A Beckman-Quarles Type Theorem for Laguerre Transformations in the Dual Plane
Calvin College, Mathematics Department

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- This is a converse showing that the rigid motions are the only transformations which are distance preserving.

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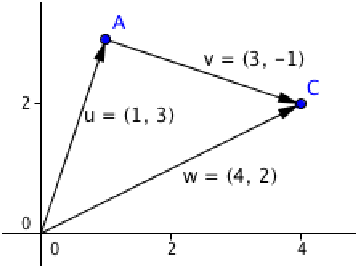

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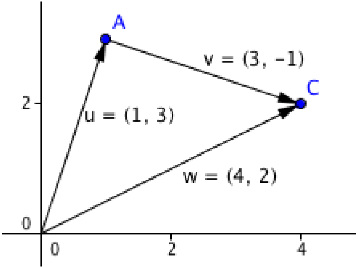

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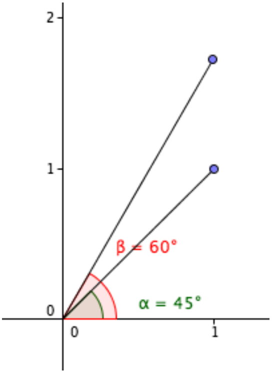

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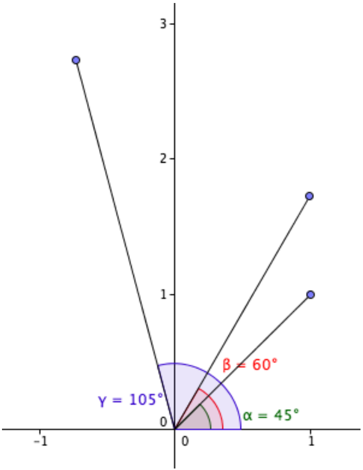

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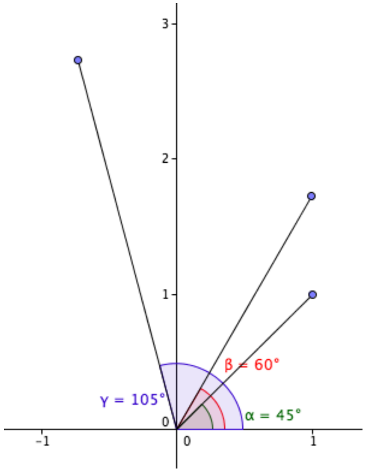

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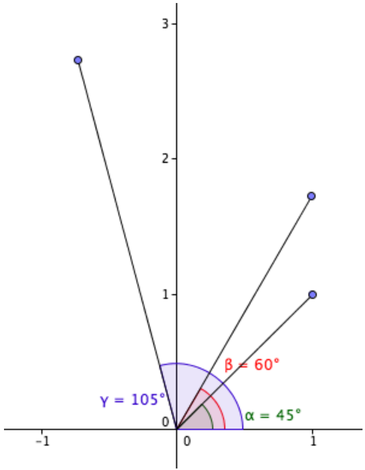

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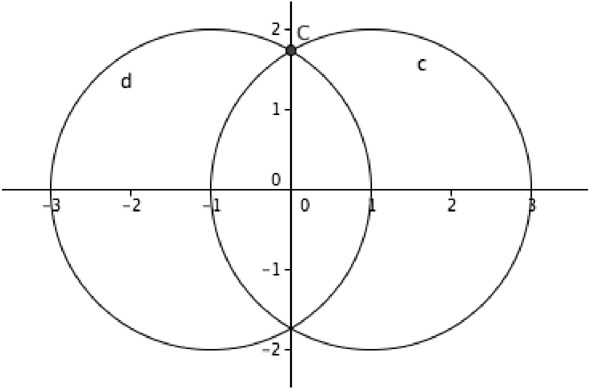

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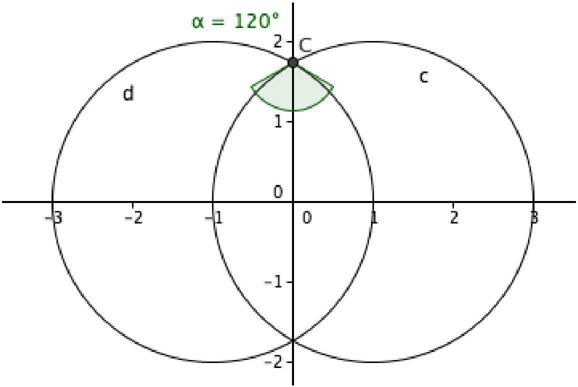

A Theorem by June Lester

Take $\mathcal{C}$ to be the space of all circles and lines and δ to represent the angle of intersection at intersection points between pairs of circles.

For a a fixed real $\rho > 0$, let $X \rightarrow \bar{X}$ be a bijective mapping from $\mathcal{C}$ to itself such that, for all A, B in $\mathcal{C}$,

$$\delta_{AB} = \rho \text{ if and only if } \delta_{\bar{A}\bar{B}} = \rho$$

Then the mapping is induced on $\mathcal{C}$ by a Möbius transformation of $\hat{\mathbb{C}}$

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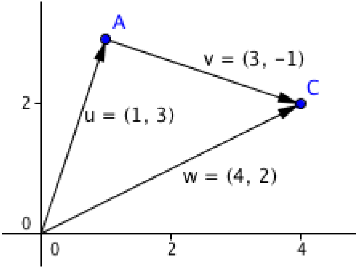

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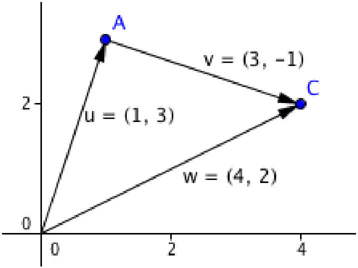

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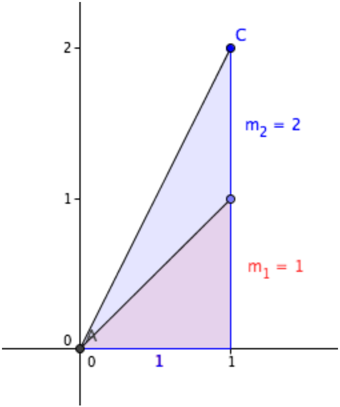

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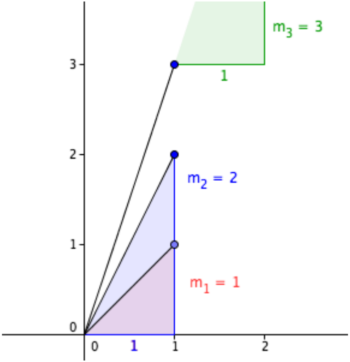

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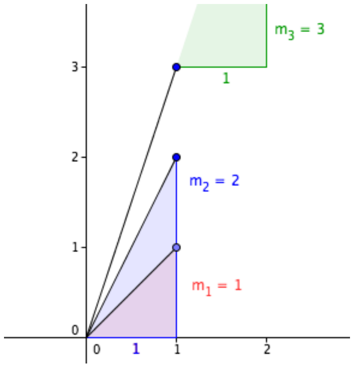

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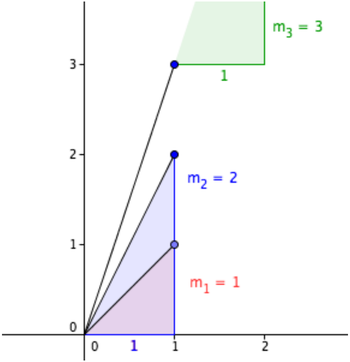

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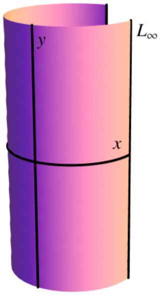

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Our Theorem

Take $\mathscr{P}$ to be the space of all vertical parabolas and non-vertical lines and δ to represent the difference of slopes at the intersection points of pairs of parabols.

Let T be a bijective mapping from $\mathscr{P}$ to itself such that, for all A, B in $\mathscr{P}$,

$$\delta(A, B) = 1 \text{ if and only if } \delta(\mathrm{T}(A), \mathrm{T}(B)) = 1$$

Then T induces a Laguerre transformation of the dual plane $\mathbb{D}$.

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- tangent: $y = 1$ and $y = -1$. Equivalently, $y = 0$ and $y = k$ for $k \neq 0$.
- intersecting: $y = kx$ and either $y = -kx$ or $y = 0$, for $k \neq 0$.

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- The transformation $\mathrm{T}(\mathscr{T}_{p,m})$ is another set $\mathscr{T}_{p',m'}$ and not just a subset.

The Induced Map $\hat{\mathrm{T}}$

Take a point $p \in \hat{\mathbb{D}}$, then construct $\mathscr{T}_{p,m}$ for $m \in \mathbb{R}$. $\mathrm{T}(\mathscr{T}_{p,m}) = \mathscr{T}_{p',m'}$ for some $p' \in \hat{\mathbb{D}}$ and $m' \in \mathbb{R}$. We define $\hat{\mathrm{T}}(p) = p'$.

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- From these properties, we can conclude that the intersection points of arbitrary figures are preserved.

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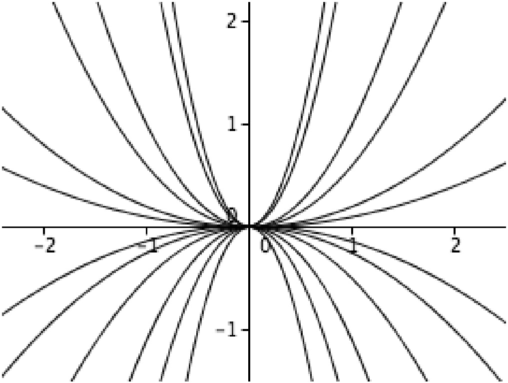

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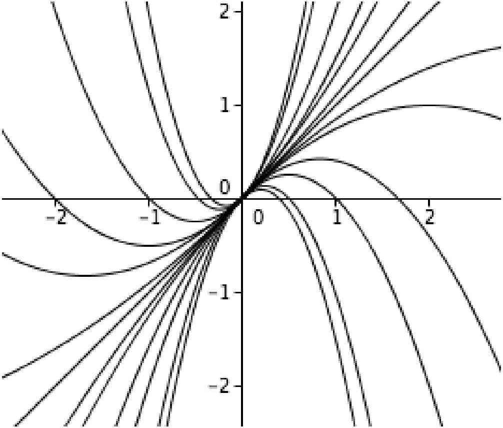
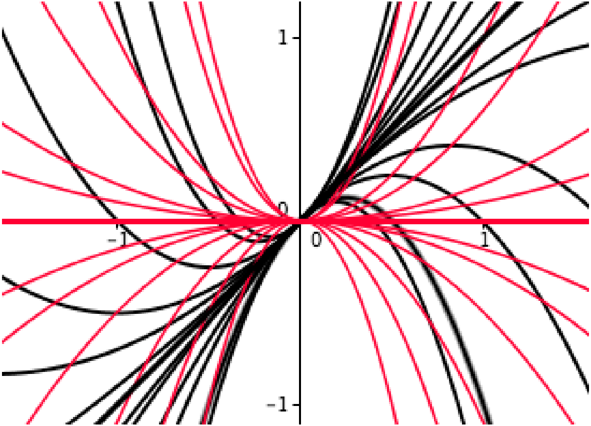


$\hat{T}$ is Well-Defined



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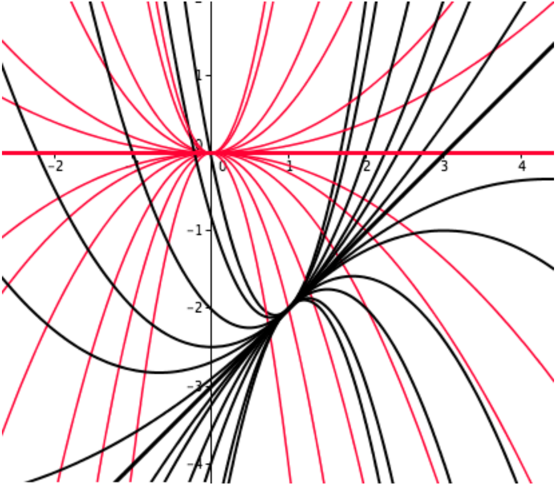

Building μ...

If $\hat{\mathrm{T}}(0) = w_1, \hat{\mathrm{T}}(1) = w_2, \hat{\mathrm{T}}(\frac{1}{0j}) = w_3$, we define

$$\mu(w) = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1}.$$

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- We show that for all $z \in \mathbb{D}$, $(\mu \circ \hat{\mathrm{T}})(z) = z$ or $(\mu \circ \hat{\mathrm{T}})(z) = \bar{z}$.

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- This implies that $\hat{\mathrm{T}}$ is either a direct or indirect Laguerre transformation due to the group structure of Laguerre transformations.