April 14

Let $P$ be a parabolic subgroup of $G$, a finite group with split BN-pair. Let $P = L_I V_I$ be the Levi decomposition where $I \subset S$ is the corresponding set of generators. We defined the truncation

$$T^G_{L_I} \xi = t^G_{P_I/V_I} \xi$$

to be the character afforded by $\text{Inv}_{V_I} \{x \in X : vx = x\} = X^{V_I}$ and Harish-Chandra induction

$$R^G_{L_I} \lambda = \text{Ind}^G_P \tilde{\lambda}$$

where $\tilde{\lambda}$ is the lift of $\lambda \in \text{ch}(L_I)$. We also showed the following.

**Proposition 1 (Reciprocity)** Let $\xi \in \text{ch}(G)$ and $\lambda \in \text{ch}(L_I)$. Then

$$\langle \xi, R^G_{L_I} \lambda \rangle_G = \langle T^G_{L_I} \xi, \lambda \rangle_{L_I}.$$ 

Recall the following theorem from representation theory.

**Theorem 1 (Mackey)** Let $K, K \leq G$, let $\theta$ be a character of $H$, and let $\eta$ be a character of $K$. Write $\theta^x$ for the character of $H^x = x^{-1} H x$ defined $\theta^x(h^x) = \theta(x)$. Let $D$ be a set of double coset representatives for $H \backslash G / K$. Then

$$\left\langle \text{Ind}^G_H \theta, \text{Ind}^G_K \eta \right\rangle_G = \sum_{x \in D} \left\langle \theta^x |_{H^x \cap K}, \eta |_{H^x \cap K} \right\rangle_{H^x \cap K}.$$ 

This gives us the following formula.

**Theorem 2 (Mackey’s Formula)** Let $I, J \subset S$ and let $P_I = L_I V_I$ be the Levi decomposition. Let $\lambda \in \text{ch}(P_I)$ and $\mu \in \text{ch}(P_J)$. Then

$$\left\langle R^G_{L_I} \lambda, R^G_{L_J} \mu \right\rangle = \sum_{x \in D_{I,J}} \left\langle T^G_{L_K} \lambda, x (T^G_{L_K^x}) \mu \right\rangle_{L_K \cap K'}.$$ 

where $D_{I,J}$ is the set of distinguished double coset representatives of $P_I \backslash G / D_J$, $K = I \cap xJ$ and $K' = xK$.

**Proof.** (Sketch)

$$\text{LHS} = \left\langle \text{Ind}^G_{L_I} \tilde{\lambda}, \text{Ind}^G_{L_J} \tilde{\mu} \right\rangle = \sum_{x \in D} \left\langle \tilde{\lambda}^x |_{P_I \cap P_J}, \tilde{\mu} |_{P_I \cap P_J} \right\rangle_{P_I \cap P_J} = \sum_{x \in D} \left\langle \tilde{\lambda} |_{P_I \cap xP_J}, x \tilde{\mu} |_{P_I \cap xP_J} \right\rangle_{P_I \cap xP_J}.$$ 

We have from the third structure theorem that

$$P_I \cap xP_J = L_K (L_I \cap xV_J) (V_I \cap xL_J) (V_I \cap xV_J)$$

and

$$O_p(L_I \cap P_J).$$

Now for each fixed $x$, we have

$$\left\langle \tilde{\lambda}, x \tilde{\mu} \right\rangle = \frac{1}{|P_I \cap xP_J|} \sum_{lyz} \tilde{\lambda}(lyz) \overline{x \tilde{\mu}(lyz)}$$
for \( l \in L_K, v \in L^I \cap \mathcal{C} V_J, y \in V_I \cap \mathcal{C} L_J, \) and \( z \in V_I \cap \mathcal{C} V_J \) so the sum comes out to

\[
|L_K| |O_p(P_K \cap L_I)| |O_p(P_K \cap L_J)| |v_I \cap \mathcal{C} V_J|.
\]

We ultimately have

\[
lvzyz \equiv lv \pmod{V_I} \equiv lx \pmod{x V_J}\]

so that \( \bar{\lambda}(lvyz) = \lambda(lv) \) and \( ^{x}p(lvyz) = ^{x} \mu(lx y) \) and finally

\[
\langle \bar{\lambda}, \bar{\mu} \rangle = \langle T_{L_K}^{L_I} \lambda, T_{L_K}^{L_J} \mu \rangle.
\]

\[\blacksquare\]

**Definition 1** Let \( \xi \in \text{Irr}(G) \). We say that \( \xi \) is cuspidal if \( T_{L_I}^{G} \xi = 0 \) for all \( I \subsetneq S \).

**Proposition 2** The following are equivalent.

1. \( \xi \) is cuspidal.
2. For all \( I \subsetneq S \), we have \( \sum_{v \in V_I} \xi(vx) = 0 \) for all \( x \in L_I \).
3. \( \langle \xi, \text{Ind}^{G}_{V_I}(1) \rangle = 0 \) for all \( I \subsetneq S \).
4. \( \langle \xi, R_{L_I}^{G} \lambda \rangle = 0 \) for all \( I \subsetneq S \) and all \( \lambda \in \text{ch}(L_I) \)

The proof follows from reciprocity.

**Proposition 3** Let \( \xi \in \text{Irr}(G) \). Then either \( \xi \) is cuspidal or there exists \( I \subsetneq S \) such that

\[
\langle \xi, R_{L_I}^{G} \lambda \rangle > 0
\]

for some \( \lambda \in \text{Irr}(L_I) \).

**Proof.** Suppose \( \xi \) is not cuspidal. Then there exists a proper subset \( I \subsetneq S \) such that \( T_{L_I}^{G} \xi \neq 0 \). Choose a minimal \( I \subsetneq S \) satisfying the above condition. Then

\[
T_{L_I}^{G} \xi = \sum_{\lambda \in \text{Irr}(L_I)} \langle T_{L_I}^{G} \xi, \lambda \rangle_{L_I} \lambda = \sum_{\lambda} \langle \xi, R_{L_I}^{G} \lambda \rangle_{G} \lambda
\]

but at least summand \( \langle \xi, R_{L_I}^{G} \lambda \rangle > 0 \) for some \( \lambda \in \text{Irr}(L_I) \). \(\blacksquare\)

We claim now that if \( \lambda \) is as in the proposition, then \( \lambda \) is cuspidal. Suppose \( \lambda \) is not cuspidal. We have \( K \subsetneq I \) and \( T_{L_K}^{L_I} \lambda \neq 0 \). By transitivity, we have

\[
T_{L_K}^{G} \xi = T_{L_K}^{G} \langle T_{L_I}^{G} \xi \rangle = T_{L_K}^{L_I} \left( \sum_{\lambda \in \text{Irr}(L_I)} \langle \xi, R_{L_I}^{G} \lambda \rangle_{L_I} \lambda \right) = \sum_{\lambda} \langle \xi, R_{L_I}^{G} \lambda \rangle T_{L_K}^{L_I} (\lambda).
\]