# First Hour Exam

1. Consider the points $P = (1, 2, 3)$, $Q = (2, 2, 4)$, and $R = (1, 3, 5)$.

   (a) Find an equation of the plane $\Pi$ containing $P$, $Q$ and $R$.

   (b) What is the area of the triangle with verticies $P$, $Q$, and $R$?

   **Answer:** The vectors $\overrightarrow{PQ} = \langle 1, 0, 1\rangle$ and $\overrightarrow{PR} = \langle 0, 1, 2\rangle$ both lie in $\Pi$ so that

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \langle -1, -2, 1\rangle$$

   is normal to $\Pi$. Therefore,

$$0 = \langle x - 1, y - 2, z - 3\rangle \cdot \langle -1, -2, 1\rangle = -x - 2y + z + 2$$

   is an equation for $\Pi$.

   We computed in class the the area of the parallelogram with sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle -1, -2, 1\rangle| = \sqrt{6}$$

   so that the triangle has ared $\frac{\sqrt{6}}{2}$

2. Find symmetric equations of the line of intersection $\ell$ of the planes $\Pi_1$ and $\Pi_2$ with equations

$$\begin{aligned} 3x + y - 2z &= 1 \\ x + y - z &= 0 \end{aligned}$$

   respectively.

   **Answer:** $\Pi_1$ and $\Pi_2$ have normals

$$\begin{aligned} \mathbf{n}_1 &= \langle 3, 1, -2\rangle \\ \mathbf{n}_2 &= \langle 1, 1, -1\rangle \end{aligned}$$

   We therefore have that

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & 1 & -1 \end{vmatrix} = \langle 1, 1, 2\rangle$$

   is orthogonal to *both* $\mathbf{n}_1$ and $\mathbf{n}_2$ and therfore must be *parallel* to the line of intersection $\ell$. We observe that the point $(1, 0, 1)$ is in both planes, and so is in the line of intersection $\ell$. We then have symmetric equations for $\ell$ given by

$$\frac{x-1}{1} = \frac{y}{1} = \frac{z-1}{2}.$$

To check this, confirm that $\ell$ lies in both $\Pi_1$ and $\Pi_2$ by substituting the parametric equations for $\ell$

$$\begin{cases} x = t+1 \\ y = t \\ z = 2t+1 \end{cases}$$

into the equations for $\Pi_1$ and $\Pi_2$.

3. The position vector of a moving particle at time $t$ is given by

$$\mathbf{r}(t) = \left\langle t - \frac{t^3}{3}, \frac{t^2}{2}, 1 + \frac{t^2}{2} \right\rangle$$

   (a) Find the velocity
   (b) Find the speed
   (c) Find the acceleration
   (d) Find the curvature
   (e) Write down an integral which gives the arc length from the point where $t = 0$ to the point where $t = 2$, but do not evaluate it.

   (a) $\mathbf{v}(t) = \langle 1 - t^2, t, t \rangle$
   (b) $|\mathbf{r}'(t)| = \sqrt{(1-t^2)^2 + t^2 + t^2} = \sqrt{1+t^4}$
   (c) $\mathbf{a}(t) = \langle -2t, 1, 1 \rangle$
   (d)

$$\kappa(t) = \frac{\left| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1-t^2 & t & t \\ -2t & 1 & 1 \end{matrix} \right|}{\sqrt{1+t^4}} = \frac{|(-1-t^2)\,\mathbf{j} + (1+t^2)\,\mathbf{k}|}{\sqrt{1+t^4}} = \frac{\sqrt{2}\,|1+t^2|}{\sqrt{1+t^4}}$$

   (e) $\displaystyle\int_0^2 \sqrt{1+t^4}\,dt$

4. Let $f(x,y) = x \sin(xy + y^3)$. Compute the partials $f_x$, $f_y$ and $f_{xy}$.

   **Answer:**

$$\begin{aligned} f_x(x,y) &= \sin\left(xy + y^3\right) + xy\cos\left(xy + y^3\right) \\ f_y(x,y) &= x\cos\left(xy + y^3\right)\left(x + 3y^2\right) \\ f_{xy}(x,y) &= \left(x + 3y^2\right)\cos\left(xy + y^3\right) + x\cos\left(xy + y^3\right) - xy\left(x + 3y^2\right)\sin\left(xy + y^3\right) \end{aligned}$$

5. Let $\dfrac{x^2 + 2y^2}{x^2 + y^2}$.

   (a) Show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(b) Sketch the level curves $k = 1, \frac{1}{2}, 2$ of the function $f$.

Along the $x$-axis, we have

$$f(x, y) = f(x, 0) = \frac{2y^2}{y^2}$$

so that $f(x, y) \to 2$, but along the $y$-axis we have

$$f(x, y) = f(0, y) = \frac{x^2}{x^2}$$

so that $f(x, y) \to 1$. This shows that the limit does not exist.

The curve $1 = f(x, y)$ simplifies to $y = 0$, the curve $\frac{1}{2} = f(x, y)$ is the point $(0, 0)$, and the curve $2 = f(x, y)$ simplifies to $x = 0$. Some more interesting curves are shown in the figure below.