Math 210 Mock Final Exam 28 April 2004

Disclaimer: All questions used below have been painstakedly researched, although their answers have not. Ambiguous, misleading, or poorly worded questions are par for the course. Students who are sticklers for correct answers should write their own exams. All opinions expressed in this Mock Exam are well-reasoned and insightful. Needless to say, they are not those of the university, its instructors, or its lackeys. Anyone who says otherwise is itching for a fight.

1. Use Green's Theorem to evaluate

$$\int_C \sqrt{1+x^3}dx + 2xydy$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$.

Answer: Let $\mathbf{F}(x,y) = \left\langle \sqrt{1+x^3}, 2xy \right\rangle$ and let D be the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_D 2y\ dA = \int_0^1 \int_0^{3x} 2y\ dy\ dx = \int_0^1 9x^2\ dx = 3$$

2. The joint density function for random variables X and Y is

$$f(x,y) = \begin{cases} C(x+y) & \text{if } 0 \le x \le 3, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.
(b) Find $P(X \le 2, Y \ge 1)$.
(c) Find $P(X + Y \le 1)$.

Answer:

$$\int_0^2 \int_0^3 C(x+y)\ dx\ dy = 15$$

so that $C = \frac{1}{15}$. Then

$$P(X \le 2, Y \ge 1) = \int_0^2 \int_1^2 \frac{x+y}{15}\ dy\ dx = \frac{1}{3}$$

and

$$P(x+y \le 1) = \int_0^1 \int_0^{1-x} \frac{x+y}{15}\ dy\ dx = \frac{1}{45}$$

3. Evaluate $\int_E y^2z^2\ dV$. where E is the region bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$.

Answer:

We have by the change of variables formula that

$$\int_E y^2 z^2 \; dV = \int_F x^2 y^2 \; dV$$

where F is the region bounded by the parabolid $z = 1 - x^2 - y^2$ and the plane $z = 0$. Formally, what I have done is mapped

$$\begin{cases} x \mapsto z \\ y \mapsto x \\ z \mapsto y \end{cases}.$$

This transformation has Jacobian determinant

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1.$$

Then using cylindrical coordinates, we have

$$\int_F x^2 y^2 \; dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r^2 \cos^2 \theta r^2 \sin^2 \theta r \; dz \; dr \; d\theta = \frac{\pi}{96}.$$

4. Rewrite the integral

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x, y, z) \; dz \; dy \; dx$$

as an iterated integral in the order $dx \; dy \; dz$.

Answer:

$$\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z) \; dx \; dy \; dz.$$

5. Use Lagrange Multipliers to find the maximum and minimum values of $f(x, y) = \frac{1}{x} + \frac{1}{y}$ subject to the constraint $\frac{1}{x^2} + \frac{1}{y^2} = 1$. **Answer:** We need to find solutions to the system

$$\frac{-1}{x^2} = \frac{-2\lambda}{x^3} \tag{1}$$

$$\frac{-1}{y^2} = \frac{-2\lambda}{y^3} \tag{2}$$

$$\frac{1}{x^2} + \frac{1}{y^2} = 1 \tag{3}$$

Then $\lambda = \pm\frac{1}{\sqrt{2}}$ by substituting (1) and (2) into (3). Substituting these values of λ back into (1) and (2) yields the points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Evaluating f at these points, we have that $(\sqrt{2}, \sqrt{2})$ is a Max and $-(\sqrt{2}, -\sqrt{2})$ is a min.

6. If $z = \cos xy + y \cos x$ where $x = u^2 + v$ and $y = u - v^2$, find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Answer:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \text{etc.}$$