Mock Quiz 2

Let
\[ f(x, y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]

This function is plotted below.

1. Do the partial derivatives \( f_x \) and \( f_y \) exist and are they continuous at all points \((x, y) \neq (0, 0)\)?

2. Is \( f \) differentiable at all points \((x, y) \neq (0, 0)\)?

3. Is \( f \) continuous at \( (0, 0) \)?

4. Is \( f \) differentiable at \( (0, 0) \)?

5. Do the partial derivatives \( f_x \) and \( f_y \) exist and are they continuous at \( (0, 0) \)?

**Answer:**

1. When \((x, y) \neq (0, 0)\), we can apply the differentiation rules from Calculus I, since holding either \( x \) or \( y \) constant produces a well defined function of one variable in the form to which the Calculus I rules apply. For example, the quotient rule applies to functions of the form \( \frac{g(x)}{h(x)} \) which is precisely what we have when we hold \( y \) fixed. Thus, we have
\[ f_x(x, y) = \frac{2xy(x^4+y^2) - x^2y(4x^3)}{(x^4+y^2)^2} \] (1)

which (obviously) exists and is continuous for all \((x, y) \neq (0, 0)\). Similarly, \( f_y(x, y) \) exists and is continuous for \((x, y) \neq (0, 0)\).

2. By (1), we have that the partials \( f_x(x, y) \) and \( f_y(x, y) \) exist and are continuous for all \((x, y) \neq (0, 0)\). This means that \( f \) is differentiable at \((x, y) \neq (0, 0)\).
3. Along the curve \( y = 0 \) we have

\[
    f(x, 0) = \frac{x \cdot 0}{x^4 + 0} \to 0
\]

but along the curve \( y = x^2 \), we have

\[
    f(x, y) = \frac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \frac{x^4}{2x^4} \to \frac{1}{2}
\]

so that \( \lim_{(x, y) \to (0,0)} f(x, y) \) does not exist.

4. \( f \) cannot be differentiable at \((0,0)\) since it is not continuous there by part (3).

5. Observe that we cannot apply the differentiation rules from Calculus I to \( f \) at \((0,0)\) because these rules don’t apply to piecewise defined functions. We must therefore apply the definition of the derivative. We have

\[
    \lim_{h \to 0} \frac{f(0 + h, 0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^2}{h^4 + 0} - 0}{h} = \lim_{h \to 0} 0 = 0
\]

which shows that \( f_0(0,0) \) exists and equals 0. Similarly, \( f_y(0,0) \) exists and equals 0.

Collecting our results from equations (1) and (2), we have that

\[
    f_x(x, y) = \begin{cases} 
        \frac{2xy(x^4 + y^2) - 4x^5y}{(x^4 + y^2)^2} & (x, y) \neq (0,0) \\
        0 & (x, y) = (0,0) 
    \end{cases}
\]

exists for all \((x, y) \in \mathbb{R}^2\). But this function can’t possibly be continuous at \((0,0)\) since it blows up as we approach \((0,0)\) from any direction. Observe that if the partials had been continuous at \((0,0)\), then \( f \) would be differentiable at \((0,0)\), contrary to our argument in (4).