MCS 548 – Mathematical Theory of Artificial Intelligence

Fall 2025

Problem Set 3

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Due: 12/1/25 at the beginning of class

Instructions: You may consult outside references, but cite all the resources used (e.g. which people or resources on the internet you consulted). You must, however, write up the answers on your own. All problems in this assignment require proof.

Problems

1. [10 pts.] Let $X = \{0,1\}^n$, $c \in \{0,1\}^n$, and U be the uniform distribution over X. Define parity functions $\chi_c : X \to \{-1, 1\}$ as

$$\chi_c(x) = (-1)^{c \cdot x}.$$

Prove the following important fact used in the computation of SQ-DIM:

$$\forall c \neq c', \ \mathrm{E}_{x \sim U}[\chi_c(x)\chi_{c'}(x)] = 0.$$

2. [10 pts.] Show that conjunctions are efficiently learnable with statistical queries.

3. [10 pts.] Consider modifying the statistical query model by letting the learner request a polynomial number of unlabeled examples from the target distribution, in addition to having access to the usual statistical query oracle. We would also then include a failure parameter δ in the learning criterion. Call this new model "SQUE" (for SQ with Unlabeled Examples). For efficient learnability, we know that

$$\text{SQ} \subseteq \eta\text{-PAC} \subseteq \text{PAC}.$$

Where would SQUE fit in this hierarchy for efficient learning?

4. [10 pts.] Suppose we are given n unit vectors in R^n divided into two sets P, Q with the guarantee that there exists a hyperplane $a \cdot x = 0$ such that every point in P is on one side of it and every point in Q is on the other. Furthermore, assume that the ℓ_2 distance of each point to the hyperplane is at least γ (this is sometimes called the "margin"). Show that a random projection (as defined in the book) to some $c \log n/\gamma^2$ dimensions will have the property that with high probability, the two sets of points will still remain separated by a hyperplane with margin $\gamma/2$.

5. [10 pts.] Show that if A is a symmetric matrix with distinct singular values, then the left and right singular vectors are the same and $A = VDV^T$.