varieties. This is done in two steps. Firstly, we show that these varieties support a trivial action of π_1 (in particular are *nilpotent* in certain range). Secondly, we use homological calculations to determine these homotopy groups and to describe cases when homotopy groups vanish.

Recall that homotopy groups $\pi_n(X,x)$ of a topological space X are $\pi_1(X,x)$-modules, with the action given by the "change of the base point" (cf. [34]). In the case when $\pi_i(X) = 0$ for $1 < i < n$, this action on $\pi_n(X)$, which is isomorphic to $H_n(\tilde{X})$ where $\tilde{X}$ is the universal cover of X, coincides with the action of the fundamental group on the homology of the universal cover via deck transformations. A topological space is called k-simple if the action of the fundamental group on $\pi_i(X)$ is trivial for $i \le k$ (cf. [34]).

Examples of k-simple spaces appearing naturally in algebraic geometry are the following. Locally, they come up when one looks at the complement to a union of germs of divisors in $\mathbf{C}^{n+1}$ forming an isolated non-normal crossing. This situation was studied in [26]. More generally (cf. [14]), instead of divisors in $\mathbf{C}^{n+1}$, one can look at the union of germs of divisors in a germ of a complex space Y having a link which is $(\dim Y - 2)$-connected. Examples of such local singularities are provided by the cones over a normal crossings divisor in $\mathbf{P}^n_{\mathbf{C}}$, in particular by cones over generic arrangements of hyperplanes. These k-simple spaces ($k = \dim Y - 2$) are, of course, Stein spaces and their theory will be reviewed in Section 4. Global k-simple examples are given by the following quasi-projective varieties:

Theorem 2.1. *Let X be a simply connected projective manifold and $D = \bigcup D_i$ be a divisor with normal crossings such that its all components D_i are smooth and ample. Then $\pi_1(X - D)$ is abelian and its action on $\pi_i(X - D)$ is trivial for $2 \le i \le \dim X - 1$.*

The proof is similar to the one presented in the local case in [26]. It uses the reduction to the case of normal crossings divisors using Lefschetz hyperplane section theorem and then surjectivity of $\pi_i(D_i - \bigcup_{j \neq i} D_j) \to \pi_i(X - D)$ which follows from ampleness of the components D_i.

This theorem reduces the calculation of the homotopy group to the calculation of *homology* of the complements. The latter can be done using the exact sequence:

$$(2)\quad H_2(X) \to H_2(X, X-D) \to H_1(X-D) \to H_1(X) \to H_1(X, X-D)$$

and the isomorphisms

$$H_j(X, X - D) = H^{2n+2-j}(D)$$

We obtain hence:

Corollary 2.2. *Let $H = \mathbf{Z}^N$ be a free abelian group generated by components of the divisor D. Let*

$$h : H_2(X, \mathbf{Z}) \to H$$

given by $a \to \sum (a, D_i) D_i$ where $a \in H_2(X), D_i \in H^2(X)$ and (a, D) is the Kronecker pairing. Then $\pi_1(X - D) = \operatorname{Coker} h$. For example, if $X = \mathbf{P}^{n+1}$ and one of the components D_i $(i = 1, .., r+1)$ is a hyperplane, then $\pi_1(X - D) = \mathbf{Z}^r$. Let X be a hypersurface in $\mathbf{P}^{n+1}$ and D be a union of $r + 1$-hyperplanes. Then $H_1(X - D) = \mathbf{Z}^r$.

The following result can be used for the calculation of the homology of some branched covers of X:

Corollary 2.3. *Let $D_i \in |\mathcal{L}_i^{m_i}| (i = 1, ..., r)$ such that D_i are divisors on X having isolated non-normal crossings and D_i is the zero set of $f_i \in H^0(X, \mathcal{O}(D_i))$. Let $s_i \in H^0(X, \mathcal{L}_i) \neq 0$. Then $\bar{U}_{m_1,..,m_r}$ given in the total space of $\oplus \mathcal{L}_i$ by $s_i^{m_i} = f_i$ is the cover corresponding the surjection: $\phi : H_1(X - D) \to G = \oplus_{i=1} \mathbf{Z}/m_i\mathbf{Z}$ The projection $\bar{U}_{m_1,..,m_r} \to X$ induces the isomorphism: $H_i(\bar{U}_{m_1,...,m_r}) \to H_i(X)$ for $i \leq n-1$.*

The next theorem is an abelian version of the result in [22] and identifies "the first non-trivial homotopy group" in the sense of [22].

Theorem 2.4. (a) *Let $X = \mathbf{P}^{n+1}$ and D be an arrangement of $r+1$ hypersurfaces as in Corollary 2.2 (i.e., such that one of the hypersurfaces has degree 1) and having finitely many non-normal crossings. Then $\pi_i(\mathbf{P}^{n+1} - D) = 0$ for $2 \leq i \leq n-1$. If all intersections are the normal crossings, then the $\pi_n(\mathbf{P}^{n+1} - D) = 0$.*
(b) *Let V be a complete intersection in $\mathbf{P}^N$ and $\dim V = n+1$. Let D be the arrangement of $r+1$ hyperplane sections of V having isolated non-normal crossings. Then $\pi_1(V - D) = \mathbf{Z}^r$ and $\pi_i(V - D) = 0$ for $2 \leq i \leq n-1$.*

Proof. Consider first (a). The claimed vanishing is a consequence of the Lefschetz hyperplane section theorem (cf. [18]) and the second part of (a). The first part follows by induction, with the inductive step being the vanishing of $\pi_n(\mathbf{P}^{n+1} - D)$ where D is an arrangement of hypersurfaces with normal crossings. Taking into account the triviality of the action of $\pi_1(\mathbf{P}^{n+1} - D)$ on π_n, the claim is a consequence of the exact sequence (cf. [6]):

$$H_{n+1}(\mathbf{P}^{n+1} - D) \to H_{n+1}(\mathbf{Z}^r) \to \pi_n(\mathbf{P}^{n+1} - D)_{\mathbf{Z}^r} \to \tag{3}$$

$$\to H_n(\mathbf{P}^{n+1} - D) \to H_n(\mathbf{Z}^r) \to 0$$

and the calculation of the homology of $\mathbf{P}^{n+1} - D$. The latter can be done using Mayer Vietoris spectral sequence (cf. [26]). The proof of (b) is similar.

§3. Characteristic varieties of homotopy groups

In this section, we study the support of the first homotopy group of quasi-projective varieties from Section 2 on which the action of π_1 fails to be trivial. This support is a subvariety of $\operatorname{Spec} \mathbf{C}[\pi_1]$, which we call *the characteristic variety.* We show that in the range $2 \leq i \leq k-1$, in which the action of π_1 on π_i is trivial, the homology H_i of the local systems, corresponding to the points of the algebraic group $\operatorname{Spec} \mathbf{C}[\pi_1]$ different from the identity, is trivial. Moreover, the first homotopy group outside this range, i.e. π_k, determines the homology H_k of the local systems. Vice versa, the (co)homology of local systems determines the support of $\pi_k \otimes \mathbf{C}$ as π_1-module. This yields, in the algebro-geometric context, a "linear" structure of the characteristic varieties.

Theorem 3.1. *Let X be a topological space such that its fundamental group $\pi_1(X) = A$ is abelian. Assume that for an ideal $\wp$ in $\mathbf{C}[A]$ the localization of the homotopy groups is trivial for $2 \leq i < k$: $\pi_i(X)_\wp = 0$. Then $H_i(\tilde{X})_\wp = 0$ for $1 \leq i < k$ and $H_k(\tilde{X})_\wp = \pi_k(X)_\wp$*

Sketch of the proof. The universal cover $\tilde{X}$ of X is a simply connected space on which A acts freely. For such a space, the group A acts on $H_j(\tilde{X}, \mathbf{C})$ for any j and on the homotopy groups $\pi_j(\tilde{X}, \tilde{x}_0) = \pi_j(X, x_0)$ $(j \geq 2)$ so that the Hurewicz map: $\pi_j(\tilde{X}) \to H_j(\tilde{X})$ is $\pi_1(X)$-equivariant (cf. [34] Ch.7, Cor. 3.7).

Let us consider a simply connected CW-complex Y on which an abelian group A acts freely. The group A then acts on the homotopy groups via composition of the map $\pi_n(Y, x) \to \pi_n(Y, a(x))$ and the identification $\pi_n(Y, a(x))$ and $\pi_n(Y, x)$, which is independent of the choice of a path connecting x and $a(x)$ due to $\pi_1(Y) = 0$. The claim is that, if $\pi_i(Y)_\wp = 0$ for $1 < i \leq n-1$, then $\pi_n(Y)_\wp = H_n(Y)_\wp$. The theorem above will follow for $Y = \tilde{X}$ and $G = \pi_1(X)$.

The claim can be obtained by induction over n as follows. Consider the fibration of path space $Maps(I, Y) \to Y \times Y$. This fibration is equivariant (where the action on $Y \times Y$ is diagonal). Space $Maps(I, Y)$ is homotopy equivalent to Y. We have the spectral sequence:

$$E^2_{p,q} : H_p(Y \times Y, H_q(\Omega Y)) \to H_{p+q}(Y)$$

This spectral sequence is equivariant. The action on the homology of fiber is given by $av = p_* a_*(v)$ where $a_* : H_i(\Omega_x Y) \to H_i(\Omega_{gx} Y)$ and p_* is the natural identification of the homology of different fibers in a fibration with a simply-connected base. Localizing at $\wp$, due to inductive assumption on Y, we obtain that the terms with $0 < p \leq n-1$ and $0 < q \leq n-2$ are zeros. In localized spectral sequence we can identify the map $H_n(Y)_\wp \to E^{n,0}_\infty = \operatorname{Ker} d^n_{n,0} : H_n(Y \times Y)_\wp \to H_{n-1}(\Omega Y)_\wp$ with the map $i_\Delta : H_n(Y)_\wp \to H_n(Y \times Y)_\wp$ corresponding to the diagonal embedding. Moreover, $d^{n,0}_n$ is surjective (since $H_{n-1}(Y)_\wp = 0$). Hence we have an exact sequence:

$$0 \to \operatorname{Im}(i_\Delta)_\wp \to H_n(Y \times Y)_\wp \to H_{n-1}(\Omega Y)_\wp \to 0$$

and since cokernel of $H_n(Y)_\wp \to H_n(Y \times Y)_\wp = H_n(Y)_\wp \oplus H_n(Y)_\wp$ is isomorphic to $H_n(Y)_\wp$, due assumed vanishing, we obtain that $H_n(Y)_\wp = H_{n-1}(\Omega Y)_\wp = \pi_n(Y)_\wp$.

We shall apply this theorem to $(n-1)$-simple spaces. For such a space the support of $\pi_i(X) \otimes_{\mathbf{Z}} \mathbf{C}$ as a $\mathbf{C}[\pi_1(X)]$ module belongs for $2 \le i \le n-1$ to the maximal ideal of the identity of the group $\operatorname{Spec} \mathbf{C}[\pi_1(X)] = \operatorname{Char}[\pi_1(X)]$. This maximal ideal is just the augmentation ideal of the group ring. Hence the localization at a prime ideal not belonging to the maximal ideal of the identity satisfies (after tensoring with $\mathbf{C}$) the assumption of Theorem 3.1. This allows, for $(n-1)$-simple spaces, to express the homology of the local systems in terms of the homotopy groups $\pi_n(X)$:

Theorem 3.2. *Let $\rho \in \operatorname{Char} \pi_1(X)$ be a character of the fundamental group different from the identity and let $\mathbf{C}_\rho$ be $\mathbf{C}$ considered as $\mathbf{C}[\pi_1(X)]$ module via the character ρ. Then*

$$H_i(X, \rho) = 0 \ (i \le n-1) \quad H_n(X, \rho) = \pi_n(X) \otimes_{\mathbf{C}[\pi_1(X)]} \mathbf{C}_\rho$$

Proof. The proof is similar to the one in the case when X is a complement to a plane curve (cf. [24]) and the local case (cf. [26]). Consider the spectral sequence (cf. [7], ch.XVI, th.8.4):

$$H_p(\pi_1(X), H_q(\tilde{X})_\rho) \Rightarrow H_{p+q}(X, \rho)$$

where $H_*(\tilde{X})_\rho$ is the homology of the complex $C(\tilde{X}) \otimes_{\mathbf{Z}} \mathbf{C}$ with the action of $\pi_1(X)$ given by $g(e \otimes \alpha) = g \cdot e \otimes \rho(g^{-1})\alpha$. We can localize this spectral sequence at the maximal ideal $\wp_\rho$ of $\operatorname{Spec} \mathbf{C}[\pi_1(X)]$ corresponding to the character ρ. The resulting spectral sequence has $E_2^{i,j} = 0$ for $1 \le j \le n-1$. The exact sequence of low degree terms yields: $H_n(X, \rho) = H_n(\tilde{X}) \otimes_{\mathbf{C}[\pi_1(X)]} \mathbf{C}_\rho$ which together with Theorem 3.1 proves the claim.

Now we are ready to define the main invariant.

Definition 3.3. *The k-th characteristic variety $V_k(\pi_n(X))$ of the homotopy group $\pi_n(X)$ is the zero set of the k-th Fitting ideal of $\pi_n(X)$, i.e. the zero set of minors of order $(n-k+1) \times (n-k+1)$ of Φ in a presentation*

$$\Phi : \mathbf{C}[\pi_1(X)]^m \to \mathbf{C}[\pi_1(X)]^n \to \pi_n(X) \to 0$$

of $\pi_1(X)$ module $\pi_n(X)$ via generators and relations. Alternatively (cf. Theorem 3.2) outside of $\rho = 1$, $V_k(\pi_n(X))$ is the set of characters $\rho \in \operatorname{Char}[\pi_1(X)]$ such that $\dim H_n(X, \rho) \ge k$.

Theorem 3.2 combined with the results of [2] yields the following strong structure property (for possibly non-essential characters):

Theorem 3.4. *The characteristic variety $V_k(\pi_n(X-D))$ is a union of translated subgroups S_j of the group* $\mathrm{Char}\,\pi_1(X-D)$ *by unitary characters ρ_j:*

$$V_k(\pi_n(X-D)) = \bigcup \rho_j S_j$$

This is an immediate consequence of the interpretation 3.2 and the following theorem applied to a resolution $\hat{X}$ of non-normal crossings of D:

Theorem 3.5 (Arapura [2]). *Let $\hat{X}$ be a projective manifold such that $H^1(\hat{X}, \mathbf{C}) = 0$. Let $\hat{D}$ be a divisor with normal crossings. Then there exists a finite number of unitary characters $\rho_j \in \mathrm{Char}\,\pi_1(\hat{X}-\hat{D})$ and holomorphic maps $f_j : \hat{X}-\hat{D} \to T_j$ into complex tori T_j such that the set $\Sigma^k(\hat{X}-\hat{D}) = \{\rho \in \mathrm{Char}\,\pi_1(\hat{X}-\hat{D}) \mid \dim H^k(\hat{X}-\hat{D}, \rho) \geq 1\}$ coincides with $\bigcup \rho_j f_i^* H^1(T_j, \mathbf{C}^*)$. In particular, Σ^k is a union of translated by unitary characters subgroups of* $\mathrm{Char}\,\pi_1(X-D)$.

The components of Σ^1 can *all* be obtained using the maps $X - D$ onto the curves with negative Euler characteristics (cf. [2]). In the case $k > 1$, maps onto quasi-projective algebraic varieties with abelian fundamental group and vanishing π_i for $2 \leq i \leq k-1$ allow one to construct components of $V(\pi_k)$ (cf. Example 7.4 below).

§4. Review of local theory of isolated non-normal crossings

Local theory of isolated singularities of holomorphic functions provides a beautiful interplay between algebraic geometry and topology and in particular the topology of (high dimensional) links (cf. [30]). The main structure is the Milnor fibration $\partial B_\epsilon - V_f^0 \cap \partial B_\epsilon \to S^1$, where V_f^0 is the zero set of a holomorphic function $f(x_1, .., x_{n+1})$ and B_ϵ is a ball of a small radius ϵ about $\mathcal{O}$ (the fibration exist even in the non-isolated case). If the singularity of f at $\mathcal{O}$ is isolated, then the fiber M_f of this fibration (the Milnor fiber) is homotopy equivalent to a wedge of spheres: $S^n \vee ... \vee S^n$. Going around the circle, which is the base of Milnor's fibration, yields the monodromy: $H^n(M_f) \to H^n(M_f)$. It has as its eigenvalues only the roots of unity $\exp(2\pi i\kappa)$ $(\kappa \in \mathbf{Q})$. Moreover, there are several ways to pick a particular value of the logarithm κ of an eigenvalue of the monodromy so that the corresponding rational number will have some geometric significance. One of the ways to do this depends on the existence of a Mixed Hodge structure (cf. [35]) on $H^n(M_f)$. The value of the logarithm is selected so that its integer part is determined by the degree of the component of $Gr_*^F H^n(M_f)$ (graded space associated with the Hodge filtration) on which particular eigenvalue of the semi-simple part of the monodromy appears.

Some of the data above can be obtained by considering the infinite cyclic cover of $\partial B_\epsilon - V_f^0 \cap \partial B_\epsilon$ instead of Milnor fibration. Such a cover is well-defined since $H_1(\partial B_\epsilon - V_f^0 \cap \partial B_\epsilon, \mathbf{Z}) = \mathbf{Z}$ for $n > 1$. For example, the universal cyclic cover is diffeomorphic to the product $M_f \times \mathbf{R}$. The monodromy can be identified with the deck transformation of the infinite cover.

With such reformulation, the Milnor theory can be extended to the case of germs of isolated non-normal crossings in $\mathbf{C}^{n+1}$ (cf. [26]), i.e. germs of functions $f_1 \cdot ... \cdot f_r$ such that the intersection points of divisors $f_1 = 0, ..., f_r = 0$ are normal crossings except for the origin $\mathcal{O}$ (more general case of germs of complex spaces with isolated singularities considered in (cf. [14]). The results, using infinite covers as a substitute for the Milnor fiber, are parallel to the above mentioned results in the isolated singularities case. Notice, however, that though the theory of Milnor fibers is applicable to germs of INNC, much less detailed information can be obtained since these singularities are not isolated for $n > 1$. For example, the Milnor fiber is not even simply-connected (cf., below however, where quite a bit of information about the Milnor fiber can be obtained as a consequence of the present approach).

Let D be a germ of INNC which belongs to a ball B_ϵ about $\mathcal{O}$ and which has r irreducible components. We have the isomorphism $H_1(\partial B_\epsilon - D, \mathbf{Z}) = \mathbf{Z}^r$ and hence the universal abelian cover of $\partial B_\epsilon - D$ has $\mathbf{Z}^r$ as the covering group. The replacement of the Milnor fiber in this abelian situation is the universal abelian cover $\widetilde{\partial B_\epsilon - D}$. Notice that a locally trivial fibration of $\partial B_\epsilon - D$ over a torus does not exist in general since typically $\widetilde{\partial B_\epsilon - D}$ has the homotopy type of an infinite complex. We have the following (cf. [26]):

Theorem 4.1. *For $n > 1$, the fundamental group $\pi_1(\partial B_\epsilon - D)$ is free abelian. The universal (abelian) cover $\widetilde{\partial B_\epsilon - D}$ is $(n-1)$-connected. In particular, $H_n(\widetilde{\partial B_\epsilon - D}, \mathbf{Z})$ is isomorphic to the homotopy group $\pi_n(\partial B_\epsilon - D)$. The latter isomorphism is the isomorphism of $\mathbf{Z}[\pi_1(\partial B_\epsilon - D)]$-modules where the module structure on the homology is given by the action of $\pi_1(\partial B_\epsilon - D)$ on the universal cover via deck transformations and the action on the homotopy is given by the Whitehead product (cf. [34]).*

Notice that the case when D is a divisor with normal crossings is "a non-singular" case since the universal cover is contractible. The simplest example of INNC is given in $\mathbf{C}^{n+1}$ by the equation $l_1 \cdot ... \cdot l_r = 0$, where

l_i are *generic* linear forms (i.e. a cone over a generic arrangement of hyperplanes in $\mathbf{P}^n$). Since the complement to a generic arrangement of r hyperplanes in $\mathbf{P}^n$ has a homotopy type of n-skeleton of the product of $r-1$-copies of the circle S^1 (in minimal cell decomposition in which one has $\binom{r-1}{i}$ cells of dimension i) one can calculate the module structure on the π_n of such skeleton. Its universal cover is obtained by removing the $\mathbf{Z}^{r-1}$ orbits of all open faces of a dimension greater than n in the unit cube in $\mathbf{R}^{r-1}$. Hence $\pi_n(\partial B_\epsilon - D) = H_n(\widetilde{\partial B_\epsilon - D}, \mathbf{Z})$ ($\widetilde{\partial B_\epsilon - D}$ is the universal cover). The chain complex of the universal cover of $(S^1)^{r-1}$ can be identified with the Koszul complex of the group ring of $\mathbf{Z}^{r-1} = \mathbf{Z}^r/(1,...,1)$ (so that the generators of $\mathbf{Z}^r$ correspond to the standard generators of $H_1(\partial B_\epsilon - D)$). The system of parameters of this Koszul complex is $(t_1-1,..,t_r-1)$. Hence $H_n(\widetilde{\partial B_\epsilon - D}, \mathbf{Z}) = \operatorname{Ker} \Lambda^n R \to \Lambda^{n-1} R$ where $R = \mathbf{Z}[t_1,..,t_r]/(t_1 \cdot ... \cdot t_r - 1)$. As a result, one has the following presentation:

$$
\begin{aligned}
(4)\quad & \Lambda^{n+1}(\mathbf{Z}[t_1,t_1^{-1},...,t_r,t_r^{-1}]/(t_1...,t_r-1)^r) \to \\
& \Lambda^n(\mathbf{Z}[t_1,t_1^{-1},...,t_r,t_r^{-1}]/(t_1...,t_r-1)^r) \to \pi_n(\mathbf{C}^{n+1} - \bigcup D_i) \to 0
\end{aligned}
$$

In particular, the support of the π_n is the subgroup $t_1 \cdot ... \cdot t_r = 1$.

We summarize the similarities between the case of hypersurfaces with isolated singularities and INNC in the table 1 in the next page (with 4.1 justifying the first three rows):

In the case of isolated singularities one has the isomorphism: $\pi_n(\partial B_\epsilon - D) = H_n(\widetilde{\partial B_\epsilon - D})$ as $\mathbf{Z}[t,t^{-1}]$-modules, where the module structure on the right is given by the monodromy action. In particular, it is a torsion module and its support is a subset of $\operatorname{Char} \mathbf{Z} = \mathbf{C}^*$ consisting of the eigenvalues of the monodromy of Milnor fibration. Monodromy theorem ([30]) is equivalent to the assertion that eigenvalues are the torsion points of $\mathbf{C}^*$. A generalization of this is the following:

Conjecture 4.2. *The support of $\pi_n(\partial B_\epsilon - D)$ is a union of translated subgroups of $\operatorname{Char} \pi_1(\partial B_\epsilon - D)$ by points of finite order.*

4.2 is a local analog of the result 3.4 in the quasi-projective case. Now let us describe a partial result in the direction of 4.2 describing some components of a characteristic variety which satisfy 4.2, and which also will explain last two rows in the above table.

As already was mentioned, the cohomology group of the Milnor fiber $H^n(M_f, \mathbf{C})$ of an isolated singularity support a Mixed Hodge structure (cf. [35]). The monodromy splits into the product of the semi-simple and

Isolated singularities	**INNC**
Milnor fiber	Infinite abelian cover
Homology of Milnor fiber	π_n
Monodromy	π_1 – module structure on π_n
Eigenvalues of monodromy	Characteristic varieties of $\pi_{\rm n}$
Monodromy theorem	Translated subgroup Property
Multiplier Ideals	Multivariable Ideals of Quasiadjucntion
Spectrum	Polytopes of quasiadjunction

Table 1

the unipotent part. The semi-simple part leaves the Hodge filtration invariant. The latter allows one to split the eigenvalues into groups corresponding to the components of $Gr^F H^n(M_f, \mathbf{C})$, depending on the graded piece on which the eigenvalue appears. As a consequence, one can assign a rational number to each eigenvalue, i.e., its logarithm so that its integer part is determined by the group to which the eigenvalue belongs (we refer to [35] for the exact description). In other words, we obtain a lift of the support of the homotopy group of the Milnor fiber into the universal cover of the subgroup of unitary characters of $\mathbf{Z}$ (the eigenvalues of the monodromy having a finite order are unitary).

In the abelian (local) case, we have the following. Let us consider the universal cover of the subgroup $\mathrm{Char}^u(\pi_1(\partial B_\epsilon - D))$ of unitary characters. It is isomorphic to $\mathbf{R}^r$ and one can take the unit cube as the fundamental domain of the covering group (i.e. $\mathbf{Z}^r$). We assign an element in the fundamental domain to a unitary character χ having finite order using the following interpretation of the unitary characters from $V_k(\pi_n(\partial B_\epsilon - D))$ (cf. [26] Prop. 4.5).

Proposition 4.3. *Let $G = \oplus_{i=1}^{i=r}\mathbf{Z}/m_i\mathbf{Z}$ be a finite quotient of $\pi_1(\partial B_\epsilon - D)$ and let $\chi \in \mathrm{Char}(\pi_1(\partial B_\epsilon - D))$ which is the image of a character of G. Then the link $X_{m_1,..,m_r}$ of the isolated complete intersection singularity:*

$$z_1^{m_1} = f_1(x_1, ..., x_{n+1}), ..., z_r^{m_r} = f_r(x_1, ..., x_{n+1}) \tag{5}$$

is a $n-1$-connected $2n+1$-manifold, which is a cover of ∂B_ϵ branched over INNC D. The condition: $\chi \in V_k(\pi_n(\partial B_\epsilon - D))$ and χ is essential (cf. 5.3) is equivalent to

$$k = \dim\{v \in H_n(X_{m_1,...,m_r}) \mid gv = \chi(g)v \forall g \in G\}$$

Note that the covering map $X_{m_1,..,m_r} \to \partial B_\epsilon$ is just a projection $(z_1, ..., z_r, x_1, .., x_{n+1}) \to (x_1, ..., x_{n+1})$. Next, we shall use the Mixed Hodge structure on the cohomology of the link (5) (cf. [36]). The Hodge filtration

$$F^0H^n(X_{m_1,...m_r}) \supset ... \supset F^nH^n(X_{m_1,...m_r}) \supset 0$$

is preserved by the group G. The logarithms of characters which appear on the subspace $F^nH^n(X_{m_1,...m_r})$ (i.e., the vectors $\log\chi = (\xi_1, .., \xi_r)$ with $0 \le \xi_i < 1, \forall i$ such that $\exp(2\pi i\xi_1), ..., \exp(2\pi i\xi_r)$ is a character χ of $H_1(\partial B_\epsilon - D)$ in coordinates given by the generators $H_1(\partial B_\epsilon - D)$) form a polytope in the sense of the following

Definition 4.4. *A polytope in the unit cube $\mathcal{U} = \{\mathbf{x} = (x_1, ..., x_n) \mid 0 \le x_i \le 1 \forall i\}$ is a subset of $\mathcal{U}$ formed by the solutions of a system of inequalities $\mathbf{a_k} \cdot \mathbf{x} \le c_k$ for some constants c_k (resp. vectors $\mathbf{a_k}$) such that $\mathbf{a_k} = (a_k^1, ..., a_k^i, ..., a_k^n), 0 \le a_k^i \in \mathbf{Q}$ and $0 \le c_k \in \mathbf{Q}, \forall i, k$. A face of a polytope $\mathcal{P}$ is a subset of its boundary $\partial\mathcal{P}$ which has the form $\partial\mathcal{P} \cap H$ for a hyperplane H different from one of $2n$ hyperplanes $x_i = 0, 1$.*

We have the following:

Theorem 4.5. *To each germ of INNC D and $l, 0 \le l \le n$ corresponds a collection $\mathbf{P}_l$ of polytopes $\mathcal{P}_{k,l} \in \mathbf{P}_l$ such that a vector $\log\chi \in \mathbf{Q}^r$ in unit cube belongs to one of the polytopes $\mathcal{P}_{k,l}$ if and only if $\dim\{v \in F^l/F^{l+1}H^n \mid gv = \chi(g)v\} = k$. In particular, $V_k(\pi_n(\partial B_\epsilon - D)) = \bigcup_k \exp\mathcal{P}_{k,l}$ where $\mathbf{R}^r \to \mathrm{Char}^u\, H_1(\partial B_\epsilon - D)$ is the exponential map.*

In the cyclic case, each of $\mathcal{P}_{k,l}$ is a rational number ξ such that $\exp(2\pi i\xi)$ is an eigenvalue of the monodromy having a multiplicity k which appears on $F^l/F^{l+1}H^n(M_f)$, i.e. is an element of the spectrum having a multiplicity k (in the case $l = n$ one obtains the constant of quasi-adjunction from [20]).

Remark 4.6. *In the case $n = 1$, i.e., the case of reducible plane curves, we have the polytopes of quasi-adjunction studied in* [25]. *In particular, these polytopes are related to the multi-variable log-canonical thresholds and multiplier ideals (cf. remark 2.6 and section 4.2 respect. in* [25]*). Similar relations exist in the case of INNC discussed here. In particular, to each face $\mathcal{F}$ of a polytope of quasiadjunction for INNC corresponds the ideal of quasiadjunction $\mathcal{A}_{\mathcal{F}}$ in the local ring of the singular point of INNC used below (cf.* (6.3)*).*

In the case of isolated singularities, there are very explicit and beautiful calculations of the eigenvalues of the monodromy and spectrum of singularities. We would like to pose the following problem:

Problem 4.7. *Calculate the characteristic varieties of INNC with $\mathbf{C}^*$-actions and in the case when D_i are generic for their Newton polytopes. What are the polytopes described in Theorem 4.5?*

This should be a generalization of the case, discussed above, of the cone over a generic arrangement and the example in [26] of the cone over a divisor with normal crossings in $\mathbf{P}^n$.

§5. Homology of abelian covers

In this section, we return to the global case of divisors with ample components having only isolated non-normal crossings.

5.1. Topology of unbranched covers

The characteristic varieties $\mathrm{Char}_i(\pi_n(X - D))$ contain information about both branched and unbranched abelian covers.

Lemma 5.1. *Let G be a finite abelian quotient of $\pi_1(X - D)$ and let U_G be corresponding unbranched covers of $X - D$. Let $\chi \in \mathrm{Char}(\pi_1(X - D))$ be a pull back of a character of G (we shall considered it as a character of the latter). Let $H_n(U_G)_\chi = \{v \in H_n(U_G) \mid g \cdot v = \chi(g)v(g \in G)\}$ Then $H_n(X - D, \mathcal{L}_\chi) = H_n(U_G)_\chi$. In particular, $\chi \in \mathrm{Char}\, G \subset \mathrm{Char}_i(\pi_n)$ if and only if $H_n(U_G)_\chi \geq i$.*

A proof can be obtained, for example, from the exact sequence of low degree non-vanishing terms in the spectral sequence of the action of the group $K = \operatorname{Ker} \pi_1(X - D) \to G$ on the universal cover $\widetilde{X - D}$ (for which we have $\widetilde{X - D}/K = U_G$):

$$H_p(K, H_q(\widetilde{X - D})) \Rightarrow H_{p+q}(U_G)$$

(cf. [7]). This is a spectral sequence of $\mathbf{C}[\pi_1(X - D)]$-modules where the $\mathbf{C}[\pi_1(X - D)]$-module structure on $\mathbf{C}[G]$ module $H_{p+q}(U_G)$ comes via surjection: $\mathbf{C}[\pi_1(X - D)] \to \mathbf{C}[G]$. The localization of this spectral sequence at a point χ of $\operatorname{Char} G \subset \operatorname{Char} \pi_1(X - D)$ yields the claim using 3.2, since the localization of $H_n(U_G)$ at χ has the same χ-eigenspace as $H_n(U_G)$.

Now, let us consider the effect of adding (ample) components to D.

Lemma 5.2. *Let D' an ample divisor such that $D \cup D'$ is a divisor with isolated non-normal crossings. Then the homomorphism of $\pi_1(X - D)$ modules: $\pi_i(X - D \cup D') \to \pi_i(X - D)$ is surjective for $1 \le i \le \dim X - 1$. In particular, if one considers $\operatorname{Spec} \mathbf{C}[\pi_1(X - D)]$ as a subset in $\operatorname{Spec} \mathbf{C}[\pi_1(X - D \cup D')]$, then the intersection of $V_k(\pi_n(X - D \cup D'))$ with $\operatorname{Spec} \mathbf{C}[\pi_1(X - D)]$ contains $V_k(\pi_n(X - D))$.*

Sketch of the proof. Let $T(D')$ be a small neighborhood of D' in X. Then by the Lefschetz theorem, $\pi_i(T(D') - D' \cap D)$ surjects onto $\pi_i(X - D)$. On the other hand, this map can be factored through $\pi_i(X - D \cup D')$ which yields the claim.

Lemma 5.2 suggests the following definition:

Definition 5.3. *The components of $V_k(X - \bar{D})$ where $\bar{D}$ is a union of a proper collection of D_i's forming D and which are considered as subsets in $\operatorname{Spec} \mathbf{C}[\pi_1(X - D)]$ called the non-essential components of $V_k(X - D)$. The remaining components are called essential.*

A character χ is called essential if $\chi(\gamma) \neq 1$ for each element $\gamma \in \pi_1(X - D)$ which is a boundary of a small 2-disk transversal to one of irreducible components of D.

We shall see in the next section that only essential characters contribute to the homology of branched covers.

5.2. Hodge theory of branched covers.

The relationship between the homology of branched and unbranched covers is more subtle in the present case than in the case of plane curves considered in [24] and the local case of Section 4. One of the reasons is that there is no prefer non-singular model for the abelian global case. Only the birational type of branched cover is an invariant of $X - D$, and hence the Betti numbers of branched covers depend upon compactification of the unbranched cover. However, the Hodge numbers $h^{i,0}$ are birational invariants (in the case $\dim X = 2$, they determine the relevant part of homology of branched cover completely due to the relation $b_1 = 2h^{1,0}$) and one can expect a relation between the Hodge numbers $h^{i,0}$ and the homology of unbranched covers.

Recall that the cohomology of unitary local systems supports a mixed Hodge structure (cf. [38]). We shall denote

$$h^{p,q,k}(\mathcal{L}) = \dim Gr_F^p Gr_{\bar{F}}^q Gr_{p+q}^W (H^k(\mathcal{L}))$$

the dimension of the corresponding Hodge space. In the case of a rank one local system having a finite order, one has the following counterpart of 5.1:

Theorem 5.4. *Let, as in 5.1, $\chi \in \mathrm{Char}(\pi_1(X - D))$ be a character of a finite quotient G of $\pi_1(X - D)$. Let $\bar{U}_G$ be a G-equivariant non-singular compactification of U_G and let $H^{p,q}(\bar{U}_G)_\chi$ be the χ-eigenspace of G acting on $H^{p,q}(\bar{U}_G)$. Then*

$$h^{n,0,n}(\mathcal{L}_\chi) = h^{n,0}(\bar{U}_G)_\chi$$

Sketch of the Proof. The functoriality of the Hodge structure on cohomology of local systems yields that the isomorphism in 3.2 is compatible with the Hodge structure: $h^{n,0,n}(X - D, \mathcal{L}_\chi) = h^{n,0,n}(U_G)_\chi$ where in the RHS are the Hodge numbers of the Deligne's MHS on the cohomology of non-singular quasi-projective manifold (cf. [11]). Let $E = \bar{U}_G - U_G$, which we assume is a divisor with normal crossings. In the exact sequence of MHS: $H^n(\bar{U}_G, U_G) \to H^n(\bar{U}_G) \to H^n(U_G)$, which splits into corresponding sequences of χ-eigenspaces, the image of right homomorphism is $W_n H^n(U_G)$ (cf. [11] 3.2.17). This result is a consequence of the identity: $\mathrm{Ker}\, H^n(\bar{U}_G) \to H^n(U_G) \cap H^{n,0} = 0$ To see the latter, notice that using the duality $H^{n+2}(E) \times H^n(\bar{U}_G, U_G) \to \mathbf{C}(-n-1)$ ($\mathbf{C}(-k)$ is the Hodge-Tate) we obtain $h^{n,0,0}(\bar{U}_G, U_G) = h^{n+1,1,n+2}(E)$. On the other hand, for each smooth component E_i of E one has $h^{i,j,n+2} \neq 0$ only when $0 \leq i, j \leq n$ and the Mayer Vietoris sequence of MHS yields the same conclusion for E. Hence $H^n(\bar{U}_G) \to H^n(U_G)$ is injective on $H^{n,0}$ and the result follows.

§6. Conjecture and results on the structure of characteristic varieties.

Now we return to the situation discussed in Section 2 and consider the complements to divisors D with isolated non-normal crossings on projective manifolds X ($\dim X = n+1$). Our goal is to calculate the components of $V_i(\pi_n(X - D))$. The procedure described below is a generalization of the one outlined in [24].

Let us assume that $H_1(X-D) = \mathbf{Z}^r$ (i.e., to avoid mainly notational complications, assume that $H_1(X - X, \mathbf{Z})$ is torsion free) and consider the covering corresponding to the homomorphism $H_1(X - D) \to G = \oplus_{i=1}^{i=r}\mathbf{Z}_{m_i}$. Let k_i be the order in G of the element of $H^{2n}(D)$ corresponding to D_i so that we have the surjective map $H^{2n}(D) \to \oplus_{i=1,..N}\mathbf{Z}_{k_i}$ and also the surjection $G' \to G$ where $G' = \oplus\mathbf{Z}_{k_i}$. Let $K = \ker G' \to G$. We have the following diagram (the left column is the part of the sequence (2)):

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ H_1(X-D,\mathbf{Z}) & \to & G \\ \uparrow & & \uparrow \\ H^{2n}(D,\mathbf{Z}) & \to & G' \\ \uparrow & & \uparrow \\ \operatorname{Im} H_2(X,\mathbf{Z}) & \to & K \\ \uparrow & & \uparrow \\ 0 & & 0 \end{array}$$

Dualizing, we obtain:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ H^1(X-D,\mathbf{R}) & \to & \operatorname{Char} H_1(X-D) & \leftarrow & \operatorname{Char} G \\ \downarrow & & \downarrow & & \downarrow \\ H_{2n}(D,\mathbf{R}) & \to & \operatorname{Char} H^{2n}(D) & \leftarrow & \operatorname{Char} G' \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Hom}(\operatorname{Im} H_2(X),\mathbf{R}) & \to & \operatorname{Char}\operatorname{Im} H_2(X) & \leftarrow & \operatorname{Char} K \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array} \tag{6}$$

with the maps from the left to the middle column on (6) induced by the universal covering map $\mathbf{R} \to S^1$. The left column itself is the part of the cohomology sequence of the pair $(X, X-D)$.

Consider the preimage of $\operatorname{Char} G' \subset \operatorname{Char} H^{2n}(D,\mathbf{Z})$ under the map $H^{2n}(D,\mathbf{Q}) \to H^{2n}(D,S^1)$ and select the fundamental domain for the action of the kernel of the latter map i.e. the action of $H^{2n}(D,\mathbf{Z})$ on $H^{2n}(D,\mathbf{Q})$. We shall assume that this domain is the unit cube $\mathcal{U}$: $\{(j_1,...,j_N) \mid 0 \le j_i < 1\}$ in $\mathbf{Q}^N$ ($N = rkH^{2n}(D,\mathbf{Q})$) with coordinates corresponding to the components of D. Selection of the fundamental domain allows to attach to each $\chi \in \operatorname{Char} G'$ unique element in $\mathcal{U}$. The preimage of $\operatorname{Char} G$ is a subgroup of $H_{2n}(D, \mathbf{R})$. The image of this subgroup in $\operatorname{Char} \operatorname{Im} H_2(X, \mathbf{Z})$ is trivial and hence belongs to

$$\operatorname{Hom}(\operatorname{Im} H_2(X, \mathbf{Z}), \mathbf{Z}) \subset H^2(X, \mathbf{Z}).$$

In particular, any character χ of G determines the element $\mathcal{L}_\chi$ of $\operatorname{Pic}(X)$. These bundles satisfy: $\mathcal{L}_\chi \otimes \mathcal{L}_{\chi^{-1}} = \otimes \mathcal{O}(D_s)$ where D_s is the collection of irreducible components of D such that $\chi(\gamma_{D_s}) \neq 1$ (γ_{D_s} is the image in $H_1(X - D)$ of the generator of the summand of $H^{2n}(D)$ corresponding to D_s). One can show that if $\tilde{X}_G \to X$ is a branched cover then the divisorial components of $f_*(\mathcal{O}_{\tilde{X}_G})$ are $\oplus_\chi L^*_\chi$. We have: $f_*(\Omega^{n+1}_{\tilde{X}_G}) = \oplus_\chi \Omega^{n+1}_X \otimes L_{\chi^{-1}}$. Also, for a given $\mathcal{D} \in \operatorname{Pic}(X)$ the collection of lifts of characters $\chi \in \operatorname{Char} H^{2n}(D)$ to $H_{2n}(D, \mathbf{R})$ such that $\mathcal{L}_{\chi^{-1}} = \mathcal{D}$ form an affine subspace $L_\mathcal{D}$ of $H_{2n}(D, \mathbf{R})$.

Example 6.1. *Let $X = \mathbf{P}^{n+1}$ and let D be an arrangement of $r+1$ hyperplanes $H_i, i = 1, \ldots, r$ (i.e., $H_1(\mathbf{P}^{n+1} - D, \mathbf{Z}) = \mathbf{Z}^r$). The characters of $H^{2n}(D, \mathbf{Z})$ which factor through $(\mathbf{Z}/n\mathbf{Z})^r$ correspond to the collections $x_i \in \mathbf{Z}, i = 1,,,, r+1, 0 \leq x_i < n$ such that $\sum x_i \equiv 0 \bmod n$. Let us consider a covering X_G with the Galois group $G = (\mathbf{Z}/n\mathbf{Z})^r$ corresponding to the homomorphism $H_1(\mathbf{P}^{n+1} - D, \mathbf{Z}) \to \mathbf{Z}/n\mathbf{Z}^{r+1}/(\mathbf{Z}/n\mathbf{Z})$ (quotient by the diagonally embedded cyclic subgroup K of $G' = (\mathbf{Z}/n\mathbf{Z})^{r+1}$) sending the boundary of a small disk transversal to H_i to a generator of the i-th summand. We have $f_*(\mathcal{O}_{X_G}) = \oplus \mathcal{L}_\chi$ with $\mathcal{L}_\chi = \mathcal{O}((-\sum \frac{x_i}{n})H)$. Moreover, $\mathcal{L}_{\chi^{-1}} = \mathcal{O}(\sum(1 - \frac{x_i}{n}))H$. Taking ramification into account, the assignment the characters of $H_1(X - D, \mathbf{Z})$ to elements of $H^{2n}(D)$ can be done so that to $(x_1, \ldots, x_{r+1})$ corresponds the character $\exp(\sum(1 - \frac{x_i+1}{n}))$ and so that: $f_*(\Omega^{n+1}) = \oplus_{x_1, \ldots, x_{r+1}} \Omega^{n+1}_{\mathbf{P}^{n+1}} \otimes \mathcal{O}((1 - \frac{x_1+1}{n})H)$ for $(x_1, \ldots, x_{r+1}), 0 \leq x_i < n$ selected so that $\sum \frac{x_i+1}{n} \in \mathbf{Z}$. We have $\operatorname{Pic}(X) = \mathbf{Z}$ and the preimage of $\mathcal{O}(l) \in \operatorname{Pic}(X)$ is the hyperplane in $H^{2n}(X, \mathbf{R})$ corresponding to the latter lift. It is given by $\frac{x_1+1}{n} + \ldots + \frac{x_r+1}{n} = l$ where x_i's are the coordinates corresponding to the basis of $H^{2n}(X, \mathbf{Z})$ given by the cycles dual to D_i's*

Now, with each $S \in \mathcal{S}$, we associate a polytope in the unit cube in $\mathbf{R}^r$ as follows. For any $S \in \mathcal{S}$, one has the map $H_1(B_\epsilon - D) \to H_1(X - D)$ and hence the map $\operatorname{Char} H_1(X - D) \to \operatorname{Char} H_1(B_\epsilon - D)$. The latter lifts to the map of universal covers: $\mathbf{R}^r \to \mathbf{R}^s$ where s is the number of components of D containing S. This can be described in coordinates as follows. A vector $\Xi : (\kappa_1, .., \kappa_r) \in \mathbf{Q}^r (0 \leq \kappa_i < 1)$ for any collection $(j_1, \ldots, j_s)$ determines the vector: $\Xi^{j_1, \ldots, j_s} = (\{\sum a_{i,j_1} \kappa_i\}, \ldots, \{\sum a_{i,j_1} \kappa_i\}) \in \mathbf{Q}^r$ ($\{\}$ is the fractional part of a rational number). For each $S \in \mathcal{S}$, we consider subsets $\mathcal{P}^{gl}_S$ consisting of vectors $\Xi = (\kappa_1, .., \kappa_r)$ such that $\Xi^{j_1,..,j_s} \in \mathcal{P}_S$ where $D_{j_1}, ., D_{j_s}$ are the components of D passing through $S \in \mathcal{S}$ and $\mathcal{P}_S \in \mathbf{Q}^s$ is a face of a polytope of quasi-adjunction of INNC formed by $D_{j_1}, .., D_{j_s}$.

Definition 6.2. *Let $\mathcal{S} \subset X$ be the collection of non-normal crossings of the divisor D. Global polytope of quasi-adjunction corresponding to $\mathcal{S}$ is $\bigcap_{S \in \mathcal{S}} \mathcal{P}_S^{gl}$. A global face of quasi-adjunction is a face of a global polytope of quasiadjunction. A divisor $\mathcal{D} = \sum \alpha_i D_i \in \operatorname{Pic} X, \alpha_i \in \mathbf{Q}$ is called contributing if the corresponding subset $L_{\mathcal{D}}$ of the elements of the universal cover (cf. definition before Example 6.1) contains a global face of quasi-adjunction $\mathcal{F}$ and $H^1(\mathcal{A}_{\mathcal{F}} \otimes \Omega_X^{n+1} \otimes \mathcal{D}) \neq 0$. Here $\mathcal{A}_{\mathcal{F}}$ is the ideal of quasiadjunction corresponding to the face $\mathcal{F}$ (cf. Remark 4.6). A global face of quasi-adjunction $\mathcal{F}$ is contributing if there is a contributing divisor such that the corresponding subspace $L_{\mathcal{D}}$ contains $\mathcal{F}$.*

Conjecture 6.3. *Zariski closure of $\exp(\mathcal{F}) \subset \operatorname{Char} H_1(U, \mathbf{Z})$ is a component of characteristic variety V_k where $k = \dim H^1(\mathcal{A}_{\mathcal{F}} \otimes \Omega_X^{n+1} \otimes \mathcal{D})$ if $\mathcal{F}$ is a contributing face of a polytope of quasiadjunction.*

I don't know if such components are *all* essential components of the characteristic variety.

The supporting evidence is the following. This conjecture is shown in [24] in the case of curves and in the case $X = \mathbf{P}^{n+1}$ and D is a hypersurface with isolated singularities in [23]. Both of these results can be generalized as follows (the proof, based on the methods used in these two papers will appear elsewhere).

Theorem 6.4. *Let $D \subset \mathbf{P}^{n+1}$ be a union of hypersurfaces D_0, D_1,..., D_r of degrees $1, d_1, ..., d_r$ respectively, which is a divisor with isolated non-normal crossings. Let $\mathcal{F}$ be a face of global polytope of quasi-adjunction, i.e. a face of an intersection of polytopes of quasi-adjunction corresponding to a collection $\mathcal{S}$ of non-normal crossings of D. Let $d_1x_1 + ... + d_rx_r = l$ be a hyperplane containing the face of quasiadjunction $\mathcal{F}$. If $H^1(\mathcal{A}_{\mathcal{F}} \otimes \mathcal{O}(l-3)) = k$, then the Zariski closure of $\exp(\mathcal{F}) \subset \operatorname{Char} H_1(\mathbf{P}^{n+1} - D)$ is a component of $V_k(\pi_n(X - D))$.*

A consequence of the conjecture is the corollary.

Conjecture 6.5. *Let $\chi \in \operatorname{Char} \pi_1(X - D)$ be a character of a finite quotient of G of the fundamental group. Let, as in Lemma 5.4, $\bar{U}_G$ be a G-equivariant compactification of the unbranched cover of $X - D$ with the Galois group G and let $h_\chi^{n,0}(\bar{U}_G)$ be the χ-eigenspace of the G acting on $H^{n,0}(U_G)$. Then $h_\chi^{n,0}(\bar{U}_G) = 0$ unless the lift of χ belongs to a contributing global face of quasi-adjunction $\mathcal{F}$ and in which case one has:*

$$h_\chi^{n,0}(\bar{U}_G) = \dim H^1(\mathcal{A}_{\mathcal{F}} \otimes \Omega_X^n \otimes \mathcal{O}(D))$$

where $D \in \mathrm{Pic}(X)$ is the divisor corresponding to the lift of character χ.

In the case when there are bundles $\mathcal{L}_i$ such that $\mathcal{L}_i^{n_i} = \mathcal{O}(D_i)$ and the cover corresponds to the group $\mathbf{Z}_{n_1} \times ... \times \mathbf{Z}_{n_r}$, using the arguments similar to those used in [39], one obtains (in agreement with the conjecture) the following: the eigenspace of the action of G corresponding to the dimension of the eigenspace corresponding to $(e^{2\pi i \frac{p_1}{n_1}}, ..., e^{2\pi i \frac{p_r}{n_r}})$ is $\dim H^1(\mathcal{A}_{\mathcal{F}} \otimes \Omega_X^{n+1} \otimes \mathcal{L}_1^{p_1} \otimes ... \otimes \mathcal{L}_r^{p_r})$. In the case when $r = 1$ the condition that $(\frac{p_1}{n_1}, ..., \frac{p_r}{n_r})$ belongs to the face of quasi-adjunction becomes the condition that $\frac{p}{n}$ is an element of the spectrum of one of the singularities of the divisor D and one obtains the result from [39].

§7. Examples

7.1. Local examples

Example 7.1. *Germs of curves.*

In the case of curves, the support of $H_1(\widetilde{\partial B_\epsilon - D}, \mathbf{C})$ is the zero set of the Alexander polynomial. There are extensive calculations of this invariant using knot-theoretical methods (cf. [15]). Hodge decomposition is considered in [25]. For example, for the singularity $x^r - y^r = 0$, the characteristic variety is $t_1 \cdots t_r = 1$ (cf. the calculation for the cone over the generic arrangement in Section 4). The faces of the polytopes of quasi-adjunction are the hyperplanes $x_1 + ... + x_r = l$, $(l = 1, .., r-2)$.

Example 7.2. *Cones.*

A generalization of the example of arrangements of hyperplanes considered in Section 4 is given by a union of non-singular hypersurfaces in $\mathbf{P}^n$ which form a divisor with normal crossings (cf. [26]). If the degrees of hypersurfaces are $d_1, .., d_r$ respectively then $V_1 = \mathrm{Supp}(\pi_n(\mathbf{C}^{n+1} - D) \otimes \mathbf{C})$ is given by $t_1^{d_1} \cdot ... \cdot t_r^{d_r} - 1 = 0$.

7.2. Global examples

Example 7.3. *Plane curves*

We refer to [24] for examples of characteristic varieties for pencils quadrics (Ceva arrangement of four lines) and pencils of cubics (arrangement of nine lines dual to inflection points of a non-singular cubic and the arrangement of 12 lines containing its inflection points). Papers [27], [28] and [10] describe a combinatorial method to detect components of characteristic variety and in [9] a generalization to arrangements of rational curves is considered. Papers [3] and [5] contain applications of characteristic varieties to geometric problems.

Example 7.4. *Arrangement in* $\mathbf{P}^3$ *with isolated non-normal crossings for which* π_2 *of the complement which support has non-trivial essential components.*

Consider the arrangement $D_{8,4}$ of hyperplanes in $\mathbf{P}^3$ which is an (8,4) configuration (cf. [17]). It includes a plane containing 4 generic points $Q_1, ..., Q_4$, six generic planes $H_{i,j}$ each passing through the line Q_iQ_j and also the plane containing the four coplanar (by Desargue theorem) points $H_{i,j} \cap H_{j,k} \cap H_{i,k}$. Recall (cf. [17]) that this configuration contains eight planes and eight points such that every plane contains four points and every point belongs to exactly four planes. Denoting eight points by $1, 2, 3, 4, 1', 2', 3', 4'$ and eight planes by $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{1}', \mathbf{2}', \mathbf{3}', \mathbf{4}'$ the incidence relation is given by the diagram:

$$\begin{array}{cc} 1 & 1' \\ 2 & 2' \\ 3 & 3' \\ 4 & 4' \end{array} \qquad \begin{array}{cc} \mathbf{1} & \mathbf{1}' \\ \mathbf{2} & \mathbf{2}' \\ \mathbf{3} & \mathbf{3}' \\ \mathbf{4} & \mathbf{4}' \end{array}$$

where the plane in position (i, j) contains all points in row i and column j except for the point in position (i, j).

This arrangement of eight hyperplanes has only isolated non-normal crossings. From 2.2, we infer that $H_1(\mathbf{P}^3 - D_{8,4}) = \mathbf{Z}^7$. Moreover we have the rational map:

$$\Pi : \quad \mathbf{P}^3 \to \mathbf{P}(H^0(\mathbf{P}^3, \mathcal{I}(2)))^* = \mathbf{P}^2$$

where $\mathcal{I}$ is the ideal sheaf of the collection of eight points in $\mathbf{P}^3$ forming this configuration. The indeterminacy points are the eight points of configuration. In order to calculate the Π-image of the hyperplanes of the arrangement, notice that the points in the target of the map correspond to the pencils of quadrics in the web, the image of a point is the pencil of quadric in the web containing this point and the lines correspond to quadrics in $H^0(\mathbf{P}^3, \mathcal{I}(2))$ i.e. are the collections of pencils containing a quadric. In particular, the image of a point P in a hyperplane $H \in D_{8,4}$ is a pencil of quadrics from $H^0(\mathbf{P}^3, \mathcal{I}(2))$ containing P. This pencil contains the quadric among the four quadrics containing P, mentioned earlier. Hence the image of P belongs to the union L of four lines in $\mathbf{P}(H^0(\mathbf{P}^3, \mathcal{I}(2)))^*$ corresponding to above four quadrics. Therefore we have a regular map: $\Pi : \mathbf{P}^3 - D_{8,4} \to \mathbf{P}^2 - L$.

Let us calculate the cohomology of local systems $\Pi^*(\mathcal{L})$, where $\mathcal{L}$ is a local system on $\mathbf{P}^2 - L$. We have the Leray spectral sequence: $H^p(\mathbf{P}^2 - L, R\Pi^q_*(\Pi^*\mathcal{L})) \Rightarrow H^{p+q}(\mathbf{P}^3 - D_{8,4}, \Pi^*\mathcal{L})$. Using $R\Pi^q_*(\mathcal{L}) = \mathcal{L} \otimes R^q_*(\mathbf{C})$ and looking at the critical set of Π, one checks that $H^0(\mathbf{P}^2 -$

$L, \mathcal{L} \otimes R\Pi^1_*(\mathbf{C})) = H^1(\mathbf{P}^2 - L, \mathcal{L} \otimes R\Pi^1_*(\mathbf{C})) = 0$ for a Zariski dense set of local systems. Hence this spectral sequence degenerates for those local systems $\mathcal{L}$ on $\mathbf{P}^2 - L$. This yields that $H^2(\mathbf{P}^3 - D_{8,4}, \Pi^*(\mathcal{L})) = H^2(\mathbf{P}^2 - L, \mathbf{C}) = \mathbf{C}^3$ for a Zariski dense set of local systems $\mathcal{L}$ on $\mathbf{P}^2 - L$.

The above calculation shows that, since

$$\pi_2(\mathbf{P}^2 - L) \otimes \mathbf{C} = \mathbf{C}[t_1, t_{,2}, t_3, t_4]/(1 - t_1 t_2 t_3 t_4)$$

the support of the homotopy group: $\pi_2(\mathbf{P}^3 - D_{8,4}) \otimes \mathbf{C}$ has a 3-dimensional component. Projections from each of eight vertices of this configuration yield linear maps of $\mathbf{P}^3 - D_{8,4}$ onto the complement in $\mathbf{P}^2$ to four lines in a general position and hence a 3-dimensional component. We obtain hence in $\operatorname{Spec} \mathbf{C}[H_1(\mathbf{P}^3 - D_{8,3})]$ nine 3-dimensional components of the support of $\pi_2(\mathbf{P}^3 - D_{8,3}) \otimes \mathbf{C}$.

The component corresponding to the web of quadrics can be detected using Theorem 6.4. Indeed local face of quasiadjucntion has the form $x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1$ where x_{i_j} correspond to the planes containing one of the above 8 points. The intersection of these hyperplanes in $\mathbf{R}^8$ has dimension 3 and belongs to the hyperplane $x_1 + ... + x_8 = 2$ since adding relations $x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1$ for all 8 points yield the relation $4x_1 + ... + 4x_8 = 8$ since each point belongs to 4 hyperplanes. This intersection will be contributing iff $\dim H^1(\mathcal{J}(8 - 4 - 2) \neq 0$ where $\mathcal{O}_{\mathbf{P}^3}/J$ has support at the above 8 points and and the stalk of J at each of those is the maximal ideal of $\mathcal{O}_{\mathbf{P}^3}$. The sheaf $\mathcal{J}$ has Koszul resolution:

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-6) \to \mathcal{O}_{\mathbf{P}^3}(-4)^3 \to \mathcal{O}_{\mathbf{P}^3}(-2)^3 \to \mathcal{J} \to 0$$

which yields $\dim H^1(\mathbf{P}^3, \mathcal{J}(2)) = 1$ i.e. the above face of quasiadjunction is contributing.

§8. Betti and Hodge realizations of multi-variable motivic zeta function

The purpose of this section is to relate the motivic zeta function of Denef and Loeser in the case of local INNC to the invariants considered in Section 4.

Recall that, to a smooth variety X over $\mathbf{C}$ and r holomorphic functions $f_i : X \to \mathbf{C}$, one associates a multi-variable motivic zeta-function $Z_{f_1,...,f_r}(T_1, .., T_r)$ which is a formal series in $\mathcal{M}_{X_0 \times \mathbf{G}_m^r}[[T_1, ..., T_r]]$. Here, as in [12], $X_0 = \bigcap_i f_i^{-1}(0)$, $\mathbf{G}_m$ is the multiplicative group of the field $\mathbf{C}$ and for a variety S the ring $\mathcal{M}_S$ is obtained from the Grothendieck group $K_0(Var_S)$ of varieties over S by inverting the class $\mathbf{L}$ of $\mathbf{A}^1_k \times S \in K_0(Var_S)$. More precisely, denote $\mathcal{L}(X)$ (resp. $\mathcal{L}_n(X)$) the arc space of

X (resp. arc space mod n) whose points are the maps $\operatorname{Spec}\mathbf{C}[[t]] \to X$ (resp. $\operatorname{Spec}\mathbf{C}[[t]]/(t^{n+1}) \to X$). Let

$$\mathcal{X}_{n_1,\dots,n_r} = \{\phi \in \mathcal{L}_n(X), n = \sum n_j \mid \operatorname{ord}_t \phi^*(f_j) = n_j \ \ j = 1, \dots, r\}$$

and $ac(f) = (ac(f_1), \dots, ac(f_r)) : \mathcal{X}_{n_1,..,m_r} \to \mathbf{G}_m^r$ assigns to an arc in $\mathcal{X}_{n_1,..,n_r}$ the vector which j-th component is the coefficient of t^{n_j} in $\phi^*(f_j)$. Together with $\pi_0 : \mathcal{X}_{n_1,..,n_r} \to X$ which assigns to an arc the image in X of its closed point $\operatorname{Spec}\mathbf{C} \to \operatorname{Spec}\mathbf{C}[[t]]$, this makes $\mathcal{X}_{n_1,..,n_r}$ into $\mathbf{G}_m^r \times X_0$-manifold. Then

(7)

$$Z_{f_1,\dots,f_r}(T_1,..,T_r) = \sum_{n_1,..,n_r, n_i \in \mathbf{N}} [\mathcal{X}_{n_1,\dots,n_r}/X_0 \times \mathbf{G}_m^r]\mathbf{L}^{(d\sum n_i)} T_1^{n_1} \cdots T_r^{n_r}$$

One has the canonical maps (resp. Betti and Hodge realizations): $e_{top} : K_0(Var_{\mathbf{C}}) \to \mathbf{Z}$ and $e_h : K_0(Var_{\mathbf{C}}) \to \mathbf{Z}[u,v]$ induced by the maps assigning to a variety V its topological euler characteristic and the E-function $\sum_i (-1)^i \dim Gr_F^p Gr_{p+q}^W H^i(V) u^p v^q$ (both F and W filtration are coming from Deligne's Mixed Hodge structure on V). We also will use the equivariant refinement of e_{top} and e_h defined for $V \in Var_{\mathbf{C}}$ supporting an action of a finite group G via biholomorphic transformations. For $\chi \in \operatorname{Char} G$, those refinements pick the corresponding eigenspaces:

(8)

$$e_{top,\chi}(V) = \sum (-1)^i \dim H^i(V)_\chi \ \ \text{and} \ \ e_{h,\chi}^m(V) = \sum_i (-1)^i Gr_F^m H^i(V)_\chi$$

The function (7) can be expressed in terms of a resolution of singularities of $f_1,,,,f_r$ as follows (cf. [12]). Let $Y \to X$ be a resolution of singularities of D, i.e. the union of the exceptional set $\bigcup_{i\in J} E_i$ and the proper preimage of an INNC D is a normal crossings divisor. For $I \subset J$, let $E_I^\circ = \cap_{i\in I} E_i - \cup_{j\in J-I} E_j$, $a_{i,k}$ (resp. c_k) is the order along the exceptional component of the pull-back on Y of function f_i (resp. the order of the pull back of the differential $dx_1 \wedge \dots \wedge dx_{n+1}$). Let U_i be the complement to the zero section of the normal bundle to E_i in Y, and U_I is the fiber product of $U_i|_{E_I^\circ}$ over E_I. Then:

(9)

$$Z_{f_1,..f_r}(T_1,..,T_r) = \sum_{I\subset J} [U_J/\mathbf{G}_m^r \times X_0] \Pi_{i\in I} \frac{\mathbf{L}^{-c_i-1} T_1^{a_{i,1}} \cdots T_r^{a_{i,r}}}{1 - \mathbf{L}^{-c_i-1} T_1^{a_{i,1}} \cdots T_r^{a_{i,r}}}$$

We have the following:

Theorem 8.1. *Betti realization of $Z_{f_1,\dots,f_r}(T_1,\dots,T_r)$ determines the essential components of the characteristic variety V_1. More precisely,* *for an essential* χ*:*

$$(10) \qquad V_1 = \{\chi \mid e_{top,\chi} \lim_{T_i \to \infty} Z_{f_1,...,f_r}(T_1,...,T_r) \neq 0\}$$

Proof. One can deduce this from C. Sabbah's results in [33] similarly to [19] since, due to the vanishing theorem 4.1, the multi-variable zeta function studied in [33] determines the support of the $\pi_1(B_\epsilon - D)$ module $\pi_n(B_\epsilon - D)$.

In the cyclic case, the Hodge realization of the motivic zeta function is equivalent to the spectrum (cf. [12]). At least in the case of curves, one has the Hodge version in the abelian case as well (as was suggested in [29]):

Theorem 8.2. *For* $n = 1$*, the Hodge realization of* (7) *determines the polytopes of quasiadjunction.*

Proof. Let $X_{m_1,...,m_r}$ be the link of an abelian cover $\mathcal{V}_{m_1.,.,.,m_r}$ given by the equations (5) with $n = 1$. A resolution of this complete intersection singularity in the category of spaces with quotient singularities (in the case of surfaces with ADE singularities) can be obtained as the normalization $\widetilde{V_{m_1,..,m_r}}$ of $\mathcal{V}_{m_1,..,m+r} \times_{B_\epsilon} Y_D$, where $Y_D \to B_\epsilon$ is an embedded resolution of the singularities of D. The exceptional locus $\tilde{E}$ of the resolution of (5) supports the action of the group $G = \mathbf{Z}_{m_1} \times ... \times \mathbf{Z}_{m_r}$. We have the following sequence of MHS (cf. [36]): $0 \to H^1_E(\widetilde{V_{m_1,..,m_r}}) \to H^1(E) \to H^1(X_{m_1,...,m_r}) \to 0$, which in the case $n = 1$ yields the equivariant isomorphism $H^1(X_{m_1,...,m_r}) = H^1(\tilde{E})$ of MHSs. Since the MHS on $H^2(\tilde{E})$ is pure, we have:

$$(11) \qquad \dim F^1 H^1(X_{m_1,..,m_r})_\chi = \dim Gr^1_F H^1(L)_\chi = e^1_{h,\chi}(\tilde{E})$$

The latter is determined by the Hodge realization of (9), since the pullback of $[U_i]$ via the map $\mathcal{M}_{\mathbf{G}^r_m \times X_0} \to \mathcal{M}_{\mathbf{G}^r_m \times X_0}$ corresponding to the map $\mathbf{G}^r_m \times X_0 \to \mathbf{G}^r_m \times X_0$ given by $z_i = u_i^{m_i}$ is equivalent to the unbranched cover of $\partial B_\epsilon - D$, which is preimage of $\mathbf{G}^r_m \subset \mathbf{C}^r$ for the projection of (5) onto the space of z-coordinates. In particular, it determines the class of the exceptional set $\tilde{E}_i$ in $\mathcal{M}_{\mathbf{C}}$. It follows from (11) that $\dim F^1 H^1(X_{m_1,..,m_r})_\chi \geq 1$ iff $e^1_{h,\chi}(\tilde{E}) \geq 1$. QED.

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