Cohomology Products in GAP, Explained in not Unbearable Detail, but Still Bad Enough to Require Being Seated while Reading

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The purpose of this document is to explain the implementation of cohomology products in the crime package for GAP including the Massey n-fold product. In this document, a composition of two functions $g \circ f$ is the function obtained by applying f first and then g. The symbol $\circlearrowright$ is used in diagrams to indicate that a polygon either commutes or anticommutes.

Let G be a finite p-group for some prime p and let $k = \mathbb{F}_p$. Also write k for the trivial kG-module. We assume that we can calculate a kG-projective resolution P_* of k, that is, for n as large as we need, we can compute the integers $\{b_m : 0 \leq m \leq n\}$, the maps $\{\partial_m : 1 \leq m \leq n\}$ and the map ϵ such that

$$P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} k \tag{1}$$

is exact, where $P_m = (kG)^{\oplus b_m}$. Later, we will assume moreover that P_* is *minimal*, that is, that $\partial_m(P_m) \leq \operatorname{Rad}(P_{m-1})$ for all $m \geq 1$.

1 Cohomology Products

The following construction is taken from [2]. We begin with two cocycles $f : P_i \to k$ and $g : P_j \to k$, that is, that $f \circ \partial_{i+1} = g \circ \partial_{j+1} = 0$. We want to compute the cup product $fg : P_{i+j} \to k$.

We first convert f into an chain map, resulting in the following commutative diagram.

$$\begin{tikzcd}
P_m \arrow[r, "\partial_m"] \arrow[d, "f_m"] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "f_{m-1}"] & P_{m-2} \arrow[r] \arrow[d, "f_{m-2}"] & \cdots \arrow[r] & P_{i+2} \arrow[r, "\partial_{i+2}"] \arrow[d, "f_{i+2}"] & P_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f_{i+1}"] & P_i \arrow[d, "f_i"] \arrow[dr, "f"] & & \\
P_{m-i} \arrow[r, "\partial_{m-i}"'] & P_{m-i-1} \arrow[r, "\partial_{m-i-1}"'] & P_{m-i-2} \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r, "\partial_2"'] & P_1 \arrow[r, "\partial_1"'] & P_0 \arrow[r, "\epsilon"'] & k \arrow[r] & 0
\end{tikzcd} \tag{2}$$

1. Define f_i such that $\epsilon \circ f_i = f$. This is possible by projectivity of P_i.

2. Define f_{i+1} such that $\partial_1 \circ f_{i+1} = f_i \circ \partial_{i+1}$. This is possible by projectivity of P_{i+1} since
$$\operatorname{im}(f_i \circ \partial_{i+1}) \leq \operatorname{im}(\partial_1) = \ker(\epsilon)$$
as $\epsilon \circ (f_i \circ \partial_{i+1}) = f \circ \partial_{i+1} = 0$.

3. Define f_{i+2} such that $\partial_2 \circ f_{i+2} = f_{i+1} \circ \partial_{i+2}$. This is possible by projectivity of P_{i+2} since
$$\operatorname{im}(f_{i+1} \circ \partial_{i+2}) \leq \operatorname{im}(\partial_2) = \ker(\partial_1)$$
as $\partial_1 \circ (f_{i+1} \circ \partial_{i+2}) = f_i \circ \partial_{i+1} \circ \partial_{i+2} = 0$.

4. Define f_m for $m > i+2$ by recursion such that $\partial_{m-i} \circ f_m = f_{m-1} \circ \partial_m$. This is possible by projectivity of P_m since
$$\operatorname{im}(f_{m-1} \circ \partial_m) \leq \operatorname{im}(\partial_{m-i}) = \ker(\partial_{m-i-1})$$
as $\partial_{m-i-1} \circ (f_{m-1} \circ \partial_m) = f_{m-2} \circ \partial_{m-1} \circ \partial_m = 0$.

Then the product fg is calculated as $g \circ f_{i+j}$. The process above is used to compute the multiplication table used by the `CohomologyRing` command and is used to find generators by the `CohomologyGenerators` command.

2 The Yoneda Cocomplex

My understanding of the purpose of the Yoneda Cocomplex is the following. The definition of the Massey product below requires a cocomplex having an associative product. The product defined above, however, is defined only for f and g cocycles in $\operatorname{Hom}(P_*, k)$. The Yoneda cocomplex Y, on the other hand, has the same cohomology as $\operatorname{Hom}(P_*, k)$, but has an associative product defined for all cochains, namely composition. Moreover, we will show that via the isomorphism $\Phi : H^*(G, k) \to H^*(Y)$, composition in Y agrees with the product defined in Section 1 up to the factor $(-1)^{\deg f \deg g}$, that is,
$$\Phi(fg) = (-1)^{\deg f \deg g} \Phi(g) \circ \Phi(f).$$

The following construction comes from [1].

Definition 1. *For* $i \geq 0$, *define*
$$Y^i = \prod_{m \geq i} \operatorname{Hom}_{kG}(P_m, P_{m-i}).$$

Then an element $f \in Y^i$ *is a collection of* kG*-homomorphisms* $\{f_m : P_m \to P_{m-i} : m \geq i\}$ *as in the following diagram.*

$$
\begin{tikzcd}
P_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & P_m \arrow[r, "\partial_m"] \arrow[d, "f_m"] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "f_{m-1}"] & P_{m-2} \arrow[r] \arrow[d, "f_{m-2}"] & \cdots \arrow[r] & P_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f_{i+1}"] & P_i \arrow[d, "f_i"] \\
P_{n-i} \arrow[r, "\partial_{n-i}"'] & P_{n-i-1} \arrow[r] & \cdots \arrow[r] & P_{m-i} \arrow[r, "\partial_{m-i}"'] & P_{m-i-1} \arrow[r, "\partial_{m-i-1}"'] & P_{m-i-2} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r, "\partial_1"'] & P_0
\end{tikzcd}
\tag{3}
$$

Diagram (3) is not required to commute.

Definition 2. *Define* $Y = \bigoplus_{i \geq 0} Y^i$. Y *is called the* Yoneda cocomplex *of* P_*. *We write* $\deg(f) = i$ *for* $f \in Y^i$. *Let* $f = \{f_m : m \geq i\} \in Y^i$ *and define*

$$
\begin{aligned}
\partial : Y^i &\to Y^{i+1} \\
f &\mapsto \left\{ f_{m-1} \circ \partial_m - (-1)^i \, \partial_{m-i} \circ f_m : m \geq 1 \right\}.
\end{aligned}
$$

We observe that cocycles in Y are those elements f for which (3) commutes if $\deg f$ is even and anticommutes if $\deg f$ is odd.

Lemma 3. Y *with differentiation* ∂ *is a cocomplex, that is,* $\partial^2 = 0$.

Proof. Let $f \in Y^i$. We will show that $\partial^2 f = 0$ at the point P_m in (3) for $m \geq i + 2 = \deg(\partial^2 f)$. Follow along in the picture.

$$
\begin{aligned}
(\partial(\partial f))_m &= (\partial f)_{m-1} \circ \partial_m - (-1)^{i+1} \, \partial_{m-i-1} \circ (\partial f)_m \\
&= \left(f_{m-2} \circ \partial_{m-1} - (-1)^i \, \partial_{m-i-1} \circ f_{m-1} \right) \circ \partial_m \\
&\quad - (-1)^{i+1} \, \partial_{m-i-1} \circ \left(f_{m-1} \circ \partial_m - (-1)^i \, \partial_{m-i} \circ f_m \right) \\
&= f_{m-2} \circ \partial_{m-1} \circ \partial_m - \partial_{m-i-1} \circ \partial_{m-i} \circ f_m \\
&= 0
\end{aligned}
$$

■

Theorem 4. *The cohomology groups of* Y *are* $H^*(G, k)$.

Proof. We will define a group isomorphism $\Phi : H^i(G, k) \to H^i(Y)$.

1. Let $f : P_i \to k$ be a cocycle in $\mathrm{Hom}^i_{kG}(P_*, k)$, that is, assume $f \circ \partial_{i+1} = 0$. Define $\Phi(f) = \{f_m : m \geq i\} \in Y^i$ as follows. The element $\Phi(f)$, together with f, is pictured in the following diagram.

$$
\begin{tikzcd}
P_m \arrow[r, "\partial_m"] \arrow[d, "f_m"] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "f_{m-1}"] & P_{m-2} \arrow[r] \arrow[d, "f_{m-2}"] & \cdots \arrow[r] & P_{i+2} \arrow[r, "\partial_{i+2}"] \arrow[d, "f_{i+2}"] & P_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f_{i+1}"] & P_i \arrow[d, "f_i"] \arrow[dr, "f"] & & \\
P_{m-i} \arrow[r, "\partial_{m-i}"'] & P_{m-i-1} \arrow[r, "\partial_{m-i-1}"'] & P_{m-i-2} \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r, "\partial_2"'] & P_1 \arrow[r, "\partial_1"'] & P_0 \arrow[r, "\epsilon"'] & k \arrow[r] & 0
\end{tikzcd}
\tag{4}
$$

(a) Define f_i such that $\epsilon \circ f_i = f$. This is possible by projectivity of P_i.

(b) Define f_{i+1} such that $\partial_1 \circ f_{i+1} = (-1)^i f_i \circ \partial_{i+1}$. This is possible by projectivity of P_{i+1} since

$$\operatorname{im}\left((-1)^i f_i \circ \partial_{i+1}\right) \leq \operatorname{im}(\partial_1) = \ker(\epsilon)$$

as $\epsilon \circ \left((-1)^i f_i \circ \partial_{i+1}\right) = (-1)^i f \circ \partial_{i+1} = 0$.

(c) Define f_{i+2} such that $\partial_2 \circ f_{i+2} = (-1)^i f_{i+1} \circ \partial_{i+2}$. This is possible by projectivity of P_{i+2} since

$$\operatorname{im}\left((-1)^i f_{i+1} \circ \partial_{i+2}\right) \leq \operatorname{im}(\partial_2) = \ker(\partial_1)$$

as $\partial_1 \circ \left((-1)^i f_{i+1} \circ \partial_{i+2}\right) = f_i \circ \partial_{i+1} \circ \partial_{i+2} = 0$.

(d) Define f_m for $m > i+2$ by recursion such that $\partial_{m-i} \circ f_m = (-1)^i f_{m-1} \circ \partial_m$. This is possible by projectivity of P_m since

$$\operatorname{im}\left((-1)^i f_{m-1} \circ \partial_m\right) \leq \operatorname{im}(\partial_{m-i}) = \ker(\partial_{m-i-1})$$

as $\partial_{m-i-1} \circ \left((-1)^i f_{m-1} \circ \partial_m\right) = f_{m-2} \circ \partial_{m-1} \circ \partial_m = 0$.

This completes the definition of Φ. The maps $\{f_m : m \geq i\}$ defined in Steps 1b-1d above satisfy

$$\partial_{m-i} \circ f_m = (-1)^i f_{m-1} \partial_m.$$

In other words, $(\partial\Phi(f))_{m+1} = 0$ for all $m \geq i+1$ so that $\partial\Phi(f) = 0$. Thus, $\Phi(f)$ is a cocycle by construction.

2. We claim than any other choice of maps $\{f'_m : m \geq i\}$ satisfying the conditions in 1a-1d above will be equivalent to $\{f_m : m \geq i\}$ in $H^i(Y)$. More precisely, if f and f' both satisfy conditions 1a-1d, then will define a map $\theta \in Y^{i-1}$ such that $\partial\theta = f - f'$. Write $g_m = f_m - f'_m$ for $m \geq i$.

$$\begin{tikzcd}
P_m \arrow[r, "\partial_m"] \arrow[d, "g_m"'] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "g_{m-1}"'] \arrow[dl, "\theta_{m-1}"'] & P_{m-2} \arrow[r, "\partial_{m-2}"] \arrow[d, "g_{m-2}"'] \arrow[dl, "\theta_{m-2}"'] & P_{m-3} \arrow[r] \arrow[d, "g_{m-3}"'] \arrow[dl, "\theta_{m-3}"'] & \cdots \arrow[r] & P_{i+2} \arrow[r, "\partial_{i+2}"] \arrow[d, "g_{i+2}"'] & P_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "g_{i+1}"'] \arrow[dl, "\theta_{i+1}"'] & P_i \arrow[r, "\partial_i"] \arrow[d, "g_i"'] \arrow[dl, "\theta_i"'] & P_{i-1} \arrow[dl, "\theta_{i-1}"'] \\
P_{m-i} \arrow[r, "\partial_{m-i}"'] & P_{m-i-1} \arrow[r, "\partial_{m-i-1}"'] & P_{m-i-2} \arrow[r, "\partial_{m-i-2}"'] & P_{m-i-3} \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r, "\partial_2"'] & P_1 \arrow[r, "\partial_1"'] & P_0 \arrow[r, "\epsilon"'] & k
\end{tikzcd} \tag{5}$$

(e) Take $\theta_{i-1} = 0$.

(f) Since $\epsilon \circ f_i = \epsilon \circ f'_i = f$, we have $\operatorname{im}(g_i) \leq \ker(\epsilon) = \operatorname{im}(\partial_1)$. Define θ_i such that $\partial_1 \circ \theta_i = (-1)^i g_i$. This is possible by projectivity of P_i. We rewrite the condition on θ_i for future reference as follows.

$$(\partial\theta)_i = 0 - (-1)^{i-1} \partial_1 \circ \theta_i = g_i \tag{6}$$

(g) By (2f), we have

$$\partial_1 \circ \theta_i \circ \partial_{i+1} = (-1)^i g_i \circ \partial_{i+1} = \partial_1 \circ g_{i+1}$$

so that

$$\operatorname{im}(g_{i+1} - \theta_i \circ \partial_{i+1}) \leq \ker(\partial_1) = \operatorname{im}(\partial_2).$$

Define θ_{i+1} such that

$$\partial_2 \circ \theta_{i+1} = (-1)^i (g_{i+1} - \theta_i \circ \partial_{i+1}),$$

and again, for future reference, we rewrite this as follows.

$$(\partial\theta)_{i+1} = \theta_i \circ \partial_{i+1} - (-1)^{i-1} \partial_2 \circ \theta_{i+1} = g_{i+1} \tag{7}$$

(h) Assume by recursion that we have computed θ_{m-2} and θ_{m-3} such that

$$\partial_{m-i-1} \circ \theta_{m-2} = (-1)^i (g_{m-2} - \theta_{m-3} \circ \partial_{m-2}).$$

Then $\partial_{m-i-1} \circ \theta_{m-2} \circ \partial_{m-1} = (-1)^i g_{m-2} \circ \partial_{m-1} = \partial_{m-i-1} \circ g_{m-1}$ so that

$$\operatorname{im}(g_{m-1} - \theta_{m-2} \circ \partial_{m-1}) \leq \ker(\partial_{m-i-1}) = \operatorname{im}(\partial_{m-i}).$$

Define θ_{m-1} such that

$$\partial_{m-i} \circ \theta_{m-1} = (-1)^i (g_{m-1} - \theta_{m-2} \circ \partial_{m-1}),$$

and again, for future reference, we rewrite this as follows.

$$(\partial\theta)_{m-1} = \theta_{m-2} \circ \partial_{m-1} - (-1)^{i-1} \partial_{m-i} \circ \theta_{m-1} = g_{m-1} \tag{8}$$

This completes the definition of θ. Then θ satisfies $\partial\theta = f - f'$ by (6), (7), and (8).

3. Suppose now that $f = \partial g$ for some cochain $g : P_{i-1} \to k$. Write $\Phi(g \circ \partial_i) = \{g_m : m \geq i\}$. We will construct θ such that $\Phi(\partial g) = \partial\theta$ for some $\theta \in Y^{i-1}$ as in the following diagram.

$$\begin{tikzcd}
P_{m+1} \arrow[r, "\partial_{m+1}"] \arrow[d, "g_{m+1}"'] & P_m \arrow[r, "\partial_m"] \arrow[d, "g_m"'] \arrow[dl, "\theta_m"'] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "g_{m-1}"'] \arrow[dl, "\theta_{m-1}"'] & P_{m-2} \arrow[r] \arrow[d, "g_{m-2}"'] \arrow[dl, "\theta_{m-2}"'] & \cdots \arrow[r] & P_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "g_{i+1}"'] & P_i \arrow[r, "\partial_i"] \arrow[d, "g_i"'] \arrow[dl, "\theta_i"'] & P_{i-1} \arrow[d, "g"] \arrow[dl, "\theta_{i-1}"'] \arrow[d, phantom, "\circlearrowright"', shift right=3ex] \\
P_{m-i+1} \arrow[r, "\partial_{m-i+1}"'] & P_{m-i} \arrow[r, "\partial_{m-i}"'] & P_{m-i-1} \arrow[r, "\partial_{m-i-1}"'] & P_{m-i-2} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r, "\partial_1"'] & P_0 \arrow[r, "\epsilon"'] & k
\end{tikzcd}$$

(i) Define θ_{i-1} such that $\epsilon \circ \theta_{i-1} = g$. This is possible by projectivity of P_{i-1}.

(j) Since $\epsilon \circ \theta_{i-1} \circ \partial_i = g \circ \partial_i = \epsilon \circ g_i$, we have that

$$\mathrm{im}\left(g_i - \theta_{i-1} \circ \partial_i\right) \leq \ker\left(\epsilon\right) = \mathrm{im}\left(\partial_1\right).$$

Thus, by projectivity of P_i, we have θ_i such that

$$\partial_1 \circ \theta_i = (-1)^i \left(g_i - \theta_{i-1} \circ \partial_i\right).$$

Then

$$(\partial\theta)_i = \theta_{i-1} \circ \partial_i - (-1)^{i-1} \partial_1 \circ \theta_i = g_i.$$

(k) Assume by recursion that we have computed the maps θ_{m-1} and θ_{m-2} such that

$$\theta_{m-2} \circ \partial_{m-1} - (-1)^{i-1} \partial_{m-i} \circ \theta_{m-1} = g_{m-1}.$$

Then

$$\partial_{m-i} \circ g_m = (-1)^i g_{m-1} \circ \partial_m = \partial_{m-i} \circ \theta_{m-1} \circ \partial_m$$

so that

$$\mathrm{im}\left(g_m - \theta_{m-1} \circ \partial_m\right) \leq \ker\left(\partial_{m-i}\right) = \mathrm{im}\left(\partial_{m-i+1}\right).$$

Define θ_m such that

$$\partial_{m-i-1} \circ \theta_m = (-1)^i \left(g_m - \theta_{m-1} \circ \partial_m\right).$$

Then

$$(\partial\theta)_m = \theta_{m-1} \circ \partial_m - (-1)^{i-1} \partial_{m-i+1} \circ \theta_m = g_m.$$

This completes the definition of θ. Then $g = \partial\theta$ by construction.

4. We will now show that Φ is a k-module homomorphism. Let $f, g : P_i \to k$ be cocycles and let $\alpha, \beta \in k$. Write $h = \alpha f + \beta g$. We want to show that $\Phi(h) = \alpha\Phi(f) + \beta\Phi(g)$. But $\epsilon \circ h_0 = \epsilon \circ (\alpha f_0 + \beta g_0) = \alpha f + \beta g$, so that we are in the situation of Step 2 above. Thus, $\Phi(h)$ and $\alpha\Phi(f) + \beta\Phi(g)$ are equivalent elements of Y.

5. By Steps 3 and 4, we have that if f and f' are equivalent in $H^*(G, k)$, then $\Phi(f)$ and $\Phi(f')$ are equivalent in $H^*(Y)$. This together with 2 shows that Φ is a well-defined k-module homomorphism.

6. Finally, Φ is a bijection, having inverse given by

$$\{f_m : m \geq i\} \mapsto \epsilon \circ f_i.$$

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3 Products in Y

Consider the following product $Y^i \otimes Y^j \to Y^{i+j}$ on Y. Let $f \in Y^i$ and $g \in Y^j$ and consider the composition of the individual component maps of f with those of g such that legitimate compositions are obtained, as in the following diagram.

$$\begin{tikzcd}
P_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & P_m \arrow[r, "\partial_m"] \arrow[d, "f_m"] & P_{m-1} \arrow[r] \arrow[d, "f_{m-1}"] & \cdots \arrow[r] & P_{i+j+1} \arrow[r, "\partial_{i+j+1}"] \arrow[d, "f_{i+j+1}"] & P_{i+j} \arrow[d, "f_{i+j}"] \\
P_{n-i} \arrow[r, "\partial_{n-i}"] \arrow[d, "g_{n-i}"] & P_{n-i-1} \arrow[r] \arrow[d, "g_{n-i-1}"] & \cdots \arrow[r] & P_{m-i} \arrow[r, "\partial_{m-i}"] \arrow[d, "g_{m-i}"] & P_{m-i-1} \arrow[r] \arrow[d, "g_{m-i-1}"] & \cdots \arrow[r] & P_{j+1} \arrow[r, "\partial_{j+1}"] \arrow[d, "g_{j+1}"] & P_j \arrow[d, "g_j"] \\
P_{n-i-j} \arrow[r, "\partial_{n-i-j}"] & P_{n-i-j-1} \arrow[r] & \cdots \arrow[r] & P_{m-i-j} \arrow[r, "\partial_{m-i-j}"] & P_{m-i-j-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] & P_0
\end{tikzcd} \tag{9}$$

Observe that we have thrown away the maps $\{f_m : i \leq m \leq i+j-1\}$. I suppose that the natural symbol for the object in (9) would be $g \circ f$, to emphasize the fact that we're talking about the component-wise composition of two elements of Y and *not* a cohomology product.

Observation 5. $\partial (g \circ f) = g \circ \partial f + (-1)^{\deg f} \partial g \circ f$.

Proof. Write $i = \deg (f)$ and $j = \deg (g)$ as in (9). We will show the claim at the point P_m in (9) for $m \geq i+j+1 = \deg (\partial (g \circ f))$. Follow along in the picture.

$$\begin{aligned}
\left(g \circ \partial f + (-1)^i \partial g \circ f\right)_m &= g_{m-i-1} \circ \left(f_{m-1} \circ \partial_m - (-1)^i \partial_{m-i} \circ f_m\right) \\
&\quad + (-1)^i \left(g_{m-i-1} \circ \partial_{m-i} - (-1)^j \partial_{m-i-j} \circ g_{m-i}\right) \circ f_m \\
&= g_{m-i-1} \circ f_{m-1} \circ \partial_m - (-1)^{i+j} \partial_{m-i-j} \circ g_{m-i} \circ f_m \\
&= (\partial (g \circ f))_m
\end{aligned}$$

■

Claim 6. *Composition in* Y *induces via* Φ *an associative binary operation*

$$H^i (G, k) \otimes H^j (G, k) \to H^{i+j} (G, k)$$

making $H^* (G, k)$ *into a ring with 1.*

4 Relationships among products on $H^* (G, k)$

Let $f \in H^i (G, k)$ and $g \in H^j (G, k)$. Consider the following products on $H^* (G, k)$.

1. The *cup product* fg defined in Section 1

2. The product induced from composition in Y

$$(f,g) \overset{\Phi}{\mapsto} \big(\Phi(f), \Phi(g)\big) \overset{\circ}{\mapsto} \Phi(g) \circ \Phi(f) \overset{\Phi^{-1}}{\mapsto} \epsilon \circ \big(\Phi(g) \circ \Phi(f)\big)_{i+j}$$

3. The *Massey 2-fold product* $\langle f, g \rangle$, defined more generally in Section 5 below,

$$(f,g) \overset{\Phi}{\mapsto} \big(\Phi(f), \Phi(g)\big) \overset{\langle \cdot \rangle}{\mapsto} (-1)^{i} \Phi(g) \circ \Phi(f) \overset{\Phi^{-1}}{\mapsto} (-1)^{i} \epsilon \circ \big(\Phi(g) \circ \Phi(f)\big)_{i+j}$$

The cup product is calculated as $g \circ f_{i+j}$, where f_{i+j} is as in (2), whereas product 2 is calculated as $g \circ f_{i+j}$, where f_{i+j} is as in (4). Comparing (2) and (4), we see that the two f_i's are the same, the f_{i+1}'s differ by $(-1)^{i}$, the f_{i+2}'s differ by $(-1)^{2i}$, and in general, the f_{i+m}'s differ by $(-1)^{im}$. Thus, products 1 and 2 differ by $(-1)^{ij}$, that is,

$$\Phi^{-1}\big(\Phi(g) \circ \Phi(f)\big) = (-1)^{ij} fg$$

so that

$$\Phi(fg) = (-1)^{ij} \Phi(g) \circ \Phi(f)$$

and therefore

$$\Phi(fg) = (-1)^{i(j+1)} \langle f, g \rangle .$$

We observe that product 1 is associative (see [2]), and that product 2 is also associative, consisting of composition of functions. The Massey product, however, is not associative in general.

5 Massey Products

The idea of the Massey product is to extend the cohomology product to an n-fold product for $n \geq 2$. The following definition is adapted from [3].

Definition 7. *For $k \geq 2$, let $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ be cocycles in Y. The* Massey k-fold product *$\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \rangle$ is defined provided that for each pair (i,j) with $1 \leq i < j \leq k$ other than $(1,k)$, the lower-degree product $\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \rangle$ is defined and vanishes as an element of $H^*(Y)$, that is, if for each qualifying (i,j), there exists $u^{i,j} \in Y$ such that $\partial u^{i,j} = \langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \rangle$. In this situation, the value of $\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \rangle$ is defined to be*

$$\sum_{t=1}^{k-1} u^{t+1,k} \circ \overline{u^{1,t}}$$

where the symbols $u^{1,1}$ and $u^{k,k}$ are taken to be $f^{(1)}$ and $f^{(k)}$ respectively and $\overline{u} = (-1)^{\deg(u)} u$.

Observe that in the case $k = 2$, the condition on (i, j) is vacuously satisfied, so that $\langle f, g\rangle = g \circ \overline{f}$.

Traditionally, one organizes the information in Definition 7 in an array, such as the following,

$$\begin{array}{cccc} f^{(1)} & u^{1,2} & u^{1,3} & \\ & f^{(2)} & u^{2,3} & u^{2,4} \\ & & f^{(3)} & u^{3,4} \\ & & & f^{(4)} \end{array}$$

and traces the top row with one hand while tracing the rightmost column with the other hand as t runs from 1 to 3. In this case, we have

$$\left\langle f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)}\right\rangle = u^{2,4} \circ \overline{f^{(1)}} + u^{3,4} \circ \overline{u^{1,2}} + f^{(4)} \circ \overline{u^{1,3}}.$$

Lemma 8. *$\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)}\right\rangle$ is a cocycle in Y.*

The reason for the sign appearing in Definition 7 becomes apparent is the following proof.

Proof. We begin by making a general observation about Y. Suppose $f \in Y^i$ and that $g = \partial\theta$ for some $\theta \in Y^{j-1}$ as in the following diagram.

$$\begin{tikzcd}
 & P_{i+j+m+1} \arrow[r, "\partial_{i+j+m+1}"] \arrow[d, "f_{i+j+m+1}"] & P_{i+j+m} \arrow[d, "f_{i+j+m}"] \\
 & P_{j+m+1} \arrow[r, "\partial_{j+m+1}"] \arrow[dl, "\theta_{j+m+1}"'] \arrow[d, "g_{j+m+1}"] & P_{j+m} \arrow[dl, "\theta_{j+m}"'] \arrow[d, "g_{j+m}"] \\
P_{m+2} \arrow[r, "\partial_{m+2}"'] & P_{m+1} \arrow[r, "\partial_{m+1}"'] & P_m
\end{tikzcd}$$

Then by Observation 5, we have

$$\begin{aligned}
(g \circ f)_{i+j+m+1} &= g_{j+m+1} \circ f_{i+j+m+1} \\
&= \theta_{j+m} \circ \partial_{j+m+1} \circ f_{i+j+m+1} - (-1)^{j-1} \partial_{m+2} \circ \theta_{j+m+1} \circ f_{i+j+m+1} \\
&= \theta_{j+m} \circ \partial_{j+m+1} \circ f_{i+j+m+1} - (-1)^{j-1} \partial_{m+2} \circ \theta_{j+m+1} \circ f_{i+j+m+1} \\
&\quad - (-1)^i \theta_{j+m} \circ f_{i+j+m} \circ \partial_{i+j+m+1} + (-1)^i \theta_{j+m} \circ f_{i+j+m} \circ \partial_{i+j+m+1} \\
&= -(-1)^i (\theta \circ (\partial f))_{i+j+m+1} + (-1)^i \partial (\theta \circ f)_{i+j+m+1}
\end{aligned}$$

so that as elements of $H^*(Y)$, we have

$$\partial\theta \circ f = -(-1)^i \theta \circ \partial f. \tag{10}$$

Now we compute the derivative of $\left\langle f^{(1)}, f^{(2)}, \dots, f^{(k)} \right\rangle$.

$$\begin{aligned}
\partial \left(\sum_{t=1}^{k-1} (-1)^{\left(\deg u^{1,t}\right)} u^{t+1,k} \circ u^{1,t} \right) &= \sum_{t=1}^{k-1} \left((-1)^{\left(\deg u^{1,t}\right)} u^{t+1,k} \circ \partial u^{1,t} + \partial u^{t+1,k} \circ u^{1,t} \right) \\
&= \sum_{t=1}^{k-1} \left(-\partial u^{t+1,k} \circ u^{1,t} + \partial u^{t+1,k} \circ u^{1,t} \right) \\
&= 0
\end{aligned}$$

■

Observation 9. *The condition* $\partial u^{i,j} = \left\langle f^{(i)}, f^{(i+1)}, \dots, f^{(j)} \right\rangle$ *forces*

$$\deg\left(u^{i,j}\right) = \sum_{t=i}^{j} \deg\left(f^{(t)}\right) + i - j$$

$$\textit{and} \qquad \deg\left\langle f^{(i)}, f^{(i+1)}, \dots, f^{(j)} \right\rangle = \sum_{t=i}^{j} \deg\left(f^{(t)}\right) + i - j + 1.$$

Troubling Observation 10. $\left\langle f^{(1)}, f^{(2)}, \dots, f^{(k)} \right\rangle$ *is not uniquely defined, unless for each* (i, j) *the condition* $\partial u^{i,j} = \left\langle f^{(i)}, f^{(i+1)}, \dots, f^{(j)} \right\rangle$ *is satisfied by exactly one cochain* $u^{i,j}$.

Suppose that we are given cocycles $f^{(1)}, f^{(2)}, \dots, f^{(k)}$ and we want to compute the map $u^{i,j}$ for some (i, j) with $1 \le i < j \le k$ other than $(1, k)$. Assume that recursively, we have computed all of the maps in the following array.

$$\begin{array}{ccccc}
f^{(i)} & u^{i,i+1} & \cdots & u^{i,j-1} & \\
 & f^{(i+1)} & & u^{i+1,j-1} & u^{i+1,j} \\
 & & & & \vdots \\
 & & & f^{(j-1)} & u^{j-1,j} \\
 & & & & f^{(j)}
\end{array}$$

The map $u^{i,j}$ will be such that

$$\partial u^{i,j} = \left\langle f^{(i)}, f^{(i+1)}, \dots, f^{(j)} \right\rangle = \sum_{t=i}^{j-1} u^{t+1,j} \circ \overline{u^{i,t}} \tag{11}$$

where $u^{i,i} = f^{(i)}$ and $u^{j,j} = f^{(j)}$. Write g for the map on the right-hand side of (11). Write

$$d = \deg(g) = \sum_{t=i}^{j} \deg\left(f^{(t)}\right) + i - j + 1.$$

The relevant maps are all pictured below.

$$
\begin{tikzcd}
P_m \arrow[r, "\partial_m"] \arrow[d, "g_m"'] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "g_{m-1}"'] \arrow[dl, "u^{i,j}_{m-1}"'] & P_{m-2} \arrow[r, "\partial_{m-2}"] \arrow[d, "g_{m-2}"'] \arrow[dl, "u^{i,j}_{m-2}"'] & P_{m-3} \arrow[r] \arrow[d, "g_{m-3}"'] \arrow[dl, "u^{i,j}_{m-3}"'] & \cdots \arrow[r] & P_{d+2} \arrow[r, "\partial_{d+2}"] \arrow[d, "g_{d+2}"'] & P_{d+1} \arrow[r, "\partial_{d+1}"] \arrow[d, "g_{d+1}"'] \arrow[dl, "u^{i,j}_{d+1}"'] & P_d \arrow[r, "\partial_d"] \arrow[d, "g_d"'] \arrow[dl, "u^{i,j}_{d}"'] \arrow[dl, phantom, shift left=3ex, "\circlearrowleft"] & P \arrow[dl, "u^{i,j}_{d-1}"'] \\
P_{m-d} \arrow[r, "\partial_{m-d}"'] & P_{m-d-1} \arrow[r, "\partial_{m-d-1}"'] & P_{m-d-2} \arrow[r, "\partial_{m-d-2}"'] & P_{m-d-3} \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r, "\partial_2"'] & P_1 \arrow[r, "\partial_1"'] & P_0 \arrow[r, "\epsilon"'] & {}
\end{tikzcd}
\tag{12}
$$

We assume now that P_* is minimal, that is, that $\partial_m(P_m) \leq \operatorname{Rad}(P_{m-1})$ for all $m \geq 1$. This implies that $\partial f = 0$ for *any* cochain f, that is, we have $\partial_{i+1} \circ f = 0$ for any kG-homomorphism $f : P_i \to k$.

The map $u^{i,j} \in Y^{d-1}$ is constructed as follows.

1. We take $u^{i,j}_{d-1} = 0$.
2. The assumption that $\left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle = g$ vanishes as an element of $H^d(Y)$ tells us that $\epsilon \circ g_d$ vanishes as an element of $H^d(G, k)$. But since P_* is *minimal*, this means that $\epsilon \circ g_d$ is actually the zero map. Then by projectivity of P_d, there exists $u^{i,j}_d$ such that $\partial_1 \circ u^{i,j}_d = (-1)^d g_d$. Observe that this means
$$\left(\partial u^{i,j}\right)_d = 0 - (-1)^{d-1} \partial_1 \circ u^{i,j}_d = g_d.$$
3. The map g is a cocycle by Lemma 8. This means that the *rectangles* in (12) either commute or anticommute, depending on whether d is even or odd. Thus,
$$\partial_1 \circ \left(g_{d+1} - u^{i,j}_d \circ \partial_{d+1}\right) = \partial_1 \circ g_{d+1} - (-1)^d g_d \circ \partial_{d+1} = 0$$
so that
$$\operatorname{im}\left(g_{d+1} - u^{i,j}_d \circ \partial_{d+1}\right) \leq \ker(\partial_1) = \operatorname{im}(\partial_2).$$
Thus, there exists $u^{i,j}_{d+1}$ such that
$$\partial_2 \circ u^{i,j}_{d+1} = (-1)^d \left(g_{d+1} - u^{i,j}_d \circ \partial_{d+1}\right).$$
Observe that this means
$$\left(\partial u^{i,j}\right)_{d+1} = u^{i,j}_d \circ \partial_{d+1} - (-1)^{d-1} \partial_2 \circ u^{i,j}_{d+1} = g_{d+1}.$$
4. Assume by recursion that we have constructed that maps $u^{i,j}_{m-2}$ and $u^{i,j}_{m-3}$ such that
$$\partial_{m-d-1} \circ u^{i,j}_{m-2} = (-1)^d \left(g_{m-2} - u^{i,j}_{m-3} \circ \partial_{m-2}\right).$$
Thus
$$\partial_{m-d-1} \circ \left(g_{m-1} - u^{i,j}_{m-2} \circ \partial_{m-1}\right) = \partial_{m-d-1} \circ g_{m-1} - (-1)^d g_{m-2} \circ \partial_{m-1} = 0$$

so that

$$\operatorname{im}\left(g_{m-1} - u^{i,j}_{m-2} \circ \partial_{m-1}\right) \leq \ker\left(\partial_{m-d-1}\right) = \operatorname{im}\left(\partial_{m-d}\right).$$

Thus, there exists $u^{i,j}_{m-1}$ such that

$$\partial_{m-d} \circ u^{i,j}_{m-1} = (-1)^{d}\left(g_{m-1} - u^{i,j}_{m-2} \circ \partial_{m-1}\right).$$

Observe that this means

$$\left(\partial u^{i,j}\right)_{m-1} = u^{i,j}_{m-2} \circ \partial_{m-1} - (-1)^{d-1}\,\partial_{m-d} \circ u^{i,j}_{m-1} = g_{m-1}.$$

This completes the construction of $u^{i,j}$. By construction, we have $\partial\left(u^{i,j}\right) = g$.

Finally, observe that in the last step in the calculation of $\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)}\right\rangle$, which is actually the *first* step, as this is a recursive process, it is only necessary to calculate $u^{1,k-1}$, but none of the maps $u^{1,m}$ for $2 \leq m \leq k-2$, and none of the maps $u^{m,k}$ for $2 \leq m \leq k-1$. In effect, the sum

$$\sum_{t=1}^{k-1} u^{t+1,k} \circ \overline{u^{1,t}} = \sum_{t=1}^{k-2} u^{t+1,k} \circ \overline{u^{1,t}} + f^{(k)} \circ \overline{u^{1,k-1}}$$

appearing in Definition 7 is calculated as

$$\sum_{t=1}^{k-2} \overset{!}{\boxed{u^{t+1,k}_{\deg u^{t+1,k}}}} \circ \overline{u^{1,t}_{\deg u^{t+1,k} + \deg u^{1,t}}} + f^{(k)}_{\deg f^{(k)}} \circ \overline{u^{1,k-1}_{\deg f^{(k)} + \deg u^{1,k-1}}},$$

But $u^{t+1,k}_{\deg u^{t+1,k}} = 0$ by construction (see Step 1 above), so the sum reduces to a single term. This is not the case with the intermediate maps $u^{i,j}$ with $j - i \leq k-2$.

References

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