Quiz 5

1. Let E be the region below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{3x^2 + 3y^2}$. Evaluate $\int_E z\ dV$ in *both* spherical *and* cylindrical coordinates.

   Answer: In spherical coordinates, the cone is given by $\rho = \frac{\pi}{6}$. Confirm this using trigonometry. Then

$$\int_E z\ dV = \int_0^{\frac{\pi}{6}} \int_0^{2\pi} \int_0^1 (\rho \cos \varphi)\, \rho^2 \sin \varphi\ d\rho\ d\theta\ d\varphi = \int_0^{\frac{\pi}{6}} \int_0^{2\pi} \frac{1}{4} \cos \varphi \sin \varphi\ d\theta\ d\varphi$$

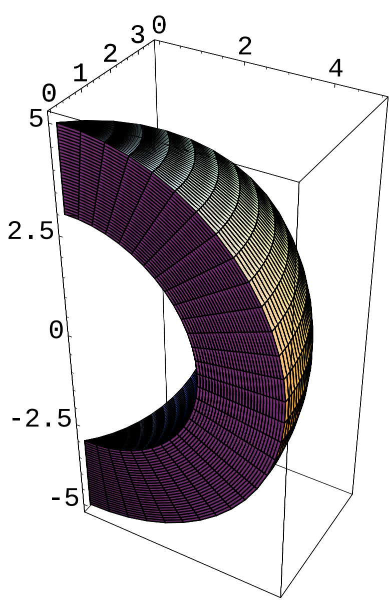
$$\int_0^{\frac{\pi}{6}} \frac{\pi}{2} \cos \varphi \sin \varphi\ d\varphi = \frac{\pi}{4} \sin^2 \varphi \Big|_0^{\frac{\pi}{6}} = \frac{\pi}{16}.$$

   In cylindrical coordinates, the two surfaces intersect in a circle of radius $\frac{1}{2}$. Confirm this using trigonometry. Then.

$$\int_E z\ dV = \int_0^{2\pi} \int_0^{\frac{1}{2}} \int_{\sqrt{3}r}^{\sqrt{1-r^2}} zr\ dz\ dr\ d\theta = \int_0^{2\pi} \int_0^{\frac{1}{2}} \left(\frac{1}{2}r - 2r^3\right)\ dr\ d\theta$$

$$= \int_0^{2\pi} \frac{1}{32}\ d\theta = \frac{\pi}{16}.$$

2. Compute the volume of the slice of cantaloupe shown below by integration. The interior angle is $\Delta\theta = \frac{\pi}{4}$.



   Answer: We have

$$\int_0^{\pi} \int_0^{\frac{\pi}{4}} \int_3^5 \rho^2 \sin \varphi\ d\rho\ d\theta\ d\varphi = \frac{49\pi}{3}.$$