Let

\[ f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]

This function is plotted below.

1. Do the partial derivatives \( f_x \) and \( f_y \) exist and are they continuous at all points \((x, y) \neq (0, 0)\)?

2. Is \( f \) differentiable at all points \((x, y) \neq (0, 0)\)?

3. Is \( f \) continuous at \((0, 0)\)?

4. Do the partial derivatives \( f_x \) and \( f_y \) exist and are they continuous at \((0, 0)\)?

5. Is \( f \) differentiable at \((0, 0)\)?

**Answer:**

1. We have by the quotient rule that

\[
f_x(x, y) = \frac{[y(x^2 - y^2) - xy(-2x)] (x^2 + y^2) - xy(x^2 + y^2) (2x)}{(x^2 + y^2)^2}
\]

which exists and is continuous for all \((x, y) \neq (0, 0)\). See the answers to the Mock Quiz for an explanation of why this is sufficient. We compute \( f_y \) similarly.

2. Yes, because by part (1) we have that \( f_x \) and \( f_y \) exist and are continuous.

3. Let \( \epsilon > 0 \) and define \( \delta = \sqrt{\epsilon} \). Suppose \( \sqrt{x^2 + y^2} < \delta \). Then

\[
|f(x, y) - f(0)| = \frac{|xy(x^2 - y^2)|}{x^2 + y^2} = \frac{|x||y||x^2 - y^2|}{x^2 + y^2} = \frac{|x||y|(x^2 - y^2)}{x^2 + y^2} = |x||y| < \delta^2 = \epsilon.
\]

This shows that \( \lim_{(x,y) \to (0,0)} f(x,y) = 0 \) so that \( f \) is continuous at 0.
4. Yes because

\[
\lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} \lim_{h \to 0} \frac{h \cdot 0 (h^2 - y^2)}{h^2 + 0^2} = 0
\]

and similarly for \( f_y \).