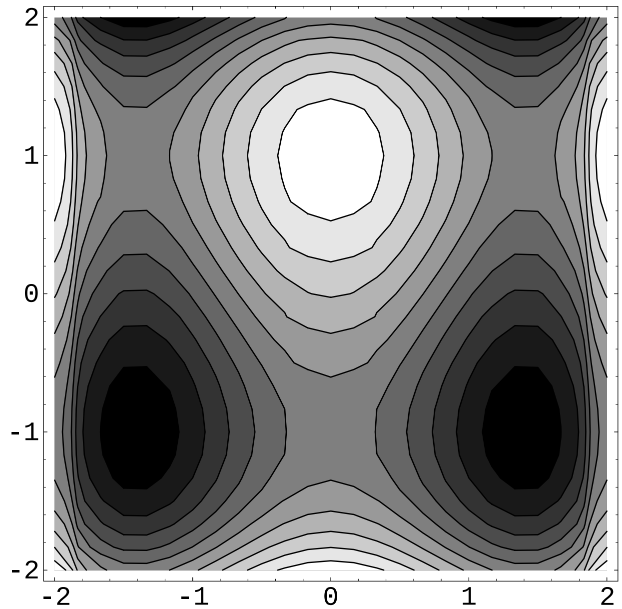
Quiz 3

1. Let $f(x, y) = 3y - y^3 - 4x^2 + x^4$. This function is plotted below.



Find and classify all critical points of f using the Second Derivative Test.

Answer:

We have

$$f_x(x, y) = -8x + 4x^3$$
$$f_y(x, y) = 3 - 3y^2.$$

Thus, we have $f_x(x, y) = 0$ iff $x = 0$ or $x = \pm\sqrt{2}$. Similarly, we have $f_y(x, y) = 0$ iff $y = \pm 1$. We therefore have critical points at

$$\left(-\sqrt{2}, -1\right), \left(-\sqrt{2}, 1\right), (0, -1), (0, 1), \left(\sqrt{2}, -1\right), \left(\sqrt{2}, 1\right)$$

We perform the following computations

$$f_{xx}(x, y) = -8 + 12x^2$$
$$f_{xy}(x, y) = 0$$
$$f_{yy}(x, y) = -6y$$
$$D = -6\left(-8 + 12x^2\right) y$$

so that we have the following values

Point	D	f_x	Type
$\left(-\sqrt{2}, -1\right)$	96	16	min
$\left(-\sqrt{2}, 1\right)$	-96	16	Saddle
$(0, -1)$	-48	-8	Saddle
$(0, 1)$	48	-8	Max
$\left(\sqrt{2}, -1\right)$	96	16	min
$\left(\sqrt{2}, 1\right)$	-96	16	Saddle

2. Find all extreme values of $f(x,y) = xy$ subject to the constraint $4x^2 + y^2 = 8$ using the Method of Lagrange Multipliers.

Answer: Let $g(x,y) = 4x^2 + y^2$ We have $\nabla f(x,y) = \langle y, x \rangle$ and $\nabla g(x,y) = \langle 8x, 2y \rangle$. Now if (x,y) is on $g(x,y) = 8$ and is an extreme value of f, then

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

for some $\lambda \in \mathbb{R}$. This ammounts to solving the following system.

$$y = 8\lambda x \tag{1}$$
$$x = 2\lambda y \tag{2}$$
$$4x^2 + y^2 = 8 \tag{3}$$

From (1) and (2), we see that if $\lambda = 0$, then $(x,y) = (0,0)$, but this solution does not satifsy the third equation. Therefore, $\lambda \neq 0$.

Solving (1) and (2) for λ, we have

$$\frac{y}{8x} = \lambda = \frac{x}{2y}$$

so that $y^2 = 4x^2$. This means that $y = \pm 4x$. Substituting this into (3), we have $4x^2 + 4x^2 = 8$ so that $x = \pm 1$ and $y = \pm 2$. Evaluating f at these points, we have

(x,y)	$f(x,y)$
$(-1,-2)$	2
$(1,-2)$	-2
$(-1,2)$	-2
$(1,2)$	2

so that f takes its maximum value along $g(x,y) = 8$ at $(-1,-2)$ and $(1,2)$ and its minimum values at $(1,-2)$ and $(-1,2)$.