Quiz 3

1. Let \( f(x, y) = 3y - y^3 - 4x^2 + x^4 \). This function is plotted below.

Find and classify all critical points of \( f \) using the Second Derivative Test.

**Answer:**

We have

\[
\begin{align*}
    f_x(x, y) &= -8x + 4x^3 \\
    f_y(x, y) &= 3 - 3y^2.
\end{align*}
\]

Thus, we have \( f_x(x, y) = 0 \) iff \( x = 0 \) or \( x = \pm \sqrt{2} \). Similarly, we have \( f_y(x, y) = 0 \) iff \( y = \pm 1 \). We therefore have critical points at

\[
\left( -\sqrt{2}, -1 \right), \left( -\sqrt{2}, 1 \right), \left( 0, -1 \right), \left( 0, 1 \right), \left( \sqrt{2}, -1 \right), \left( \sqrt{2}, 1 \right)
\]

We perform the following computations

\[
\begin{align*}
    f_{xx}(x, y) &= -8 + 12x^2 \\
    f_{xy}(x, y) &= 0 \\
    f_{yy}(x, y) &= -6y \\
    D &= -6 \left( -8 + 12x^2 \right) y
\end{align*}
\]

so that we have the following values

<table>
<thead>
<tr>
<th>Point</th>
<th>( D )</th>
<th>( f_x )</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (-\sqrt{2}, -1) )</td>
<td>96</td>
<td>16</td>
<td>min</td>
</tr>
<tr>
<td>( (-\sqrt{2}, 1) )</td>
<td>-96</td>
<td>16</td>
<td>Saddle</td>
</tr>
<tr>
<td>( (0, -1) )</td>
<td>-48</td>
<td>-8</td>
<td>Saddle</td>
</tr>
<tr>
<td>( (0, 1) )</td>
<td>48</td>
<td>-8</td>
<td>Max</td>
</tr>
<tr>
<td>( (\sqrt{2}, -1) )</td>
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<td>Saddle</td>
</tr>
</tbody>
</table>
2. Find all extreme values of \( f(x, y) = xy \) subject to the constraint \( 4x^2 + y^2 = 8 \) using the Method of Lagrange Multipliers.

**Answer:** Let \( g(x, y) = 4x^2 + y^2 \) We have \( \nabla f(x, y) = \langle y, x \rangle \) and \( \nabla g(x, y) = \langle 8x, 2y \rangle \). Now if \( (x, y) \) is on \( g(x, y) = 8 \) and is an extreme value of \( f \), then

\[
\nabla f(x, y) = \lambda \nabla g(x, y)
\]

for some \( \lambda \in \mathbb{R} \). This amounts to solving the following system.

\[
\begin{align*}
y &= 8\lambda x \quad \text{(1)} \\
x &= 2\lambda y \quad \text{(2)} \\
4x^2 + y^2 &= 8 \quad \text{(3)}
\end{align*}
\]

From (1) and (2), we see that if \( \lambda = 0 \), then \( (x, y) = (0, 0) \), but this solution does not satisfy the third equation. Therefore, \( \lambda \neq 0 \).

Solving (1) and (2) for \( \lambda \), we have

\[
\frac{y}{8x} = \lambda = \frac{x}{2y}
\]

so that \( y^2 = 4x^2 \). This means that \( y = \pm 4x \). Substituting this into (3), we have \( 4x^2 + 4x^2 = 8 \) so that \( x = \pm 1 \) and \( y = \pm 2 \). Evaluating \( f \) at these points, we have

\[
\begin{array}{c|c}
(x, y) & f(x, y) \\
\hline
(-1, -2) & 2 \\
(1, -2) & -2 \\
(-1, 2) & -2 \\
(1, 2) & 2
\end{array}
\]

so that \( f \) takes its maximum value along \( g(x, y) = 8 \) at \((-1, -2)\) and \((1, 2)\) and its minimum values at \((1, -2)\) and \((-1, 2)\).