MCS 441 – Theory of Computation I
Spring 2013
Problem Set 10

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Due: 5/1/13 at the beginning of class

Related reading: Chapters 8.1-8.3
Instructions: Atop your problem set, write your name, clearly list your collaborators (see syllabus for the collaboration policy), and indicate whether you are an undergraduate or graduate student.

1. [5 pts] Let $A, B \in$ PSPACE. Prove that this implies that $\left(\overline{(A \cup B)}\right)^* \in$ PSPACE.
Hints: 1) Consider the operations separately. 2) For part of this question, it may help to use non-determinism (but it is not necessary).

2. [5 pts] Let B be a language such that TQBF $\leq_P B$. Prove that B is NP-Hard.

3. [9 pts] Say a Boolean formula is minimal if there is no shorter (i.e. using fewer symbols to write) Boolean formula equivalent to it. Let

$$\text{MBF} = \{\langle\phi\rangle \mid \phi \text{ is minimal}\}.$$

a. Prove MBF $\in$ PSPACE.

b. Consider the following argument that MBF $\in$ coNP:

> *If $\phi \notin$ MBF, then there exists a smaller equivalent formula ψ. Use ψ as the certificate (considering the verifier definition of* NP*) to* $\overline{\text{MBF}}$*, showing* $\overline{\text{MBF}} \in$ NP*.*

MBF, however, is actually not known to be in coNP. So, What was wrong with the argument above?

4. [7 pts] Consider the following generalized version of the game Tic-Tac-Toe: players 1 and 2 take turns placing Xs and Os, respectively, on empty spaces on an $n \times n$ board, until one of the players wins by placing k symbols contiguously in a row, column, or diagonal. (The classic variant of Tic-Tac-Toe has $k = n = 3$.) If no move can be made by the player whose turn it is, the game ends in a draw. The problem is to decide whether player 1 will win, if both players play optimally, given an arbitrary board configuration and player turn.

a. Formulate this problem as the language TIC-TAC-TOE. (i.e. TIC-TAC-TOE = ...)

b. Show TIC-TAC-TOE $\in$ PSPACE.

5. **[4 pts]** Prove that if $A \in$ PSPACE then $A \leq_P A_{TM}$.

Extra Credit [3 pts] (for undergraduates): Write-up your solutions using LaTeX. This is not easy and will require downloading, installing, and learning a new type-setting language. However, the future benefits of knowing LaTeX may make this exercise worthwhile. (This guide may be of help: `http://ftp.math.purdue.edu/mirrors/ctan.org/info/lshort/english/lshort.pdf`) You may consult online sources for this task.