Cohomology of Milnor fiber of arrangements and algebraic equations over function fields

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Terao Conference PIMS
August 11 2011

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References:

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Milnor fiber of an arrangement: If $\mathscr{A} = \bigcup\{\mathbf{x} \in \mathbf{P}^n | l_i(\mathbf{x}) = 0 \; i = 1, ...d\}$ *and*

$$F(\mathbf{x}) = \Pi_{i=1}^{i=d} l_i(\mathbf{x})$$

then

$$M_{\mathscr{A}} = \{X \in \mathbf{C}^{n+1} | F(X) = 1\}$$

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$M_{\mathscr{A}}$ *is cyclic cover of* $\mathbf{P}^n - \mathscr{A}$ *corresponding to the kernel* $\pi_1(\mathbf{P}^n - \mathscr{A}) \to \mathbf{Z}/d\mathbf{Z}$ *given by the linking number with* $\mathscr{A}$ *i.e. its cohomology is a topological invariant.*

Basic properties of $H_1(M_{\mathscr{A}}, \mathbf{C})$.
It depend only on $\pi_1(\mathbf{P}^n - \mathscr{A})$ *and its surjection on* $\mathbf{Z}/d\mathbf{Z}$
(by Hurewicz $H_1(M_{\mathscr{A}}, \mathbf{C}) = K/K'$ *where* $K = ker\pi_1(\mathbf{P}^n - \mathscr{A}) \to \mathbf{Z}/d\mathbf{Z}$.
If H is a generic plain than:

$$\pi_1(\mathbf{P}^n - \mathscr{A}) = \pi_1(H - H \cap \mathscr{A})$$

Hence it is enough to consider the case of line arrangements.
On the other hand methods of this talk apply to union of algebraic curves of any degree (arrangements of curves of arbitrary degree).

Monodromy of Milnor fiber

Milnor fiber $M_{\mathscr{A}} = \{(x, y, z) \in \mathbf{C}^3 \Pi l_i(x, y, z) = 1\}$ *has the action of the group* μ_N *given by*

$$(x, y, z) \to (\omega_N x, \omega_N y, \omega_N z)$$

This is action of the monodromy. Rather than to be interested in $H_1(M_{\mathscr{A}}, \mathbf{C})$ *as a vector space we would like to have the characteristic polynomial of the monodromy of a generator of the group* μ_N

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For an arrangement in $\mathbf{C}^2$ *with* N *lines the multiplicity of the root* $t = 1$ *is equal to* $N - 1$*:*

$$\Delta_{\mathscr{A}}(t) = (t-1)^{N-1} \bar{\Delta}_{\mathscr{A}}(t) \quad \bar{\Delta}_{\mathscr{A}}(1) \neq 0$$

The problem hence is to understand the structure of $\bar{\Delta}(t)$*.*

For each singular point one has **local** *Alexander polynomial:* $\Delta(C, P)(t)$*: the Alexander polynomial of the link of singularity at each multiple point also equal to the characteristic polynomial of the monodromy.*

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Example

For ordinary triple point (intersection of three lines):

$$\Delta(t) = (t-1)^2(t^2+t+1)$$

For r-fold point:

$$\Delta(t) = (t-1)^{r-1}(t^{r-1}+....+1)^{r-2}$$

Divisibility theorem (A.Libgober, 1982) provides restrictions on the roots of $\bar{\Delta}(t)$ (i.e the eigenvalues of the monodormy) in terms of multiplicities of the multiple points of the arrangement.

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Theorem

*(***Dependence of local type: Divisibility***)*
$\Delta_C(t)$ divides product of local Alexander polynomials of all singularities. For irreducible curve $\Delta_C(1) \neq 0$.

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Theorem

(Dependence on behavior at infinity: Divisibility) *Let S^3 be a sphere of a large radius in $\mathbb{C}^2$. Then*

$$\Delta_C(t)$$

divides $\Delta(S^3 - S^3 \cap C)$.

Example

For an arrangement of lines with triple and double points only:
$\Delta = (t-1)^{r-1}(t^2+t+1)^s$.
Main Problems 1. What are possible values of s? (realization problem for the arrangements with double and triple points only)
2. Is s a combinatorial invariant?

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The multiplicity s of t^2+t+1 satisfies: $s \leq N-2$ since $\bar{\Delta}(t)|(t^{N-1}+\ldots+1)^{N-2}$

Dependence of homology of Milnor fiber on geometry of multiple points. *For an arrangements of r lines with double and triple points only*

$$\Delta_C(t) = (t-1)^{N-1} \text{ if } N \not\equiv 0(3) \text{ and otherwise}$$

$$\Delta_C(t) = (t-1)^{N-1}(t^2+t+1)^{\bar{s}}$$

where $\bar{s}$ is difference between actual and expected dimensions of curves of degree $N-3-\frac{N}{3}$ passing through the triple points. Equivalently

$$\bar{s} = dim H^1(\mathbf{P}^2, \mathcal{I}(N-3-\frac{N}{3}))$$

.

Example

Ceva Arrangement: 6 lines containing the sides and medians on a triangle. number of double points is 2, number of triple points is 4: complete intersection of two quadrics.
$\bar{s}$ depends on linear system of curves of degree $6-3-\frac{6}{3}=1$ passing through triple points.
Cayley-Bacharach: the above difference for $S \subset \mathbf{P}^2$ which is a complete intersection of curves of degree m, n with respect to the curves of degree $m+n-3$ is equal to 1.
Hence set of triple points is complete intersection of two quadrics and $2+2-3-1$ we get $s=1$.

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Example

Consider arrangement consisting of 9 lines dual to 9 inflection points. It has 12 triple points. The characteristic polynomial of the monodromy of Milnor fiber is $(t-1)^7(t^2+t+1)^2$ (the largest value of s known).

Dependence of global information for arbitrary singularities: *(Libgober -1983)*

$$\Delta_C(t) = \Pi_\alpha[(t - exp(2\pi i\alpha)(t - exp(-2\pi i\alpha)]^{s(\alpha,C)}$$

where α runs through all roots of local Alexander polynomials and $s(\alpha, C)$ is the difference between the actual and expected dimensions of the linear system of curves passing through the singularities of C and defined using α and the data of singularities of C.

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Linear systems are specified by ideal in the local rings of singular points (ideals of quasi-adjunction -close "relatives" of multiplier ideals). For singularities of arrangements (ordinary multiple points of multiplicity m) local condition is to belong to $\mathcal{M}^{m-2}$. Condition for curves with multiple points are more involved. If only double and quadruple points then one looks at curves of degree $N - 3 - \frac{3N}{4}$ with local equations in the square of the maximal ideal.

Theorem

(Cogolludo-L)

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Proof

Depends on use of Hodge theory and relation to Mordell-Weil groups.

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Let $G = \pi_1(\mathbf{C}^2 - \mathscr{A})$. *Then*

$$0 \to G'/G'' \to G/G'' \to G/G' \to 0$$

$G'/G'' \otimes \mathbf{C}$ *is a* $\mathbf{C}[G/G']$*-module. Consider*

$$Spec\mathbf{C}[G/G'] = Hom(G, \mathbf{C}^*)$$

It has as connected component of identity the torus $\mathbf{C}^{*\,rkG/G'}$. $\mathbf{C}[G/G']$*-module defines algebraic subvariety of* $Spec\mathbf{C}[G/G']$ *called* **support of the module** *(can be described as the zero set of minors of a presentation by generators and relations).*

Definition

Characteristic variety V_1 of an arrangement $\mathscr{A} \subset \mathbf{C}^2$ is support of $\mathbf{C}[G/G']$-module $A = G/G' \otimes \mathbf{C}$.
k-th characteristic variety is support of $\Lambda^k A$.
Using work of D.Arapura it was deduced that V_k is a union of translated subgroups.

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Theorem

(Libgober, 92) Let ξ be an eigenvalue of the monodromy of Milnor fiber of $\mathscr{A}$ and s_ξ be its multiplicity. Denote by χ_ξ the character $\pi_1(\mathbf{C}^2 - \mathscr{A}) \to \mathbf{C}^*$ sending meridian about each line of $\mathscr{A}$ to ξ. Then

$$s_\xi = max_k | \chi_\xi \in V_k$$

In particular the characteristic varieties determine the cohomology of Milnor fiber.

Theorem

(Libgober-Yuzvinski) Components of characteristic variety which are sub-tori of Char$\pi_1(\mathbf{C}^2 - \mathscr{A})$ containing identity are combinatorially defined.

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They correspond to components of resonance variety. **Resonance varieties of arrangements of hyperplanes (Falk)**.
Let H be an arrangement in $\mathbb{C}^{n+1}$. The cohomology algebra:

$$H^*(\mathbb{C}^{n+1} - H, \mathbb{C})$$

is generated by $H^1(\mathbb{C}^{n+1} - H, \mathbb{C})$. (Orlik-Solomon algebra)

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Aomoto complex $\mathcal{A}(a), a \in H^1$

$$.... \to H^i \to H^{i+1} \to ....$$

where the differential is the cup product with a.

Resonance variety:

$$\mathcal{R}_k(H) = \{a \in H^1(\mathbb{C}^{n+1} - H, \mathbb{C}) | dim H^1(\mathcal{A}(a)) \geq l\}$$

The map

$$exp : H^1(\mathbf{C}^2 - \mathscr{A}, \mathbf{C}) \to H^1(\mathbf{C}^2 - \mathscr{A}, \mathbf{C}^*) = Char\pi_1$$

provides one to one correspondence between the components of resonance variety and component of characteristic variety containing the identity.
The multiplicity of the eigenvalue ξ of the monodromy of Milnor fiber is greater or equal than k where k is such that χ belongs to the component V_k containing the identity.

Strategy

Let $\mathscr{A}, \mathscr{A}'$ be combinatorially equivalent arrangements. If ξ is an eigenvalue of monodromy of $M(\mathscr{A})$ and **if χ_ξ belongs to a component of characteristic variety of positive dimension containing the identity** *then ξ is also the eigenvalue of monodromy of $\mathscr{A}'$.*

$$s_{\mathscr{A}}(\xi) > 0 \Rightarrow s_{\mathscr{A}'}(\xi) > 0$$

For this implication one need different interpretation of multiplicity of eigenvalues. Diophantine equations over function fields provide this.

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For this implication one need different interpretation of multiplicity of eigenvalues. Diophantine equations over function fields provide this.

More precisely, one associate with an arrangement with triple points elliptic curve over $\mathbf{C}(x, y)$. The rank of the group of rational points on it is related to the eigenvalues of the monodromy on the Milnor fiber of arrangement. So we shall discuss rational points on elliptic curves.

Typical fields: $\mathbb{Q}, \mathbb{F}_q, \mathbb{F}_q(t), \mathbb{C}, \mathbb{C}(X)$ *where* X *is an algebraic variety.*

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If $\mathrm{char}K \neq 2, 3$*, can think that* C *is the set of solutions in* K *to*

$$y^2 = x^3 + ax + b \quad a, b \in K$$

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(Weierstrass form)

K*-points of elliptic curve form a group.*
Over $\mathbb{Q}$*: rational solutions to Diophantine equation.*
Over a function field $\mathbb{C}(X)$*: a curve is* $Y \to X$ *with generic fiber an elliptic curve over* $\mathbb{C}$ *(elliptic surfaces, threefolds etc.) and points are maps* $X \to Y$*.*

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Theorem

(Lang-Neron) Let k be an algebraically closed. Let K be field of functions on X. Let A_K be an abelian variety over K. Then there is an abelian variety B over k (Chow trace) and map $\tau : B \to A$ over K such that $A_K/\tau B_k$ is finitely generated.

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Remarks

For $E_{\mathbb{P}^1} = E_{\mathbb{C}} \times \mathbb{P}^1$ Chow trace is $E_{\mathbb{C}}$.

$C \subset \mathbb{C}^2$ with equation $f(x, y) = 0$ consider

$$u^3 + v^3 = f(x, y) \subset \mathbb{C}^4$$

This elliptic threefold is isotrivial elliptic curve over $\mathbb{C}(x, y)$. (replacing $f(x, y)$ by $g(t)$ gives isotrivial elliptic surface).

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Change of coordinates:

$$u \to uf^{\frac{1}{3}}, v \to vf^{\frac{1}{3}}$$

shows that this curve is isomorphic to $u^3 + v^3 = 1$ over $\mathbb{C}(x,y)(f^{\frac{1}{3}})$ which is the function field of the surface:

$$z^3 = f(x,y)$$

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Over $\mathbb{C}(t)$ the largest known rank is 68 (Shioda).

Theorem

(Cogolludo-L) Let $f(x,y) = \Pi l_i = 0$ *be the product of the equations of a line arrangement with double and triple points only. Let* W_f *be the elliptic threefold* $u^3 + v^3 = f(x,y)$*. Then the rank* $MW(W_f) = 2\bar{s}$ *where* $\bar{s}$ *is the multiplicity of the factor* $(t^2 + t + 1)$ *in the characteristic polynomial of the monodromy of the Milnor fiber of the arrangement. Both do not exceed* $\frac{N}{2} - 2$*.*

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It is surprising that $rkMW(W_f)$ is topological i.e. invariant in equisingular deformations of f.
In general MW_f is closely related to Neron-Severi group of a smooth model of W_f (Shioda-Tate for surfaces, Wasir and others in higher dimensions) and configuration of algebraic divisors changes in deformations.

Main reasons:
Isotriviality of W_f.
Decomposibility of Albanese variety of cyclic multiple planes.

V algebraic variety $\Rightarrow$ abelian variety $Alb(V)$ and map $V \to Alb(V)$ such that for any $V \to A$ corresponds factorization $V \to Alb(V) \to A$.

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Explicit construction:

$$Alb(V) = H^0(\Omega^1_V)^* / H_1(V, \mathbf{Z})$$

Albanese map:

$$P \to (\omega \to \int_{P_0}^{P} \omega)$$

Theorem

(Cogolludo-L, 2010) Let V_f be a cyclic multiple plane $z^n = f(x, y)$ branched over a curve with nodes and cusps only then there is isogeny:

$$Alb(V_f) = E_0 \times ... \times E_0$$

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Idea of the proof: to a singularity of plane curve corresponds local Albanese variety. For cusp the local Albanese is E_0.
Theorem: $Alb(V_f)$ is quotient of product of local Albanese varieties
Poincare complete reducibility theorem yields the claim.

In general the local Albanese is isogeneous to the product of Jacobians of cyclic covers of $\mathbf{P}^1$*. It may depends on moduli i.e. not just of the topological type of singularity. For unibranched singularity it is abelian variety of CM type depending explicitly on the Puiseux pairs of the branch i.e. does depend only on the topological type.*

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It is interesting to compare such complete decomposability with the problem posed by T.Ekedahl and J.P.Serre (1993)
Do exist curves of arbitrary large genus for which the Jacobian is a product of elliptic curves?

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More closely related question is:
How many component E_0 *the Jacobian of a cyclic cover of* $\mathbf{P}^1$ *can have (in particular 3-fold cyclic cover).*

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Remarks

E_0^g is not a Jacobian if $g > 1$.

Theorem

(Cogolludo-L.) Let $f(x, y)$ be equation of a curve C_f with only **simple** *singularities or single type (or nodes). Then*

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This follows from:
1. $\bar{s}$ is equal to the dimension of Albanese of V_f and hence the number of factors E_0.
2. The pullback of W_f onto V_f via $V_f \to \mathbf{P}^2$ trivializes W_f i.e. is birational to $V_f \times E_0$.
3.Elements of $MW(W_f)$ correspond to the equivariant maps $V_f \to E_0$ (elliptic pencils similar to rational pencils)
4.$Maps(V_f, E_0) = Hom(Alb(V_f), E_0) = Hom(E_0^{\bar{s}}, E_0) = \mathbb{Z}[\omega_6]^{\bar{s}}$

Remarks

Equivariant elliptic pencils for the group $\mathbf{Z}_3$ can be thought of as rational pencils with ramification locus of V_f, i.e. the arrangement, belonging to fibers of multiplicity 1 or 3.

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Equivariant elliptic pencils for the group $\mathbf{Z}_3$ can be thought of as rational pencils with ramification locus of V_f, i.e. the arrangement, belonging to fibers of multiplicity 1 or 3.

Corollary

The elliptic pencils correspond to the orbifold pencils: birational maps $\mathbb{P}^2 \to \mathbb{P}^1$ having multiple fibers of multiplicities $(3,3,3)$. This is equivalent to the following: the equation of f can be split so that

$$A_1 f_1^3 + A_2 f_2^3 + A_3 f_3^3 = 0$$

for $f_i \in \mathbb{C}(x,y)$ We view this equation over $\mathbb{C}(x,y)$ with unknowns f_i. (For fixed A_i and arbitrary integer exponents of f_i this is called Catalan equation).

Case $A_1 f^p + A_2 g^q + A_3 h^r = 0$ when genus is greater than one considered by J.Silverman (at most finitely many solutions). In the case $A_i \in \mathbb{C}$ (Fermat theorem) one has no solutions. It has solutions only if arrangement $A_1 A_2 A_3 = 0$ composed of a pencil. In the latter case the number of solutions is infinite (i.e. the rank of Mordell Weil group of this elliptic curve is positive).

Corollary

Solutions of the set of Catalan equations with fixed f i.e. fixed arrangement form a group i.e. the Mordell Weil group of associated threefold. The problem of combinatorial invariance of the cohomology of Milnor fiber is the problem of combinatorial invariance of the rank of this group.

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Main step is enumeration of solutions of Catalan equation:

$$A_1 f^3 + A_2 g^3 + A_3 h^3 = 0$$

where A_i are products of linear forms and $A_1 A_2 A_3 = 0$ is an arrangement with double and triple points only.

Theorem

Let $A_i = \Pi_{j \leq r_i} L_{i,j}$, $i = 1, 2, 3$, be products of linear forms $L_{i,j} \in \mathbf{C}[x_1, x_2, x_3]$ (r_i are non negative integers). Assume that $L_{i,j}$ is not a multiple of $L_{i',j'}$ for any pair two pairs $(i, j), (i', j')$ and that lines $L_{i,j} = 0$ form an arrangement of lines with points of multiplicity 2 and 3 only. 1) The equation

$$A_1 f^3 + A_2 g^3 + A_3 h^3 = 0 \tag{1}$$

has solution only if $A_1 A_2 A_3 = A'_1 A'_2 A'_3$ where A'_1, A'_2, A'_3 are linear dependent over $\mathbf{C}$. In this case the solutions of (1) with linearly dependent A_i, are the pullbacks of solutions of Catalan equation over $\mathbf{C}(t)$.
2) In the case A_i are linearly dependent over $\mathbf{C}$ the set of solutions to (1) is infinite.

if $\mathscr{A}$ is combinatorially equivalent to $\mathscr{A}'$ and if $\bar{s}(\mathscr{A}) > 0$ then the rank of corresponding Mordell Weil group is positive $\Rightarrow$ a Catalan equation for $\mathscr{A}$ has solutions $\Rightarrow$ $\mathscr{A}$ is composed of pencil $\Rightarrow$ exist component of characteristic variety of $\mathscr{A}$ containing identity, $\Rightarrow$ exist component of characteristic variety of $\mathscr{A}'$ containing identity $\Rightarrow$ $\bar{s}(\mathscr{A}') > 0$.

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Conjecture

One can relate the number of the pencils if which an arrangement is composed to the number of points of minimal height in the Mordell Weil group. The former is combinatorially invariant and hence this would lead to the combinatorial invariance of $\bar{s} = rkMW$.

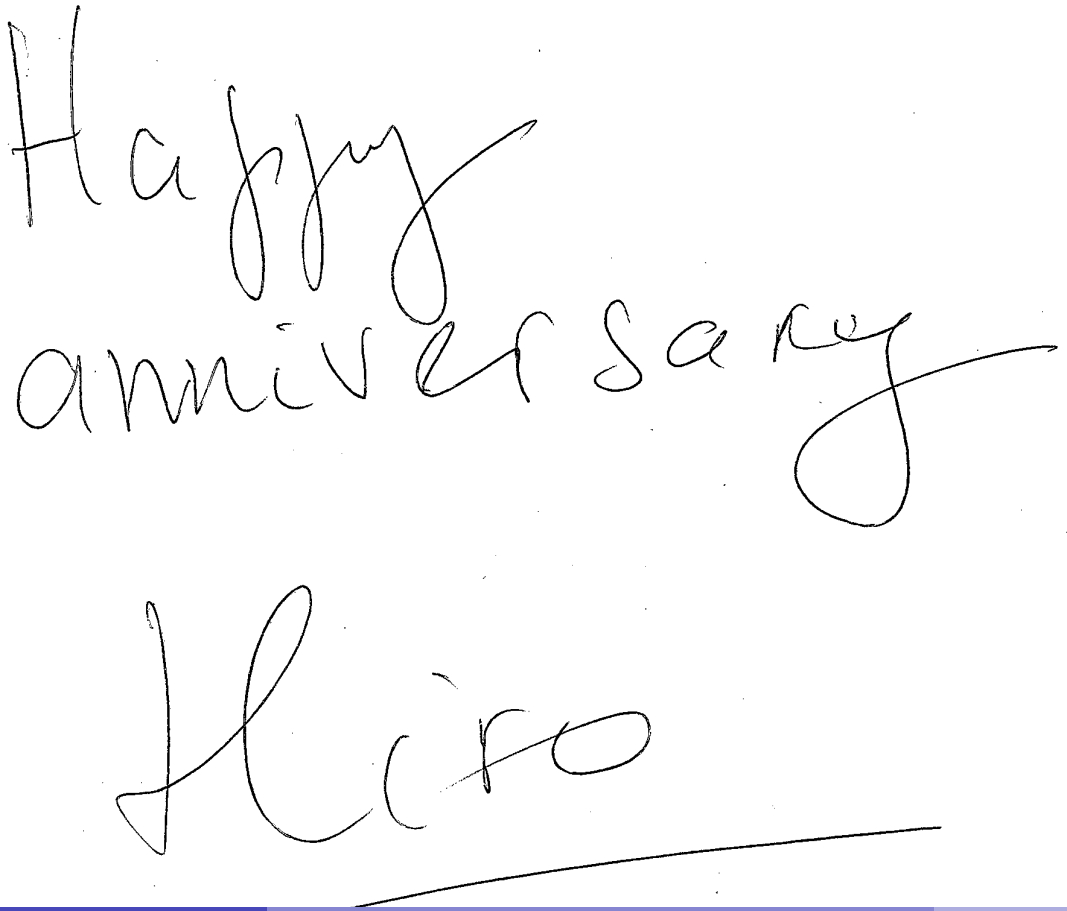
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Remarks

For arrangement of 9 lines dual to nine inflection points of a smooth cubic one has four pencils.


Happy
anniversary
Hiro