Alexander invariants of the complements to algebraic curves

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This talk:

1. Parallels between knot theory and the complements to algebraic curves.

2. Multivariable Alexander invariants.

3.Curves with fundamental groups of the complement having free groups of large ranks as quotients.

Parallels between knot theory and geometry of the complements to algebraic curves

The discovery of a close relation between plane curves and knots goes back to Zariski in the 1930's and relation between algebraic geometry and knots even earlier (Wirtinger etc.).

R. Kirby's list has several problems on topology of plane curves,

L.Rudoplh: Knot theory of complex plane curves. 2005 Handbook of knot theory.

L.Kauffman thesis: CYCLIC BRANCHED-COVERS, O(N)-ACTIONS AND HYPERSURFACE SINGULARITIES studies the periodicity of the diffeomorphism types of certain manifolds with group action where links of singularities e.g. $z_1^a + z_2^b + z_3^2 + \cdots + z_n^2$ are the key examples.

Germs of singularities $f(x, y) = 0$

If the germs $f(x, y)$ has an isolated singularity at the origin, then $\{(x, y)|f(x, y) = 0\} \cap \partial B_\epsilon$ is an iterated torus link, the number of components is the number of branches, linking number of branches is the intersection multiplicity of branches, type of each branch is determines by the Puiseux expansion of the branch, etc.

Example

$f = x^n - y^n \Rightarrow$ Hopf link
$x^2 = y^3 \Rightarrow$ trefoil etc.

Projective curves

Global case: Topology of $\mathbb{P}^2 \setminus C$, C is the zero set of $f(x, y, z) = 0$.

Let $C \subset \mathbb{P}^2(C)$ be zero set of $f(x, y, z) = 0$. If C is smooth and irreducible then the diffeomorphism type of $\mathbb{P}^2 \setminus C$ depends only on the degree of f, any two smooth curves can be deformed one into another through smooth curves.

If C has singularities then one wants to consider **equisingular deformations**:
each singularity has local type: the germ of the singularity or its link. One wants deformations that do not "loose singularities" and do not acquire new ones. Such deformations do not change topological type of the complement or of the pair $(\mathbb{P}^2, C)$ (Thom's isotopy theorem).

Main problems

What are possible fundamental groups of the complement $\pi_1(\mathbb{P}^2 \setminus C)$ and what is their relation to geometry of C (local type of singularities, their number, position...).

What are possible homeomorphisms types of the complements types of curves (for fixed degree there are finitely many deformation types and hence finitely many homeomorphism types, though the number of possible singularities types grows doubly exponential).

$S^3 \setminus L$	$\mathbb{P}^2 \setminus C$
$2-$ *complex*	$2-$ *complex*
classification up to isotopy	*up to equisingular isotopy*
link is a closed braid	$C \Longleftrightarrow \beta_1 \cdots \beta_N = \Delta^2$
$H_1(S^3 \setminus L, \mathbb{Z}) = \mathbb{Z}^r$	$H_1(\mathbb{P}^2 \setminus C, \mathbb{Z}) = \mathbb{Z}^r/(d_1, \cdots d_r)$
$\pi_1(S^3 \setminus L)$ *very interesting*	$\pi_1(\mathbb{P}^2 \setminus C) =?$
Alexander polynomial $\Delta(t)$ *defined*	*Alexander polynomial defined*
$\Delta(t) -$ *mild restrictions*	$\Delta(t)$ *cyclotomic*

No analogs of skein relations (difficult to construct interesting singular curves), Seifert forms, ...

Multivariable Invariants

Definition of Alexander module Let $G \to G/G' = \mathbb{Z}^m$ be a surjection. Then G'/G'' has a structure of $\Lambda_r = \mathbb{Z}[G/G'] = \mathbb{Z}[t_1, t_1^{-1}, \cdots t_r, t_r^{-1}]$-module. **Characteristic variety of** G is the zero set of the 0-th elementary ideal $E_0(G'/G'' \otimes \mathbb{C})$ in $Spec\mathbb{C}[t_1, t_1^{-1}, \cdots t_r, t_r^{-1}] = (\mathbb{C}^*)^r$.

Example

Let $G = \pi_1(S^3 \setminus \bigcup_1^r L_i)$. Crowell-Strauss showed that $E_0 = (\Delta_1)M_{id}^{\frac{(r-2)(r-3)}{2}}$. This implies that the characteristic variety of a link group has codimension 1 in $\mathbb{C}^r$. This is NOT the case for fundamental groups of the most of algebraic curves when $r \geq 2$.

Linear structure of characteristic varieties of algebraic links.

Consider link with p-components:

$$z_1^p + z_2^{pq} = 0$$

Then

$$\Delta(t) = \frac{((t_1 \cdots t_p)^q - 1)^{p-1}}{(t_1 \cdots t_p - 1)}$$

The zero set is the union of

$$t_1 \cdots t_p = \omega_q^i \quad i = 1, \cdots q-1$$

Translated subgroups theorem

$t_1 \cdots t_p = \omega_q^i \quad i = 1, \cdots q-1$ are cosets of $(\omega_q^i, 1, \cdots 1)$ for the subgroup $t_1 \cdots t_p = 1$ of the torus $(\mathbb{C}^*)^p$ (translated subgroups by an element of finite order).

The latter is the case for the links of algebraic plain curve singularities as follows from the explicit formulas in terms of a resolution of singularity. For 1-variable Alexander polynomial this is equivalent to roots being roots of unity (also follows from identification with the roots of monodromy of fibering of the complement i.e. the Milnor fibration and "Monodomry Theorem" in algebraic geometry.).

In fact this is the case for fundamental groups of the complements to plane curves and more generally smooth quasi-projective varieties.

Theorem Let $C \subset \mathbb{P}^2$ be a plain algebraic curve. Then support of the Alexander module is a union translated subgroups of the torus by an element of a finite order.

The same conclusion holds for the characteristic varieties of $\pi_1(X)$ where X is a smooth quasi-projective manifold such that $\pi_1(X)$ admits a free quotient.

This result follows from a much more specific theorem due to D. Arapura but the case of one-variable Alexander polynomials follows from divisibility theorem (Libgober, 1982).

Maps corresponding to the components.

Theorem (D. Arapura) For each irreducible component of characteristic variety

$$\rho\Sigma_i$$

($\rho \in CharT$ a torsion character, Σ_i a subgroup of $CharH_1(\mathbb{P}^2 \setminus \bigcup_{i=1}^{N} C_i, \mathbb{Z})$) having the dimension greater or equal than 2) there is holomorphic map

$$\phi_i : \mathbb{P}^2 \setminus \bigcup_{i=1}^{i=N} C_i \to \mathbb{P}^1 \setminus [d_i] \ \ (d_i = \dim\Sigma_i + 1)$$

such that

$$\Sigma_i = \phi_i^*(CharH_1(\mathbb{P}^1 \setminus [d_i], \mathbb{Z})$$

Local systems

Arapura theorem actually deals with local systems and based on Hodge theory.

Let $\chi : \pi_1(X) \to \mathbb{C}^*$ be a character of a group having surjection onto $\mathbb{Z}^r$ i.e. $\chi \in Hom(\mathbb{Z}^r, \mathbb{C}^*) = (\mathbb{C}^*)^r$. Let $Char\pi_1(X)$ be the characteristic variety of the fundamental group. One can show that (E. Hironaka, with deRham (1942), Burde (1967), for $r = 1$ case) that one has cohomology $H^i(X, \chi)$ such that

$$\dim H^1(X, \chi) \neq 0 \Longleftrightarrow \chi \in Char(\pi_1(X))$$

Remark

Silver-Williams show a similar result for twisted Alexander polynomials. There is a twisted version of the above theory of the complements using twisted characteristic varieties etc. (L. 2008)

How to see a map of the complement to a curve C?

Example

Let $f_0 = 0, f_\infty = 0$ be two homogeneous polynomials of the same degree. Let $f_i = a_i f_0 + b_i f_\infty, i = 1, \cdots r$ be a sequence of polynomials obtained from f_0, f_∞ for some $a_i, b_i \in \mathbb{C}$. Then

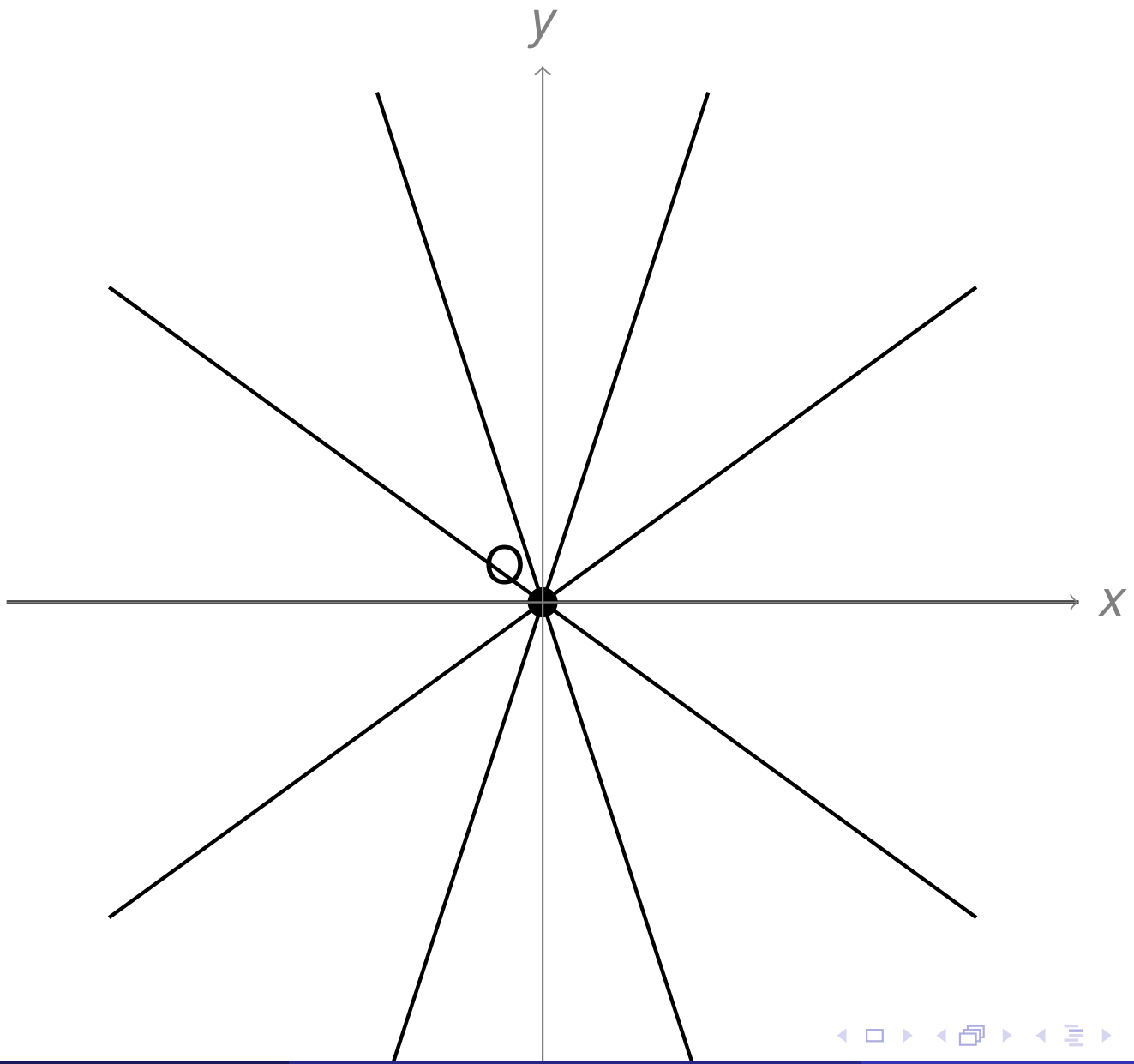
$$\Phi_{f_0, f_\infty} : [x : y : z] \to [f_0(x, y, z) : f_\infty(x, y, z)]$$

is well defined map $\mathbb{P}^2 \to \mathbb{P}^1$ unless $f_0(x, y, z) = f_\infty(x, y, z) = 0$. In particular, denoting the zero set of $f_i, i = 0, 1, \cdots, r, \infty$ by C_i one has a well defined map;

$$\mathbb{P}^2 \setminus \bigcup C_i \to \mathbb{P}^1 \setminus [r + 2]$$

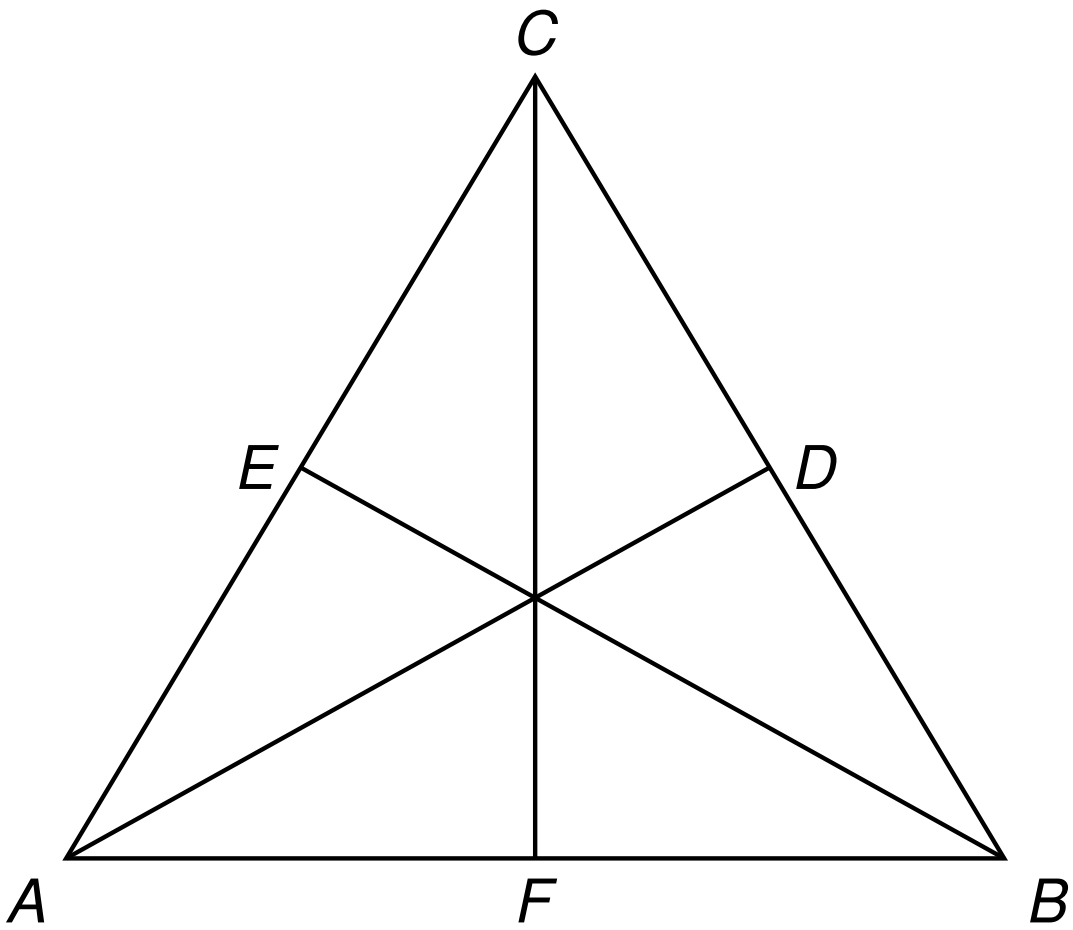
$[x, y, z] \to [x, y] \Rightarrow \mathbb{P}^2 \setminus [n \text{ lines through origin}] \to \mathbb{P}^1 \setminus [n]$

$$\pi_1(\mathbb{P}^2 \setminus 6 \text{ lines}) = F_5$$



Ceva arrangement

5 components of characteristic variety of π_1 of the complement.



Free quotients of the fundamental groups of the complements.

Problem: What are the ranks of free quotients of the fundamental groups of complements to a curve such that all irreducible components have degree at most d?

Example

What are possible ranks of free quotients of the fundamental groups of the complements to arrangements of lines (i.e. $d = 1$)?

The same problem on arbitrary surface.

Pencils and free quotients

$\pi_1(\mathbb{P}^2 \setminus C) \to F_r, r \geq 2$ implies that the characteristic variety of $\pi_1(\mathbb{P}^2 \setminus C)$ has dimension at least 2. Above theorem implies there is there is holomorphic map

$$\mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus [r]$$

which by removable singularity theorem can be extended to regular map

$$\Phi : \mathbb{P}^2 \setminus \textit{finite set} \to \mathbb{P}^1$$

If has the form: $(x, y, z) \to [f, g]$ (i.e. is a pencil of curves).

It follows that fibers of Φ are unions of components of C.

A component C_1 of $C = \bigcup_1^N C_i$ is **redundant** if $\pi_1(\mathbb{P}^2 \setminus C) \to F_r$ factors as

$$\pi_1(\mathbb{P}^2 \setminus \bigcup_{i=2}^{N} C_i) \to \pi_1(\mathbb{P}^2 \setminus C) \to F_r$$

Clearly the question is only about curves **without redundant components**, the condition which we assume now.

Trichtomy: infinite, finite, part of uniform construction.

Theorem

(J.I.Cogolludo-Libgober) Let $\mathcal{C}_d$ be the class of curves with irreducible components having degree at most d, $2 \leq r \in \mathbb{N}$.
There exist constants $A(d), B(d)$ such that if $C \in \mathcal{C}_d$ is such that $\pi_1(\mathbb{P}^2 \setminus C)$ admits a surjection onto F_r then:

a) if $r > A(d)$ then the set of non-redundant components are the elements of a pencil of the curves of the same degree.

b) If $B(d) < r \leq A(d)$ then there are finitely many curves without redundant components $C \in \mathcal{C}_d$ and admitting a free quotient of rank r.

c) If $r \leq B(d)$ then there are infinitely many curves without redundant components $C \in \mathcal{C}_d$ and admitting a free quotient of rank r.

Example

Class $\mathcal{C}_1$-Arrangements of lines (Yuzvinski-Libgober, Yuzvinski-Pereira). If $\mathcal{A}$ is and arrangement of lines and $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ admits surjection onto F_r then the union $\mathcal{A}'$ of non-redundant lines is

a) if $r \geq 4$ a collection of $r+1$ concurrent lines.

b) if $r = 3$ then $\mathcal{A}'$ is the Hesse arrangement of 12 lines and conjecturally no other possibilities.

c) If $r = 2$ then infinitely many $\mathcal{A}$ e.g. for any N there is pencil of curves of degree N with 3 completely reducible fibers:

$$a(x^N - y^N) + b(y^N - z^N) = 0$$

(at $(a, b) = (1, 0), (0, 1), (1, 1)$) inducing map

$$\mathbb{P}^2 \setminus \mathcal{A} \to \mathbb{P}^1 \setminus 3pts$$

Example

Class $\mathcal{C}_2$: arrangements of lines and quadrics For $d = 2$, $B(2) = 5$. Extremal example:

$$\lambda_0 x_0(x_1^2 - x_2^2) + \lambda_1 x_1(x_2^2 - x_0^2) + \lambda_2 x_2(x_0^2 - x_1^2) = 0$$

is the pencil of cubics and gives reducible curve $\mathcal{D}$ which is the union of 6 lines and 6 quadrics.

One does have surjection:

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{D}) \to F_5$$

and if $\pi_1(\mathbb{P}^2 \setminus \mathcal{D}) \to F_r, r > 5$ then $\mathcal{D}$ is union of quadrics belonging to the same pencil.

Comments on distribution.

There are infinitely many curves in each class $\mathcal{C}_d$ but for each sufficiently large r ($r > B(d)$) there are only finitely many classes of equisingular isotopy of curves $C \in \mathcal{C}_d$ such that $\pi_1(\mathbb{P}^2 \setminus C)$ admits a surjection onto F_r.

The number of isotopy classes of curves in C_d with $\pi_1(\mathbb{P}^2 \setminus C)$ admitting surjection onto $F_r, r \geq 10$ is:

$$\begin{array}{ll} d = 1 & 1 \\ d = 2 & 4 \\ d = 3 & \leq 341 \end{array}$$

The number of curves with surjection onto F_r grows at least doubly exponentially in d (as the number of equisingular strata of curves of degree d)

Arbitrary surfaces

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Let X be a smooth simply connected surface. C is an **AMPLE** curve. Then $X \setminus C$ has the homotopy type of a 2-complex, one can write presentation in terms of braid group (of a surface possibly having a positive genus).

Fix an ample class $[C]$ of numerical equivalence in $H^2(X, \mathbb{Z})$ and consider "enhanced" collection of classes $[\mathcal{C}]$ (corresponding to the curves "having degree less than $[C]$). Then for r sufficiently large, there are only finitely many curves for which $\pi_1(X \setminus C)$ admits surjection onto F_r. Moreover, for sufficiently large r, these curves belong to a pencil of curves in the class $[C]$.



Happy Birthday, Lou!