MCS 441 – Theory of Computation I
Spring 2016
Problem Set 4

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Due: 3/16/2016 at the beginning of class

Instructions: Atop your answers, write your name and indicate whether you are an undergraduate or graduate student. Answer all questions in the order they are assigned.
Important note: Problems labeled "(**U**)" and "(**G**)" are assigned to undergraduate and graduate students, respectively. Undergraduate students can get a small bonus for solving the graduate problems. Graduate students are encouraged to solve the undergraduate problems for practice and are responsible for understanding the answers to those questions.

1. [7 pts] Let L_1 be the language $\{0^n1^n \mid n \geq 1\}$ over $\Sigma = \{0, 1\}$.

a. [5 pts.] Draw the state diagram for M_1, a deterministic Turing Machine that recognizes L_1. It will be assumed that all the transitions not depicted go to the reject state, which may or may not be drawn.
Note: knowing how to explicitly program TMs is not an important skill. However, doing it once is a worthwhile exercise.

b. [2 pts] Is M_1, as constructed in the previous part, a decider?
Note: you might have constructed a machine for which it is very hard to answer this question. In that case, consider changing your machine.

2. [6 pts] Read (at least) the Introduction, Sections 1 and 2, and Section 9 up to the end of 9.I of Turing's 1936 paper: `http://homepages.math.uic.edu/~lreyzin/s16_mcs441/Turing36.pdf`.

a. [3 pts] Give an example of a computation that Turing would have probably believed a human would be unable to do without paper. Justify your answer.

b. [3 pts] Is Turing more concerned with creating a machine that can simulate human computation or with creating a machine that humans can simulate? Justify your answer.

Note: answer the two questions above briefly, referring to and/or quoting the text for support. *Do not write long essays answering these questions – a well thought-out paragraph should be sufficient for each.*

3. [5 pts] Imagine a modified Turing Machine that has a tape that is infinitely long in *both* directions. The machine starts at the beginning of the input, and the tape has blank space going infinitely both before the beginning and after the end of the input. Are there any languages that this machine can recognize that a Turing Machine cannot? Why or why not?

4. [10 pts] Read the article "*Who Can Name the Bigger Number*?" by Scott Aaronson (now a professor at MIT): `http://homepages.math.uic.edu/~lreyzin/s16_mcs441/Aaronson99.pdf`. This article may include concepts we have not yet covered, but the introduction to them should be sufficiently self-contained.

a. [5 pts] Going back to 10,000 B.C., name some advances in mathematics over time that have effectively allowed people to express bigger numbers than before the advent of those advances. You may do outside research to answer this question.[1] Include at least one advance from Aaronson's article.

b. [5 pts] Would Aaronson say Turing machines are important even if they didn't have the immense real-world impact that they currently have? Why or why not?

5. [12 pts] The goal of this problem is to find an uncomputable (or "incalculable") function. This function, $S(n)$, will grow so fast that even Turing machines cannot keep up with it. Let

$$\varepsilon\text{-HALT}_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } M \text{ halts on the input } \varepsilon \text{ (i.e. no input)}\}.$$

You can assume the input alphabet to be $\Sigma = \{0, 1\}$, which is sufficient to encode M.

a. [2 pts] Is ε-HALT_{TM} recognizable? Why or why not?

b. [5 pts] Prove that ε-HALT_{TM} is not decidable.
Hint: you can use the fact that HALT_{TM} is not decidable and use that to show that ε-HALT_{TM} being decidable would lead to a contradiction.

c. [5 pts] Let $\mathcal{H}_n$ be the set of n-state TMs that eventually halt when run on the empty input. Let $S(n)$ be the maximum number steps a TM in $\mathcal{H}_n$ can take.

(**U**) Let

$$L_{5.c.U} = \{m \mid m \in \{0,1\}^* \text{ s.t. } m = S(n) \text{ for some } n \geq 1\}.^2$$

Prove that $L_{5.c.U}$ is not decidable.

(**G**) Let $S_{-1}(n)$ be the maximum number of steps that a TM in $\mathcal{H}_n$ can take *that is not equal to* $S(n)$. In essence, we are considering the second longest running machine that halts. Let

$$L_{5.c.G} = \{m \mid m \in \{0,1\}^* \text{ s.t. } m = S_{-1}(n) \text{ for some } n \geq 1\}.^2$$

Prove that $L_{5.c.G}$ is not decidable.

6. [5 pts] Consider the language L_6, defined as follows:

$$L_6 = \begin{cases} \{1\} & \text{if } L_{5.c.U} \text{ contains infinitely many primes} \\ \{0\} & \text{otherwise.} \end{cases}$$

Is L_6 decidable? Why or why not?

[1] A few examples will suffice; do not go overboard and write a research paper.
[2] Interpret m as a binary number.