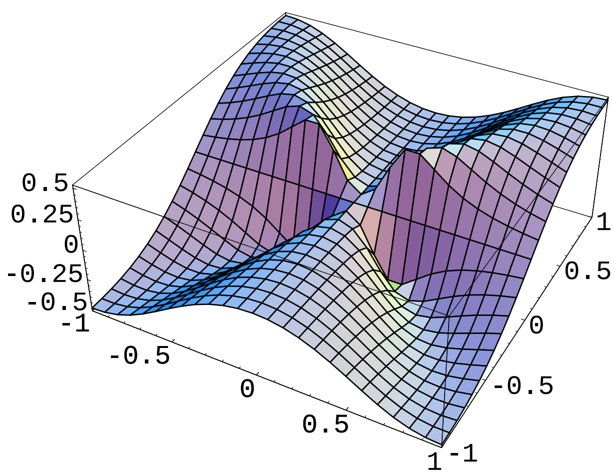
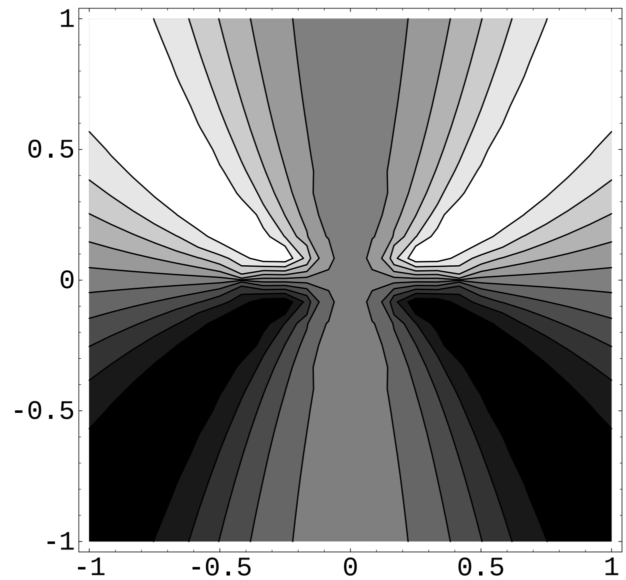
Mock Quiz 2

Let

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

This function is plotted below.





1. Do the partial derivatives f_x and f_y exist and are they continuous at all points $(x,y) \neq (0,0)$?

2. Is f differentiable at all points $(x,y) \neq (0,0)$?

3. Is f continuous at $(0,0)$?

4. Is f differentiable at $(0,0)$?

5. Do the partial derivatives f_x and f_y exist and are they continuous at $(0,0)$?

Answer:

1. When $(x,y) \neq (0,0)$, we can apply the differentiation rules from Calculus I, since holding either x or y constant produces a well defined function of one variable in the form to which the Calculus I rules apply. For example, the quotient rule applies to functions of the form $\frac{g(x)}{h(x)}$ which is *precisely* what we have when we hold y fixed. Thus, we have

$$f_x(x,y) = \frac{2xy\left(x^4+y^2\right) - x^2y\left(4x^3\right)}{\left(x^4+y^2\right)^2} \tag{1}$$

which (obviously) exists and is continuous for all $(x,y) \neq (0,0)$. Similarly, $f_y(x,y)$ exists and is continuous for $(x,y) \neq (0,0)$.

2. By (1), we have that the partials $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous for all $(x,y) \neq (0,0)$. This means that f is differentiable at $(x,y) \neq (0,0)$.

3. Along the curve $y = 0$ we have

$$f(x, 0) = \frac{x \cdot 0}{x^4 + 0} \to 0$$

but along the curve $y = x^2$, we have

$$f(x, y) = \frac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \frac{x^4}{2x^4} \to \frac{1}{2}$$

so that $\lim_{(x,y)\to(0,0)} f(x, y)$ *does not* exist.

4. f cannot be differentiable at $(0, 0)$ since it is not continuous there by part (3).

5. Observe that we *cannot* apply the differentiation rules from Calculus I to f at $(0, 0)$ because these rules don't apply to piecewise defined functions. We must therefore apply the definition of the derivative. We have

$$\lim_{h\to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{\frac{h^2 \cdot 0}{h^4 + 0} - 0}{h} = \lim_{h\to 0} 0 = 0 \qquad (2)$$

which shows that $f_0(0, 0)$ exists and equals 0. Similarly, $f_y(0, 0)$ exists and equals 0.

Collecting our results from equations (1) and (2), we have that

$$f_x(x, y) = \begin{cases} \frac{2xy(x^4+y^2)-4x^5y}{(x^4+y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

exists for all $(x, y) \in \mathbb{R}^2$. But this function can't possibly be continuous at $(0, 0)$ since it blows up as we approach $(0, 0)$ from any direction. Observe that if the partials *had* been continuous at $(0, 0)$, then f would be differentiable at $(0, 0)$, contrary to our argument in (4).