

# An Example CRIME calculation: The cohomology ring of $Q_8$

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Let  $G = Q_8 = \langle x, y \mid x^2 = y^2 = (xy)^2, x^4 = 1 \rangle = \langle x, y, z \mid x^2 = y^2 = z = (xy)^2, x^4 = 1 \rangle$ . Observe that  $z$  in the second presentation is redundant, but simplifies the notation later. In **GAP**, we execute the following commands.

```
gap> G:=SmallGroup(8,4);
<pc group of size 8 with 3 generators>
gap> Pcgs(G);
Pcgs([ f1, f2, f3 ])
```

Then a little manipulation in **GAP** reveals that  $f1$ ,  $f2$ , and  $f3$ , correspond with  $x$ ,  $y$ , and  $z$  from the presentation above, and with  $i$ ,  $j$ , and  $-1$  from the standard presentation of  $Q_8$ .

Let  $k = \mathbb{F}_2$ . It's well known that  $k$  has a periodic minimal  $kG$ -projective resolution. To see this, we start with the following commands.

```
gap> C:=CohomologyObject(G);
<object>
gap> ProjectiveResolution(C,10);
[ 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2 ]
```

`ProjectiveResolution` returns the  $kG$ -ranks of the terms of the minimal projective resolution. These numbers are called the *Betti numbers* of the resolution. Therefore, this tells us that  $k$  has a minimal  $kG$ -projective resolution

$$P_* : \quad \cdots \longrightarrow kG \xrightarrow{\partial_4} kG \xrightarrow{\partial_3} (kG)^2 \xrightarrow{\partial_2} (kG)^2 \xrightarrow{\partial_1} kG \xrightarrow{\epsilon} k \longrightarrow 0 \quad (1)$$

We can see from (1) that  $P_*$  appears to be periodic, but we confirm this below by looking at the boundary maps. The map  $\epsilon$  is the usual augmentation  $\epsilon \left( \sum_g \alpha_g g \right) = \sum_g \alpha_g$ .

Since  $P_*$  is minimal, the cohomology groups  $H^i(G) = \text{Ext}^i(k, k)$  are simply

$$\text{Hom}_{kG}(P_i, k) = k^{b_i}.$$

Here,  $b_i$  is the  $(i+1)$ st element in the list returned by `ProjectiveResolution`, so the first element in this list is the dimension of  $P_0$ . Thus, the Betti numbers give the ranks of the cohomology groups as well.

To look at the boundary maps, we need some notation. Recall that for  $G$  a  $p$ -group of size  $p^n$  and  $k$  a field of characteristic  $p$ , which is exactly the situation that we're in in this example, the group algebra  $kG$  has a basis

$$\mathcal{B}' = \left\{ x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \mid 0 \leq a_1, a_2, \dots, a_n \leq p-1 \right\} \quad (2)$$

where  $x_1, x_2, \dots, x_n$  is a polycyclic generating set for  $G$ . In fact, the fact that  $\mathcal{B}'$  is a basis merely expresses the fact the  $x_1, x_2, \dots, x_n$  is a polycyclic generating set. In the example  $G = Q_8$ , arranging the  $(a_1, a_2, \dots, a_n)$ 's in reverse lexicographic order, we have

$$\begin{aligned} \mathcal{B}' &= (1, x, y, xy, z, xz, yz, xyz) \\ &= (1, i, j, k, -1, -i, -j, -k). \end{aligned}$$

However, a more computationally efficient basis of  $kG$  is the following.

$$\mathcal{B} = \left\{ (x_1 - 1)^{a_1} (x_2 - 1)^{a_2} \dots (x_n - 1)^{a_n} \mid 0 \leq a_1, a_2, \dots, a_n \leq p-1 \right\} \quad (3)$$

Let  $I = x + 1$ ,  $J = y + 1$ , and  $K = xy + 1$ . Observe that  $I^2 = J^2 = z + 1$ . Observe also that  $K = I + J + IJ$ . The element  $K$  was included to make the boundary maps below look more symmetric. Then in the example  $G = Q_8$  we have

$$\mathcal{B} = (1, I, J, IJ, I^2, I^3, I^2J, I^3J)$$

The boundary maps returned by `BoundaryMaps` are with respect to the basis  $\mathcal{B}$ .

```
gap> Display(BoundaryMap(C,1));
. 1 . . . . .
. . 1 . . . . .
gap> Display(BoundaryMap(C,2));
. 1 . . . . . 1 . . . . .
. . 1 . . . . . 1 1 1 . . . . .
gap> Display(BoundaryMap(C,3));
. . 1 . . . . . 1 1 1 . . . . .
gap> Display(BoundaryMap(C,4));
. . . . . 1
gap> Display(BoundaryMap(C,5));
. 1 . . . . .
. . 1 . . . . .
```

Observe first that  $\partial_5 = \partial_1$ , so we see that  $P_*$  is in fact periodic as mentioned above. The matrices for  $\partial_n$  give only the image of  $1_G$  from each direct factor of  $P_n$ , since the images of the the other elements of  $P_n$  are determined by linearity.<sup>1</sup> For example, since

$$\partial_1 : P_1 = kG \oplus kG \rightarrow P_0 = kG,$$

the matrix returned above tells us that  $\partial_1(1_G, 0) = I$  and  $\partial_1(0, 1_G) = J$ . Summarizing the information above, we have the following.

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<sup>1</sup>Note to users: if the matrices giving the action of  $kG$  on itself with respect to  $\mathcal{B}$ , or the full matrices for the  $\partial_n$ 's would be useful to users, please let me know. I could include functions to return them, but I hesitate to overload the user with superfluous information.

$$\partial_n = \begin{cases} \begin{pmatrix} I \\ J \end{pmatrix} & \text{if } n \equiv 1 \pmod{4} \\ \begin{pmatrix} I & J \\ J & K \end{pmatrix} & \text{if } n \equiv 2 \pmod{4} \\ \begin{pmatrix} J & K \end{pmatrix} & \text{if } n \equiv 3 \pmod{4} \\ \begin{pmatrix} I^3 J \end{pmatrix} & \text{if } n \equiv 0 \pmod{4} \end{cases} \quad (n \geq 1) \quad (4)$$

The matrices in (4) are meant to be multiplied on the right as usual in **GAP**.

Now since  $H^1(G) = \text{Hom}_{kG}(P_1, k)$ , we have a natural basis  $\{\eta_1, \eta_2\}$  of  $H^1(G)$  where  $\eta_1$  is the map sending  $(1_G, 0) \mapsto 1_k$  and  $(0, 1_G) \mapsto 0$  and  $\eta_2$  is the other way around.

Then the following are chain maps representing  $\eta_1$  and  $\eta_2$ .

$$\begin{array}{ccc} P_3 \xrightarrow{(J \ K)} P_2 \xrightarrow{\begin{pmatrix} I & J \\ J & K \end{pmatrix}} P_1 & & P_3 \xrightarrow{(J \ K)} P_2 \xrightarrow{\begin{pmatrix} I & J \\ J & K \end{pmatrix}} P_1 \\ \downarrow (0 \ 1) \quad \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \searrow \eta_1 & & \downarrow (1 \ 1) \quad \downarrow \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \searrow \eta_2 \\ P_2 \xrightarrow{\begin{pmatrix} I & J \\ J & K \end{pmatrix}} P_1 \xrightarrow{\begin{pmatrix} I \\ J \end{pmatrix}} P_0 \xrightarrow{\epsilon} k & & P_2 \xrightarrow{\begin{pmatrix} I & J \\ J & K \end{pmatrix}} P_1 \xrightarrow{\begin{pmatrix} I \\ J \end{pmatrix}} P_0 \xrightarrow{\epsilon} k \end{array} \quad (5)$$

In the rows of the diagrams in (5) we have copies of  $P_*$ , while in the columns, we have maps making the diagrams commute. These maps were produced by inspection and by ... well, let's just say that I used **GAP** a tiny bit. Fortunately, this is exactly what the **CRIME** package does for us, as we will see below.

For the purpose of multiplication, the pictures in (5) represent  $\eta_1$  and  $\eta_2$ , so the composition of the two pictures represents the product, as in the following picture.

$$\begin{array}{ccccc} P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 \\ \downarrow (0 \ 1) & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \searrow \eta_1 \\ P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \xrightarrow{\epsilon} k \\ \downarrow \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \searrow \eta_2 & & \\ P_1 & \longrightarrow & P_0 & \xrightarrow{\epsilon} & k \end{array} \quad (6)$$

From (6), we can see that  $\eta_1 \eta_2 = \zeta_2$  where  $\{\zeta_1, \zeta_2\}$  is the natural basis of  $H^2(G)$ . This is the map going from  $P_2$  in the top row to  $k$  in the bottom, as in the diagrams in (5).

By composing the first diagram with itself, we find that  $\eta_1^2 = \zeta_1$ . Similarly, by more chain map production and composition, we find that  $\eta_2 \zeta_2$  is a nonzero element of degree 3, but that no product of elements of degree  $< 4$  produces a nonzero element of degree 4.

Let  $\{\xi\}$  be the natural basis of  $H^4(G)$ . We lift  $\xi$  to a chain map.

$$\begin{array}{ccccccc} P_8 & \xrightarrow{(I^3 J)} & P_7 & \xrightarrow{\begin{pmatrix} J \\ K \end{pmatrix}} & P_6 & \xrightarrow{\begin{pmatrix} I & J \\ J & K \end{pmatrix}} & P_5 \xrightarrow{(I \ J)} P_4 \\ \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \downarrow 1 \searrow \xi \\ P_4 & \xrightarrow{(I^3 J)} & P_3 & \xrightarrow{\begin{pmatrix} J \\ K \end{pmatrix}} & P_2 & \xrightarrow{\begin{pmatrix} I & J \\ J & K \end{pmatrix}} & P_1 \xrightarrow{(I \ J)} P_0 \xrightarrow{\epsilon} k \end{array} \quad (7)$$

This time, the production of the chain map is easy because of the periodicity of  $P_*$ . From (7), we see that all the elements of degree 4–7 arise as products of  $\xi$  with elements of degree 0–3, which in turn are products of  $\eta_1$  and  $\eta_2$ .

Thus, by recursion, we find that  $\eta_1$ ,  $\eta_2$ , and  $\xi$  generate the entire ring  $H^*(G)$ . This is precisely what **GAP** tells us from the following commands.

```
gap> CohomologyGenerators(C,10);
[ 1, 1, 4 ]
gap> A:=CohomologyRing(C,10);
<algebra of dimension 17 over GF(2)>
gap> LocateGeneratorsInCohomologyRing(C);
[ v.2, v.3, v.7 ]
```

`CohomologyGenerators` merely tells us the degrees of the generators, and they agree with those which we computed above.

The ring returned by `CohomologyRing` has basis  $[A.1, A.2, \dots, A.17]$  corresponding with the concatenation of the natural bases of the  $H^i(G)$ 's. Thus,  $A.1$  is the identity element,  $A.2$  and  $A.3$  correspond with  $\eta_1$  and  $\eta_2$ ,  $A.4$  and  $A.5$  correspond with  $\zeta_1$  and  $\zeta_2$ , etc. Observe that  $17 = \sum_{i=0}^{10} b_i$  which explains the dimension of  $A$ . The true cohomology ring is infinite-dimensional, so that  $A$  can be seen as a degree-10-truncation, that is,  $A \cong H^*(G)/J_{>10}$  where  $J_{>10}$  is the subring of all elements of degree  $> 10$ .

The following commands verify the calculations mentioned above.

```
gap> A.2^2;
v.4
gap> A.2*A.3;
v.5
gap> A.3*A.5;
v.6
```

The command `LocateGeneratorsInCohomologyRing` tells us that  $\eta_1$ ,  $\eta_2$ , and  $\xi$  correspond with  $A.2$ ,  $A.3$ , and  $A.7$ , which we had already deduced by degree considerations, but if  $\dim H^4(G)$  had been greater than 1, we wouldn't have known which element corresponded with  $\xi$ .

Finally, **GAP** gives us a presentation of  $H^*(G)$  with the following command.

```
gap> CohomologyRelators(C,10);
[ [ z, y, x ], [ z^2+z*y+y^2, y^3 ] ]
```

This tells us that

$$H^*(G) \cong k[z, y, x] / (z^2 + yz + y^2, y^3)$$

is a polynomial ring in the variables  $z$ ,  $y$  and  $x$ , modulo the ideal generated by  $z^2 + yz + y^2$  and  $y^3$ .