Science in China Series A: Mathematics
2009, Vol. 52, No. , 1–
www.SciChina.com
www.springerlink.com

Cohomology of bundles on homological Hopf manifolds

Anatoly Libgober[1,*]

[1]Department of Mathematics, University of Illinois, Chicago, IL 60607
(email: libgober@math.uic.edu)

Abstract We discuss the properties of complex manifolds having rational homology of $S^1 \times S^{2n-1}$ including those constructed by Hopf, Kodaira and Brieskorn-van de Ven. We extend certain previously known properties of cohomology of bundles on such manifolds.As an application we consider degeneration of Hodge-deRham spectral sequence in this non Kahler setting.

1 Introduction

The goal of this paper is to obtain information on the cohomology of bundles on generalizations of Hopf manifolds applying methods of [16] in the bundle setting. Roughly, the cohomology of the bundles $\Omega^p_{\mathcal{H}}(E)$ on a topological Hopf manifold $\mathcal{H}$ where E is a bundle with a trivial pull back on the universal cover, controlled by the "Alexander modules" naturally identified with the appropriate local cohomology associated with the universal cover (cf. theorem 1.2).

It is an interesting problem to find a reasonable classification of manifolds with fixed topological, differential or almost complex type. In the case of appropriately generalized Hopf manifolds (see section 2.1 below) the analytic type of the universal cover and the action of the fundamental group on it provide discrete invariants of "the components of the moduli space" of complex structures (cf. 2.4). Here we make explicit many examples of such types of discrete invariants. The main point of this note is that regardless of most specific properties of the analytic structure on the universal cover one has strong restrictions on the cohomology of bundles generalizing results of [17], [18] (in primary case) and [10] (in non primary Hopf manifolds case).

To put things in perspective we start with a review of some results and viewpoints on manifolds which are natural generalizations of the classical construction of Hopf and Kodaira (cf. [14] [15]). Kodaira had shown that a surface with universal cover biholomorphic to $\mathbb{C}^2 - 0$ is the quotient of the latter by a group having an infinite cyclic group as a normal subgroup of finite index. Moreover, a compact surface with $\pi_1 = \mathbb{Z}$ and vanishing second Betti number has $\mathbb{C}^2 - 0$ as its universal cover. In higher dimensions fixing the homological type leaves much more possibilities for the analytic type of the universal cover: for example Brieskorn and van de Ven found infinitely many examples of manifolds homeomorphic or even diffeomorphic to $S^1 \times S^{2n-1}$ but having analytically distinct universal covers. These universal covers are the non singular loci of affine hypersurfaces with isolated singularities given by equations $z_1^{a_1} + ... + z_n^{a_r} = 0$ and having as their links (possibly exotic) odd dimensional spheres. More generally, we consider generalized Hopf manifolds defined as quotients [1] of the non-singular loci of affine varieties which are weighted homogeneous complete intersection with isolated singularity having a $\mathbb{Q}$-sphere as its link. If this link is a $\mathbb{Z}$-sphere and $n > 2$ then the quotient is homeomorphic to $S^1 \times S^{2n-1}$. In the case when the universal cover of a primary [2] Hopf manifold is $\mathbb{C}^n - 0$, Haefliger (cf. [12]) described actions of the fundamental group on the universal cover (extending the case $n = 2$ studied by Kodaira). In the case when the universal cover of a homological Hopf manifold $\mathcal{H}$ is biholomorphic to non singular locus $V - 0$ of a complete intersecion (5) we show that, $\mathcal{H}$ is biholomorphic to a finite quotients of $V - 0/\mathbb{Z}$ with the action of $\mathbb{Z}$ given by (4) provided that the degree of such complete intersection is sufficiently large.

More precisely we will see the following:

Theorem 1.1. *Let $\mathcal{H}$ be a homological Hopf manifold having as its universal cover the non-singular locus of an affine hypersurface $z_1^{a_1} + ... + z_n^{a_r} = 0$ such that $\sum \frac{1}{a_i} < 1$ or a complete intersection (5) ($k \geqslant 2, \sum \frac{1}{a_i} < k$). Then $\mathcal{H}$ is a quotient by a finite group of the Brieskorn van deVen manifold (or respectively $(V - 0)/\mathbb{Z}$ where V is a complete intersection (5) with the action of a generator of $\mathbb{Z}$ given by (4)).*

Received March 7, 2008; accepted June 6, 2008
DOI:
*Corresponding author
The author supported by National Science Foundation grant.

[1]and with action more complicated than the actions (4)

[2]i.e. such that $\pi_1 = \mathbb{Z}$

This deduced from a result on the automorphism group of affine hypersurfaces and complete intersections (5) or (3) (cf. 2.5).

The main, rather technically looking result of this note, is the following:

Theorem 1.2. *Let $\mathcal{H}$ be a homological Hopf manifold with the universal covering space $p: V \to \mathcal{H}$ where V is a complement to compact set A is a Stein space $\bar{V}$ and $i: V \to \bar{V}$. Let f be an element of the infinite order in $\pi_1(\mathcal{H})$ and let $f_V: V \to V$. be the corresponding biholomorphic automorphism of V. Let E be a bundle on $\mathcal{H}$. We denote f_V^k the corresponding automorphism of $H^k(V, p^*(E))$. Assume that the local cohomology group $H_A^r(\bar{V}, i_*(\pi^*(E)))$ are finite dimensional. Then:*

a) $\dim H^k(\mathcal{H}, E) = \dim \mathrm{Ker} f_V^k + \dim \mathrm{Coker} f_V^{k-1}$.

b) In particular if $H^k(V, p^*(E)) = 0$ for $a \leqslant k \leqslant b$ then $H^k(\mathcal{H}, E) = 0$ for $a+1 \leqslant k \leqslant b-1$.

This yields the following result for the Hopf and Brieskorn Van de Ven manifolds and some homological Hopf manifolds (see next section for explanation of terminology):

Theorem 1.3. *Assume $n \geqslant 3$.*
a) (cf.[10]) Let $\mathcal{H}$ be a Hopf manifold which is the quotient of $\mathbb{C}^n - 0$ by the group containing an element of infinite $f \in Aut(\mathbb{C}^n)$ such that $f(0)$ has eigenvalues ξ such that $|\xi| < 1$. Let E be a bundle on $\mathcal{H}$ such that pullback on the universal cover is trivial. Then:

$$H^q(\mathcal{H}, \Omega^p(E)) = 0 \quad \text{for } q \neq 0, 1, n-1, n \tag{1}$$

$$H^0(\mathcal{H}, \Omega^p(E)) = H^1(\mathcal{H}, \Omega^p(E)) \quad \text{and} \quad H^{n-1}(\mathcal{H}, \Omega^p(E)) = H^n(\mathcal{H}, \Omega^p(E)) \tag{2}$$

b) Let $\mathcal{H}$ be Brieskorn van deVen manifold or a quotient $V - 0/\mathbb{Z}$ where V is the complete intersection (5) and $\mathbb{Z}$ is the cyclic group generated by the automorphism (4). For a pair of finite dimensional vector spaces A, B, denote by $\mathcal{V}_n(A, B) = \oplus_{i=0}^{i=n} V_i$ the graded vector space such that $V_0 = V_1 = A, V_{n-1} = V_n = B$ and $V_i = 0$ for $i \neq 0, 1, n-1, n$, Let $\mathcal{W}_{n,q} = \oplus_{i=0}^{i=n} W_i$ denote a graded vector space such that $\mathrm{rk} W_{q-1} = 1, \mathrm{rk} W_q = 2, \mathrm{rk} W_{q+1} = 1, W_i = 0$ $(i \neq q, q \pm 1)$. Let E be a vector bundle on $\mathcal{H}$ such that its pull back on the universal cover is trivial. If the multiplicity of V at the origin is greater than one, then for $p \geqslant 1$ one has an isomorphism of graded spaces:

$$\oplus_q H^q(\mathcal{H}, \Omega^p(E)) = \oplus \mathcal{V}_n(H^0(\Omega^p(E), H^n(\Omega^p(E)))) \oplus \mathcal{W}_{n,n-p-1}$$

Other results of this note include a calculation of the cohomology of local systems on homological Hopf manifolds (section 3) and a study of degeneration of Hodge-deRham spectral sequence (section 5).

My interest in this material stemmed from the lecture of Prof. X.Y.Zhou on Xiamen conference. I want to thank organizers of Xiamen conference for their hospitality and Prof. Steven Yau for providing useful references related to material of this paper.

2 Generalizations of Hopf manifolds and cohomology of bundles on their universal covers

2.1 Main definitions

Notations: Below, for a compact manifold $\mathcal{H}$, b_i denotes $\mathrm{rk} H^i(\mathcal{H}, \mathbb{Q})$.

Definition 2.1. A ($\mathbb{Q}$)-*homological Hopf* manifold is a compact complex manifold of dimension n with $b_1 = b_{2n-1} = 1, b_i = 0, i \neq 0, 1, 2n-1, 2n$. Such a manifold $\mathcal{H}$ is called an integral homological Hopf manifold if it has the integral cohomology groups isomorphic to $H^*(S^1 \times S^{2n-1}, \mathbb{Z})$. A $\mathbb{Z}$-homological Hopf manifold is called primary if its fundamental group is isomorphic to $\mathbb{Z}$.

$\mathbb{Q}$-homological Hopf manifolds (which may not be $\mathbb{Z}$-homological), in the case $n = 2$, were considered in [8].

Definition 2.2. A *topological Hopf* manifold is a compact complex manifold with universal cover biholomorphic to a complement to a point in a contractible Stein space.

Definition 2.3. A *Hopf* manifolds is a compact complex manifold $\mathcal{H}$ for which the universal cover is biholomorphic to the complement in $\mathbb{C}^n$ to the origin $O \in \mathbb{C}^n$. A Hopf manifold is called primary if the Galois group of the universal covering (or equivalently the fundamental group of $\mathcal{H}$) is isomorphic to $\mathbb{Z}$.

In the last decade, many diverse constructions of non-Kahler manifolds were proposed (cf. [2]). An interesting problem is to classify homological or topological (primary or non primary) Hopf manifolds. [3] More precisely, one would like to describe the topological, differentiable and almost complex manifolds [4] which admit a complex structure yielding a homological or topological Hopf manifold. Moreover, one would like to describe the moduli space parametrizing the complex structures on such an almost complex manifold.

The main results of Kodaira on the classification of Hopf surfaces can be summarized as follows (cf. [14], [15]). A Hopf surface is a quotient by a group G which has $\mathbb{Z}$ as its center and such that $G/\mathbf{Z}$ is finite (cf. [15],

[3] the terminology is suggested by more studied problem of classification of homological projective spaces

[4] cf. [22] for a discussion of invariants of almost complex structures on Brieskorn van deVen manifolds

theorem 30). Any Hopf surface contains a curve. A homological Hopf surface having algebraic dimension equal to zero and containing at least one curve is a Hopf surface ([15], theorem 34; case of algebraic dimension one considered in [8]).

The first examples of topological Hopf manifolds, (which are not Hopf) were found by Brieskorn and Van de Ven (cf. [3]). The contractible Stein spaces, in which the complement to a point serves in [3] as the universal cover of the topological Hopf manifold are the affine hypersurfaces $V \subset \mathbb{C}^{n+1}$ where V is a zero set of weighted homogeneous polynomial

$$z_0^{a_0} + ... + z_n^{a_n} = 0 \tag{3}$$

Here the integers a_i must satisfy conditions which assure that the link of a singularity (3) are topological (and possibly exotic) spheres. For example this is the case when n is odd and $a_1 = 3, a_2 = 6r - 1, a_i = 2$ (varying r yields all exotic spheres bounding a parallelizable manifold as the links of such singularities (3)). The examples of primary topological Hopf manifolds are obtained as the quotients by the action of the restriction of the following automorphism of $\mathbb{C}^{n+1}$:

$$T \cdot (z_0, ..., z_n) = (\lambda^{\frac{1}{a_a}} z_0, ..., \lambda^{\frac{1}{a_i}} z_i, ..., \lambda^{\frac{1}{a_n}} z_n) \quad (|\lambda| < 1) \tag{4}$$

(leaving the hypersurface (3) invariant).

More generally, consider a complete intersection:

$$\sum_{\nu=1}^{n+k} \alpha_{\mu,\nu} z^{a_\nu} \quad \mu = 1, ..., k \tag{5}$$

with generic coefficients $\alpha_{\mu,nu}$. The latter assures that (5) has an isolated singularity at the origin. Under the appropriate conditions (cf. [13] Satz 1.1) the link of this singularity (5) is a homology sphere (over $\mathbb{Q}$ or $\mathbb{Z}$).

Note that Zaidenberg conjectures that if V is set of zeros of a polynomial then contractible V with one isolated singularity up to an automorphism of $\mathbb{C}^n$ is the zero set of a weighted homogeneous polynomial (cf. [34]). This would imply that the non singular loci of hypersurfaces (3) are the only covering spaces of topological Hopf manifolds having an affine hypersurface as its closure.

2.2 Topological properties of $\mathbb{Q}$-Hopf manifolds

Kodaira's result on the fundamental groups of Hopf surfaces can be extended to topological Hopf manifolds.

We shall start by considering the question when a quotient of the complement $V - O$ in a Stein space V to a point is a homological Hopf manifold.

Proposition 2.4. *(i) The fundamental group $\pi_1(\mathcal{H})$ of a topological Hopf manifold $\mathcal{H}$ is a central extension of $\mathbb{Z}$ by a finite group.*

(ii) The class of biholomorphic equivalence of the universal cover is an invariant deformation type of a topological Hopf manifold. Deformation type of a topological Hopf manifold is given by the type of $V - O$ and the conjugacy class of the subgroup $G \subset Aut_O V$ which is a central extension $\mathbb{Z}$ by a finite group.

(iii) Let V be an affine hypersurface with $\mathbb{C}^*$ action or a complete intersection (5) and an isolated singularity at the origin. The quotient $V - O/\pi_1$ is a $\mathbb{Q}$-homology Hopf manifold if and only if the invariant subgroup of the action of π_1 on $H^{n-1}(V - O, \mathbb{Q})$ is trivial. In particular if the link of V is a $\mathbb{Q}$-sphere (resp. $\mathbb{Z}$-sphere) then $V - O/\pi_1$ is a $\mathbb{Q}$-homology (resp. integral) Hopf manifold.

Proof. The proof (i) is a direct generalization of Kodaira's argument. Since V is Stein, by Remmert embedding theorem (cf. [23]) we can assume that V is a subspace of $\mathbb{C}^N$ and select a ball $B \subset \mathbb{C}^N$ centered at the image of O. Let g be an element of infinite order in π_1 acting properly discontinuously on $V - O$. By Hartogs theorem g extends to an automorphism of V fixing O. Either g or g^{-1} takes $\partial B \cap V$ into $B \cap V$ and $\cap_n g^n(B \cap V) = O$ since existence of $z \neq O$ in the boundary of this intersection contradicts proper discontinuity of the action of π_1. Hence the quotient of $V - O$ by the subgroup $\{g\}$ generated by g is compact. The index of the cyclic subgroup $\{g\}$ in $\pi_1(\mathcal{H})$ is equal to the degree of the cover $V - O/\{g\}$ and hence is finite. The subgroup of $\{g\}$ which is normal in $\pi_1(\mathcal{H})$ yields claimed presentation of the latter as a central extension.

(ii) is a a consequence of result of Andreotti and Vesentini on pseudo-rigidity of Stein manifolds (cf. [1]).

To see (iii) consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} : H^p(\pi_1/\mathbb{Z}, H^q(\mathbb{Z}, W)) \to H^{p+q}(\pi_1, W) \tag{6}$$

where W is a $\mathbb{Q}$-vector space with a structure of a π_1-module and $\mathbb{Z}$ is the center generated by an element $g \in \pi_1$. Since $\pi_1/\mathbb{Z}$ is finite and $H^q(\mathbb{Z}, W)$ has no $\mathbb{Z}$-torsion we have $E_2^{p,q} = 0$ for $p \geqslant 1$. Moreover $H^q(\mathbb{Z}, W)$ is the subspace of g-invariants (resp. g-covariants) of W for $q = 0$ (resp. $q = 1$) and is trivial for $q > 1$. Hence in the spectral sequence (6) there are at most two non trivial terms and so

$$\dim H^q(\pi_1, W) = \begin{cases} W^g & q = 0 \\ W_g & q = 1 \\ 0 & q > 1 \end{cases} \tag{7}$$

Using a homotopy equivalence between $V - O$ and the link of isolated singularity of V we obtain $H^q(V - O, \mathbb{Q}) = 0$ for $q \neq 0, n-1, n, 2n-1$ (cf. [19]). Next consider the spectral sequence of the action of π_1 on the universal cover $V - 0$:

$$H^p(\pi_1, H^q(V - O)) \Rightarrow H^{p+q}(V - O/\pi_1) \tag{8}$$

Applying (7) for $W=\mathbb{Z}$ for $q=0,2n-1$, $W=H^{n-1}(V-O)$ or $W=H^n(V-O)$ if $q=n-1,n$ and using a π_1-equivariant identification of $H^n(V-O)$ with $H^{n-1}(V-O)^*$ [5] (which is a consequence of the Poincare duality for the link of singularity of V) we obtain the result. ◻

2.3 Biholomorphic automorphisms of universal covers

Next we shall consider the question of existence of automorphisms of $V-O$ different than (4) which generate an infinite cyclic group acting properly discontinuously with compact quotient i.e. automorphisms yielding primary topological Hopf manifolds. We shall assume that V is a zero set of an arbitrary weighted homogeneous polynomial (i.e. a sum of monomials $z_1^{i_1}\cdots z_n^{i_n}$ such that $\sum i_k b_k = d$ where b_k are the weights and d is the degree) [6] with isolated singularities. In the case when $V=\mathbb{C}^n$ the automorphisms generating infinite properly discontinuously acting groups with compact quotients were described by Haefliger (cf. [12]).

Lemma 2.5. *Let V be a zero set of a weighted homogeneous polynomial f having weights a_i and the degree d. Then one has the extension:*

$$0\to\mathbb{C}^*\to Aut(V-0)\to G\to 0$$

where the group G is finite if $\sum_i \frac{1}{a_i}<1$. More generally, one has the same type exact sequence if V is a complete intersection (5) and $\sum\frac{1}{a_i}<k$.

Proof. The affine hypersurface V is a open subset in a $\mathbb{P}^1$-bundle $\bar V\to V_\infty$ over a hypersurface in a weighted projective space which one can identify with the hyperplane section at infinity. $AutV$ is the subgroup of the group $Aut^\circ\bar V$ of automorphisms of $\bar V$ fixing invariant the two sections of $\bar V$ (corresponding to the exceptional set of the weighted blowup of projective closure of V and the hyperplane section at infinity V_∞). [7] Clearly, the $\mathbb{C}^*$ action provides a normal subgroup $\mathbb{C}^*$ of $Aut^\circ\bar V$ and the quotient is the subgroup of the group of automorphisms of V_∞. The latter has ample canonical class if the condition on the weights is met and hence V_∞ has a finite automorphisms group. The argument in the case of complete intersection is the same. ◻

On the other hand for the quadric hypersurface $z_1^2+....+z_n^2$ one has many non-trivial automorphisms (cf. [30]). For example one has the automorphisms of

$$X_1X_2+X_3^2+X_4^2+....+X_n^2 \tag{9}$$

given by Danilov and Gizatullin (cf. [6] p.101) defined by the change of variables:

$$X_1'=\beta X_1,\quad X_2'=\beta(\alpha^2X_2+2\alpha X_3 f(X_1)+X_1f^2(X_1)), \tag{10}$$
$$X_3'=\sqrt{-1}(\alpha\beta X_3+\beta X_1 f(X_1))\quad X_i'=(\alpha\beta)X_i\quad (4\leqslant i\leqslant n)$$

Such an automorphism preserves quadric (9) for all f and no power of (10) has a fixed point on $\mathbb{C}^n-0$. The jacobian of (10) at the origin has the form:

$$\begin{pmatrix} \beta & 0 & 0 & \cdot & \cdot \\ 0 & \beta\alpha^2 & 2\alpha f(0) & \cdot & \cdot \\ \sqrt{-1}\beta f(0) & 0 & \sqrt{-1}\alpha\beta & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \end{pmatrix} \tag{11}$$

Hence for $f(x)$ such that $f(0)=0$ the eigenvalues of (11), which are equal to $\beta,\sqrt{-1}\alpha\beta,\alpha^2\beta$, are in the unite circle provided that $|\beta|<1$ and $|\alpha|<1$. Therefore the action of the infinite cyclic group generated by (10) is proper discontinuous (cf. [12]) and we obtian many examples of $\mathbb{Q}$-homological Hopf manifolds having different deformation type.

2.4 Cohomology of bundles on universal covers

The results of sections 4 and 5 below deal primarily with the cohomology of bundles on topological Hopf manifolds. They depend on vanishing of the cohomology of the bundles on universal covers which we now review.

In the case of topological Hopf manifolds the vanishing of the cohomology of bundles on the universal cover $V-O$ follows from the vanishing of cohomology of coherent sheaves on Stein spaces in positive dimensions (Cartan's theorem B) and from the following two results:

Theorem 2.6. *(Scheja; cf [26], [29] p. 129) Let V be a complex space, A a subvariety of dimension $\leqslant d$ and $\mathcal{F}$ a coherent sheaf on V such that* $\mathrm{codh}\mathcal{F}\geqslant d+q$. *Then $H^k(V,\mathcal{F})\to H^k(V-A,\mathcal{F})$ is an isomorphism for $0\leqslant k<q-1$ and injective for $k=q-1$.*

and

[5] * denotes dual space

[6] for the hypersurface (3), one has $b_i=\frac{\text{l.c.m.}(a_1,...,a_n)}{a_i}, d=\text{l.c.m.}(a_1,...,a_n)$

[7] Automorphisms of ruled surfaces were studied in [20].

Theorem 2.7. *(Siu, cf. [28]) Let V be a complex space, A a subvariety of dimension $\leqslant d$, $i : V - A \to V$ and $\mathcal{F}$ and coherent analytic sheaf on $V - A$. If $codh\mathcal{F} \geqslant d + 3$ then $i_*(\mathcal{F})$ is a coherent sheaf (on V).*

Corollary 2.8. *Let V be a Stein space with isolated singularity O and let $\mathcal{F}$ be a coherent sheaf on $V - O$. Assume that $\dim V \geqslant 3$. Then $\mathcal{F}$ extends to a coherent sheaf on V and $H^k(V - O, \mathcal{F}) = 0$ for $0 < k \leqslant d - 1$ where d is the cohomological codimension of the stalk of this extension at O. In particular if $\mathcal{F}$ is a locally trivial bundle on $V - O$ which admits a locally trivial extension to V then $H^k(V - O, \mathcal{F}) = 0$ for $k \neq 0, n - 1$.*

Proof. Indeed, $hd_{V-O}\mathcal{F} = 0$ since $\mathcal{F}$ is locally free and hence $codh\mathcal{F} = \dim V$. Therefore $i_*(\mathcal{F})$ is coherent. Taking $X = V, A = O$ in theorem 2.6 we see that $H^k(V - 0, \mathcal{F}) \to H^k(V, \pi_*(\mathcal{F}))$ is injective for $0 < k \leqslant d - 1$ and Cartan's theorem B yields the first claim. In the case when extension is locally trivial we have $d = \dim V$ and hence the second assertion. ☐

Remark 2.9. Consider the case when bundle $\mathcal{F}$ on $\mathbb{C}^n - 0$ is a pullback of a bundle $\mathcal{F}'$ on $\mathbb{P}^{n-1}$ via Hopf map $\pi : \mathbb{C}^n - 0 \xrightarrow{\mathbb{C}^*} \mathbb{P}^{n-1}$. Then $\mathcal{F}$ extends to a locally trivial bundle on $\mathbb{C}^n$ if and only if $\mathcal{F}'$ is a direct sum of line bundles ([27]). The cohomology of $\mathcal{F}$ can be found using Leray spectral sequence $H^p(\mathbb{P}^{n-1}, R^q\pi_*(\mathcal{F})) \to H^{p+q}(\mathbb{C}^n - O, \mathcal{F})$. Since by projection formula $R^q\pi_*(\pi^*(\mathcal{F}')) = R^q\pi_*(\mathcal{O}_{\mathbb{C}^n-O}) \otimes \mathcal{F}'$ and since the fibers of π are Stein (in fact just $\mathbb{C}^*$) we have $R^q\pi_*(\mathcal{O}_{\mathbb{C}^n-O}) = 0$ for $q \neq 0$. Hence the above Leray sequence degenerates and vanishing $H^q(\mathbb{C}^n - O, \mathcal{F}) = 0$ for $0 < q \leqslant n - 2$ follows from a well known vanishing of cohomology of line bundles on $\mathbb{P}^{n-1}$ (i.e. in all dimensions $\neq 0, n - 1$.) Note that vanishing of cohomology of a bundle $\mathcal{F}'$ on $\mathbb{P}^{n-1}$ in indicated range is closely related to existence of a splitting of the bundle (cf. [24] p.39).

Remark 2.10. It follows from the result of extension of bundles $\pi^*(\mathcal{F})$ to $\mathbb{C}^n$ mentioned in the last remark that $\pi^*(\Omega^p_{\mathbb{P}^{n-1}})$ cannot be extended to a *locally trivial* bundle on $\mathbb{C}^n$ and hence corollary 2.8 does not yield information on the cohomology of this bundle.

Remark 2.11. Note that the canonical class of the $\mathbb{Z}$-quotients of $\mathbb{C}^{n+1} - O$ or hypersurfaces (3) is given by the divisor (if $n = 2$ cf. [15]):

$$K_{\mathcal{H}} = -\sum_{i=0}^{i=n} D_i \tag{12}$$

where D_i divisor on $\mathcal{H}$ biholomorphic to Hopf (resp. Brieskorn van de Ven for quotients of (3)) manifold which is the image of affine hypersurface in $\mathbb{C}^{n+1}$ (resp. in (3)) given by $z_i = 0$ $(i = 0, ..., n)$. Indeed, the form $\frac{dz_0}{z_0} \wedge ... \wedge \frac{dz_n}{z_n}$ is a meromorphic form with poles at $z_0 \cdots z_n = 0$ on the universal cover invariant under the deck transformations and hence descending to the quotient. For a the quotient of the hypersurface (3) one has:

$$K_{\mathcal{H}} = a_0 D_0 - \sum_{i=0}^{n} D_i \tag{13}$$

Indeed, the restriction to V of invariant under the action of (4) meromorphic form $\omega_1 = \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_n}{z_n}$ descents to a meromorphic form on $\mathcal{H}$. On the other hand the form $\omega_2 = Res \frac{dz_0 \wedge .... \wedge dz_n}{z_0^{a_0} + ... z_n^{a_n}} = \frac{dz_1 \wedge ... \wedge dz_n}{a_0 z_0^{a_0 - 1}}$ in holomorphic and non-vanishing on $V - 0$. One has $\omega_1 = \frac{z_0^{a_0-1}\omega_2}{z_1 \cdots z_n}$ and hence the zeros and poles of ω_1 restricted to V are given by (13).

In the study of the cohomology of bundles $\Omega^p(E)$ on Hopf manifolds we restrict our-self to the case when $V - O$ is the complement to the fixed point in a hypersurface in $\mathbb{C}^{n+1}$ which supports a $\mathbb{C}^*$ action or complete intersection (5). The needed results on the cohomology of sheaves of differential forms are essentially contained in [33] in the case of hypersurfaces and in [32] in the case of complete intersections.

Lemma 2.12. *Let V be a hypersurface (3) or a complete intersection (5). Then one has the following:*

$$\mathrm{rk} H^q(V - 0, \Omega^p) = \begin{cases} 0 & p + q \leqslant n - 2 \quad 1 \leqslant q \leqslant n - 2 \\ \tau & p + q = n - 1, n \quad 1 \leqslant q \leqslant n - 2 \\ 0 & p + q \geqslant n + 1 \quad 1 \leqslant q \leqslant n - 2 \end{cases}$$

From this it follows:

Lemma 2.13. *If the multiplicity of V at the origin is greater than one then for the automorphism $T^*_{H^q(V-0,\Omega^p)}$ of $H^q(V - 0, \Omega^p)$ induced by (4) one has*

$$\dim \mathrm{Ker}(T^*_{H^q(V-0,\Omega^p)} - I) = 1$$

Proof. In [33], S.Yau obtained, for a hypersurface V, an isomorphism $H^q(V - 0, \Omega^p)$ and the vector space:

$$M_f = \mathbb{C}[z_1, ..., z_{n+1}]/(... \frac{\partial f}{\partial z_i} ...)$$

(or its dual). One has splitting $M_f = \mathbb{C}\cdot 1 \oplus \mathcal{M}$ where $\mathcal{M}$ is the image in the quotient ring of the maximal ideal of the local ring at the origin. This image is a vector space which has the monomials $\ldots \cdot z_i^{j_i} \cdot \ldots$ with $0 \leqslant j_i < a_i, \sum_i j_i > 0$ as its basis. The action of automorphism T is given by $z_i \to \lambda_i z_i$ and the above monomials are eigenvectors of this action with eigenvalues all different from 1. Hence $\mathrm{Ker}(T^*_{H^q(V-0,\Omega^p)} - I)$ corresponds to the summand $\mathbb{C}\cdot 1$. A similar argument works in the case of complete intersections using Prop. 1.3 (d) of [32]. □

3 Cohomology of local systems on topological Hopf manifolds

Proposition 3.1. *Let $\mathcal{L}$ be a local system on a topological Hopf manifold $\mathcal{H}$. Then*

$$H^i(\mathcal{L}) = 0 \quad for \quad i \neq 0, 1, 2n-1, 2n$$

$$\dim H^0(\mathcal{L}) = \dim H^1(\mathcal{L}) = \dim H^{2n-1}(\mathcal{L}) = \dim H^{2n}(\mathcal{L}) = 1$$

Proof. Let $\tilde{\mathcal{H}}$ be the universal covering space. We have the spectral sequence for the action of a subgroup $\mathbb{Z}$ (generated by T) of the center of $\pi_1(\mathcal{H})$ on $\tilde{\mathcal{H}}$ yields:

$$H^p(\tilde{\mathcal{H}}) \stackrel{T-I}{\rightarrow} H^p(\tilde{\mathcal{H}}) \rightarrow H^p(\tilde{\mathcal{H}}/\mathbb{Z}) \rightarrow H^{p+1}(\tilde{\mathcal{H}}) \tag{14}$$

Since

$$H^p(\tilde{\mathcal{H}}, \mathbb{Q}) = \begin{cases} 0 & p \neq 0, 2n-1 \\ \mathbb{Q} & p = 0, 2n-1 \end{cases}$$

and the action of T is trivial for both $p = 0, 2n-1$ we obtain the claim for the cohomology of the primary Hopf manifold $\tilde{\mathcal{H}}/\mathbb{Z}$. The claim for the local systems on $\tilde{H}/\pi_1(\mathcal{H})$ follows from the spectral sequence (below $\sigma : \tilde{\mathcal{H}}/\mathbb{Z} \to \mathcal{H}$)

$$H^p(\pi_1(\mathcal{H})/\mathbb{Z}, H^q(\sigma^*(\mathcal{L})) \rightarrow H^{p+q}(\mathcal{H}, \mathcal{L}) \tag{15}$$

since the group $\pi_1(\mathcal{H})/\mathbb{Z}$ is finite. □

Remark 3.2. One can obtain the conclusion of Proposition 3.1 for homological Hopf manifolds, if one makes additional assumptions on the fundamental group e.g. for $\mathcal{H}$ such that $\pi_1(\mathcal{H})$ satisfies the conclusions of Proposition 2.4 (i) or more generally on characteristic varieties of the group $\pi_1(\mathcal{H})$ (cf. [16]).

4 Cohomology of vector bundles

Let X be a complex space on which a group G acts via holomorphic maps freely and let $\pi : X \to X/G$. Let $\mathcal{F}$ be a coherent sheaf on X/G. Let $\bullet^G : V \to V^G$ be the functor assigning to a $\mathbb{C}$-vector space with a G-action the subspace of invariants. One has canonical isomorphism (cf [9]): $\Gamma(X, \pi^*(\mathcal{F}))^G = \Gamma(X/G, \mathcal{F})$. The corresponding spectral sequence of the composition of functors is given by (cf. [9]):

$$E_2^{p,q} = H^p(G, H^q(X, \pi^*\mathcal{F})) \Rightarrow H^{p+q}(X/G, \mathcal{F}) \tag{16}$$

In particular this spectral sequence can be applied to the case when X be a complex manifold, $\pi : \tilde{X} \to X$ is its universal covering space and $G = \pi_1(X)$. Let $\mathcal{F}$ be a coherent sheaf on X. The above spectral sequence becomes:

$$E_2^{p,q} = H^p(\mathbb{Z}, H^q(\tilde{X}, \pi^*\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F}) \tag{17}$$

In the case when $G = \mathbb{Z}$ and hence has the cohomological dimension equal to 1, this spectral sequence degenerates in term E_2 (i.e. due to the vanishing $H^i(\mathbb{Z}, V) = 0, i \geqslant 2$ for a $\mathbb{Z}$-module V). Since $E_2^{0,q} = H^q(X, \pi^*(\mathcal{F}))^{\mathbb{Z}} = \mathrm{Ker}(T - Id) : H^q(X, \pi^*\mathcal{F}) \to H^q(X, \pi^*\mathcal{F})$ and $E_2^{1,q} = \mathrm{Coker}(T - Id) : H^q(X, \pi^*\mathcal{F}) \to H^q(X, \pi^*\mathcal{F})$.

Hence we have the exact sequence:

$$\begin{aligned} \rightarrow H^{q-1}(X, \pi^*(\mathcal{F})) \stackrel{T-1}{\rightarrow} H^{q-1}(X, \pi^*(\mathcal{F})) \rightarrow H^q(X/G, \mathcal{F}) \\ \rightarrow H^q(X, \pi^*(\mathcal{F})) \stackrel{T-1}{\rightarrow} H^q(X, \pi^*(\mathcal{F})) \end{aligned} \tag{18}$$

Note that we have $H^i(\tilde{X}, \mathcal{O}) = 0$ for $i \neq 0, n-1, n$, if $\tilde{X} = \mathbb{C}^n - 0$.

Proof of the theorem 1.2. The finiteness assumption and the exact sequence (for a sheaf $\mathcal{F}$ on $\bar{V}$):

$$H^q(\bar{V}) \rightarrow H^q(V, \mathcal{F}) \rightarrow H_A^{q+1}(\bar{V}, \mathcal{F}) \tag{19}$$

yields that $H^q(V, p^*(E))$ is finite dimensional for $q \leqslant r$ and is infinite dimensional for $q = 0$. Moreover the space $H^0(V, p^*(E))$ is Montel with the semi-norm given by the compact subsets $\bigcup_i (f^i(U) \cup f^{-i}(U))$ where U is the closure of the fundamental domain of an infinite order transformation f on V in the center of $\pi_1(H)$. In particular f^* is compact and hence the index of $I - f^*$ acting on $H^0(V, p^*(E))$ is zero. Therefore the dimensions of kernel and cokernel are equal. This and the exact sequence relating the cohomology of cover discussed above yield the claim. □

Proof of the theorem 1.3. We have $H^q(V-O,\Omega^p_{V-O}\otimes\pi^*(E))=0$ unless $q=0,n-1-p,n-p,n-1,p\geqslant 2$. In the case $q=0,n-1$ spaces are infinite dimensional but the operators on H^q induced by T are Fredholm with zero index. Hence the exact sequence (18) yields:

$$\begin{gathered}\dim H^q(V-0/G,\Omega^p(E))=\dim H^{q+1}(V-0/G,\Omega^p(E))\quad q=0,n-1\\ H^q(V-0/G,\Omega^p(E))=0\quad q\neq n-1-p,n-p,0,1,n-1,n\\ \dim H^{n-2-p}(V-0/G,\Omega^p(E))=\tau,\\ \dim H^{n-1-p}(V-0/G,\Omega^p(E))=\tau\\ \dim H^{n-p}(V-0/G,\Omega^p(E))=\tau\quad p\neq 0,1\end{gathered}\tag{20}$$

Indeed, for $q=0,1,n-1,n$ are cohomology of bundles on Hopf manifold are kernel and cokernel of Fredholm operators which, as was mentioned earlier, has zero index. For $q\neq 0,1,n-1,n$ contribution from $H^q(V-0,\Omega^p\otimes\pi^*(E))$ contributes τ into $\dim H^q$ and $\dim H^{q+1}$ since the action of the covering group on the cohomology $H^{n-2-p}(V-0,\Omega^p(E))$ is trivial and the remaining groups are zeros. Now the result follows from lemma 2.13.

Corollary 4.1. *Let $\mathcal{H}$ be a homological Hopf manifold as in 1.3 Then $Pic(\mathcal{H})=\mathbb{C}^*$.*

Proof. We have $H^0(\mathcal{H},\mathcal{O})=H^1(\mathcal{H},\mathcal{O})=\mathbb{C}$ from theorem 1.3. Hence the cohomology sequence of the exponential sequence $0\to\mathbb{Z}\to\mathcal{O}\to\mathcal{O}^*\to 0$ has the form $H^1(\mathcal{H},\mathbb{Z})\to\mathbb{C}\to H^1(\mathcal{H},\mathcal{O}^*)\to H^2(\mathcal{H},\mathbb{Z})$. Since $H^2(\mathcal{H},\mathbb{Q})=0$, this yields the claim. □

5 Hodge deRham spectral sequence

Recall that the category of local systems on a manifold is equivalent to the category of locally constant sheaves which in turn is equivalent to the category of locally free sheaves with integrable connection (cf. [7]). For a local system, $\mathcal{L}$, let $E_{\mathcal{L}}$ be the corresponding locally free sheaf and and $\nabla_{\mathcal{L}}$ be the corresponding flat connection on $E_{\mathcal{L}}$.

Proposition 5.1. *Let $\mathcal{H}$ be a Hopf manifold, E be a bundle on $\mathcal{H}$ with a trivial pullback on the universal cover and $\mathcal{L}$ be the corresponding local system. Then the Hodge deRham spectral sequence:*

$$H^q(\mathcal{H},\Omega^p(E))\to H^{p+q}(\mathcal{H},\mathcal{L})\tag{21}$$

degenerates in the term E_2.

Proof. In order to calculate the differential d_1 i.e. the map $E_1^{p,q}\to E_1^{p,q}$ for $q=0,1$ recall that the terms $E_1^{p,0}$ and $E_1^{p,1}$ are respectively invariants and covariants of the selfmap of $\Gamma(\mathbb{C}^n,\Omega^p\otimes p^*(E))$ induced by a map $T:\mathbb{C}^n\to\mathbb{C}^n$ (such that $\mathbb{C}^n-O/T=\mathcal{H}$). On the other hand the holomorphic deRham complex on $\mathbb{C}^n$:

$$...\to\Gamma(\mathbb{C}^n,\Omega^p)\xrightarrow{d}\Gamma(\mathbb{C}^n,\Omega^{p+1})\to...\tag{22}$$

is exact. The explicit construction of the holomorphic form η such that for a given close form ω one has $\omega=d\eta$ shows that η is T-invariant if ω is. Hence d induces exact sequence of T-invariant forms. i.e. the term E_2 is zero (in particular one can recover the vanishing results of Prop. 3.1). □

References

[1] A.Andreotti, E.Vesentini, On the pseudo-rigidity of Stein manifolds. Ann. Scuola Norm. Sup. Pisa (3) 16 1962 213–223.

[2] F.Bosio, L.Meersseman, Real quadrics in $\mathbf{C}^n$, complex manifolds and convex polytopes. Acta Math. 197 (2006), no. 1, 53–127.

[3] E.Brieskorn and A. van de Ven, Some complex structures on products of homotopy spheres. Topology 7 1968 389–393.

[4] F.Catanese, Automorphisms of rational double points and moduli spaces of surfaces of general type. Comp. vol.61, (1987).

[5] K.Dabrowski, Examples of Hopf manifolds with arbitrary admissible algebraic dimensions, Bull, Polish Acad. of Science, 31 (1983) no. 9-12, 325 (1984).

[6] V.Danilov, M.Gizzatulin, Automorphisms of affine surfaces. II. Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 1, 54–103, 231.

[7] P.Deligne, Equations differentielles e points singuliers reguliers. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[8] W.Ebeling, C.Okonek, Homology Hopf surfaces, Compositio Math. 91 (1994), no. 3, 277–304.

[9] A.Grothendieck. Sur quelques points d'algbre homologique. Thoku Math. J. (2) 9 1957 119–221.

[10] Nin Gang, Xiang-Yu Zhou The cohomology of bundles on general non primary Hopf manifolds. Recent progress on some problems in several complex variables and partial differential equations, 107–115, Contemp. Math., 400, Amer. Math. Soc., Providence, RI, 2006.
[11] Hasegawa, K. Deformations and diffeomorphism types of Hopf manifolds. Ill. J. of Math 37(1993) p.643.
[12] A.Haefliger, Deformations of transversely holomorphic flows on spheres and deformations of Hopf manifolds. Compositio Math. 55 (1985), no. 2, 241–251.
[13] H.Hamm, Exotische Sphren als Umgebungsrnder in speziellen komplexen Rumen. Math. Ann. 197 (1972), 44–56
[14] K. Kodaira, On the structure of compact complex analytic surfaces. I, II. Proc. Nat. Acad. Sci. U.S.A. 50 (1963), 218–221;
[15] K.Kodaira, On the structure of compact complex analytic surfaces. II. Amer. J. Math. 88 1966 682–721.
[16] Libgober, A. Characteristic varieties of algebraic curves. Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), 215–254, NATO Sci. Ser. II Math. Phys. Chem., 36, Kluwer Acad. Publ., Dordrecht, 2001.
[17] D.Mall, The cohomology of line bundles on Hopf manifolds. Osaka J. Math. 28 (1991), no. 4, 999–1015.
[18] D.Mall, On holomorphic vector bundles on Hopf manifolds with trivial pullback on the universal covering. Math. Ann. 294 (1992), no. 4, 719–740.
[19] J.Milnor, Singular points of complex hypersurfaces. Annals of Mathematics Studies, No. 61 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968
[20] M.Maruyama, On automorphism groups of ruled surfaces. J. Math. Kyoto Univ. 11 1971 89–112.
[21] A.Mizuhara, Three dimensional Hopf manifolds. Acta Human. Sci. Univ. Sangio Kyotien. Natur. Sci. Ser. No. 8 (1978), 5–21.
[22] S.Morita, A topological classification of complex structures on $S^1 \times S^{2n-1}$. Topology 14 (1975), 13–22.
[23] R.Narasimhan, Imbedding of holomorphically complete complex spaces. Amer. J. Math. 82 1960,f 917–934.
[24] C.Okonek, M. Schneider, Spindler, Vector bundles on Complex Projective spaces. Progress in Mathematics, 3. Birkhuser, Boston, Mass., 1980
[25] Palamodov, V. P. Deformations of Hopf manifolds and the Poincar-Dulac theorem. Funktsional. Anal. i Prilozhen. 17 (1983), no. 4, 7–16.
[26] G.Scheja, Riemannsche Hebbarkeitsstze für Cohomologieklassen. Math. Ann. 144 1961 345–360.
[27] J.P.Serre, Prolongement de faisceaux analytiques cohrents. Ann. Inst. Fourier (Grenoble) 16 1966 fasc. 1, 363–374.
[28] Siu, Yum-tong, Extending coherent analytic sheaves. Ann. of Math. (2) 90 1969 108–143.
[29] Siu, Yum-tong, Techniques of extension of analytic objects. Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, New York, 1974.
[30] B.Totaro, The automorphism group of an affine quadric. Math. Proc. Cambridge Philos. Soc. 143 (2007), no. 1, 1–8.
[31] M.Verbitsky, Holomorphic bundles on diagonal Hopf manifolds, Izv. Ross. Akad. Nauk. Ser. Mat, 70, (2006) 867-882.
[32] H.Vosegaard, Generalized Tjurina numbers of an isolated complete intersection singularity. Math. Ann. 329 (2004), no. 2, 197–224.
[33] Stephen S. T. Yau, Kohn-Rossi cohomology and its application to the complex Plateau problem. I. Ann. of Math. (2) 113 (1981), no. 1, 67–110.
[34] M.Zaidenberg, Selected Problems, math.AG/0501457