Homework 1, Solutions

1. Let $V$ be a three-dimensional vector space over $\mathbb{C}$ with basis $\{v_1, v_2, v_3\}$ and let $G = S_3$ act on $V$ by permuting the indicies. Let $V_1 = \langle v_1 + v_2 + v_3 \rangle$ and $V_2 = \{\alpha v_1 + \beta v_2 + \gamma v_3 : \alpha + \beta + \gamma = 0 \rangle = \langle v_1 - v_2, v_2 - v_3 \rangle$

be $\mathbb{C}G$-submodules.

(a) Show that $V = V_1 \oplus V_2$.

(b) Show that $V_2$ is simple.

(c) Let $H = \langle (123) \rangle \leq G$ and let $G$ act on $G/H$ by multiplication. Show that $\mathbb{C}G/H$ is reducible as a $\mathbb{C}G$-module.

(d) Let $F$ be a field of characteristic 3 and let $V_1$ and $V_2$ be the $FG$-submodules of $V$ defined above. Do we still have $V = V_1 \oplus V_2$?

(a) \[
\begin{pmatrix}
1 & 1 & 0 \\
1 & -1 & 1 \\
1 & 0 & -1
\end{pmatrix}
\]

is invertible iff $\{v_1 + v_2 + v_3, v_1 - v_2, v_2 - v_3 \}$ is a basis for $V$ iff every element of $V$ can be expressed uniquely as a linear combination of $v_1 + v_2 + v_3, v_1 - v_2,$ and $v_2 - v_3$. If every element of $V$ can be expressed uniquely as $u_1 + u_2$ for $u_1 \in V_1 = \langle v_1 + v_2 + v_3 \rangle$ and $u_2 \in V_2 = \langle v_1 - v_2, v_2 - v_3 \rangle$ iff $V = V_1 \oplus V_2$.

(b) If $W$ is a non-zero submodule of $V_2$, then there must be some $w = \alpha v_1 + \beta v_2 + \gamma v_3 \in W$ with $w \neq 0$. Also, $\alpha + \beta + \gamma = 0$ as $W \subset V_2$. Now since $W$ is a $\mathbb{C}S_3$-submodule, we have that

\[
\begin{align*}
w - (12) \cdot w &= (\alpha - \beta) (v_1 - v_2) \\
w - (23) \cdot w &= (\beta - \gamma) (v_2 - v_3) \\
w - (13) \cdot w &= (\alpha - \gamma) (v_1 - v_3)
\end{align*}
\]

are all in $W$. If any two of the scalars $\alpha - \beta, \beta - \gamma, \alpha - \gamma$ are both zero, then $\alpha = \beta = \gamma$ so that $w \notin V_2$, a contradiction. We therefore have that two of the above scalars, say $\alpha - \beta$ and $\beta - \gamma$ nonzero. But this means that both $v_1 - v_2$ and $v_2 - v_3$ are in $W$ so that $W = V_2$. This shows that $V_2$ is irreducible.

(c) We have that $\langle H + (12)H \rangle$ is a $\mathbb{C}G$-module of $\mathbb{C}G/H$ as $g \cdot (H + (12)H) = H + (12)H$ for all $g \in G$. Since this submodule has dimension 1, $\mathbb{C}G/H$ is reducible.

(d) If $F$ is a field of characteristic 3, then $V_1 = \langle v_1 + v_2 + v_3 \rangle$ is a submodule of $V_2 = \{\alpha v_1 + \beta v_2 + \gamma v_3 : \alpha + \beta + \gamma = 0 \rangle$ so that $V \neq V_1 \oplus V_2$. Also note that the matrix given above has determinant 3.

2. Let $D$ be a division ring and let $A$ be the submodule of $M_n(D)$ of upper triangular matrices.

(a) Describe $J(A)$.
(b) Describe the decomposition of $A/J(A)$ as a direct sum of simple modules.

(a) Define $L_k = \{ A = (a_{i,j}) \in A : a_{k,k} = 0 \}$. To show that $L_k$ is an ideal, it suffices to show that $E_{i,j} A \in L_k$ for all $i \leq j$. But left multiplying $A$ by $E_{i,j}$ results in a matrix with the $j$th row of $A$ in the $i$th row and zeros elsewhere. Also, since $i < j$, we have, that this operation has the effect of moving rows upward and clobbering the rest so that the $k, k$ entry of $E_{i,j} A$ is 0.

To show that $L_k$ is a maximal ideal, let $B \in A \setminus L_k$. Then $(b_{k,k}) \neq 0$. But then $E_{k,k}^{-1} b_{k,k} B = E_{k,k} \in L_k$ so that $(L_k, B) = A$.

We therefore have that $\bigcap_{k=1}^{n} L_k = \{ A = (a_{i,j}) \in A : a_{k,k} = 0, 1 \leq k \leq n \}$.

However, since the righthand side is a nilpotent ideal, we have the reverse inclusion since $J(A)$ contains all nilpotent ideals.

Remark: The final answer is that $J$ is precisely the set of upper triangular matrices with 0 along the main diagonal. This is the right hand side above.

(b) We have that $A_1$ and $A_2$ will be in the same coset of $J(A)$ if and only if they have the same diagonal elements. Therefore, if we define

$$M_j = \{ \alpha_j E_{j,j} + J(A) : \alpha_j \in D \},$$

then $M_j$ is irreducible as $D$ is a division ring and

$$\mathcal{A}/(JA) = \bigoplus_{j=1}^{n} M_j.$$

Homework 2 Solutions
Added to Marcus’ Homework

Here are some general comments on the decomposition of $F[x]$-modules, $F$ a field. Let $V$ be a cyclic $F[x]$-module, i.e. $V$ has a basis $\{ v, xv, \ldots x^{n-1}v \}$ and the minimal polynomial $f(x)$ of $x$ (as a linear transformation of $V$) has degree equal to the dimension $n$ of $V$. Then in fact $V$ is isomorphic to $F[x]/(f(x))$ and the latter can be written as $\bigoplus_{i}^{n} F[x]/(p_i(x))^{r_i}$, where each factor $F[x]/(p_i(x))^{r_i}$ is cyclic and the $p_i(x)$ are distinct and irreducible. (Note that in general, if we have a decomposition of an $F[x]$-module according to the “rational form” as in DF, p.455, we have to apply the above to each cyclic factor.) Now consider a cyclic module $V = F[x]/(p(x)^r)$. Then $V$ is indecomposable as $F[x]$-module and is irreducible if $r = 1$. To see that such a $V$ is indecomposable, suppose $V = V_1 \oplus V_2$. Then on $V_1$ (resp. $V_2$), the minimal polynomial of $x$ would be of the form $p(x)^k$ (resp. $p(x)^l$) where $k, l < r$ and then the minimal polynomial on $V$ would have degree less than $deg(p).r$. Similarly if $r = 1$ and $V$ had a proper submodule $V_1$ (not zero) then the minimal
polynomial of \( x \) on \( V_1 \) would be a factor of \( p(x) \) of smaller degree, but \( p(x) \) is irreducible. We can apply these arguments to the problems below. In Problem 1 the modules are not cyclic, but we apply the above criteria to each cyclic factor. In Problem 2 the module \( Q[x]/(x^n - 1) \) is cyclic.

1. Let \( A = \mathbb{Q}[x] \), \( V \) an \( A \)-module where \( xv = Tv \) for \( v \in V \) with \( T \) the linear transformation given by

\[
\begin{pmatrix}
2 & -2 & 14 \\
0 & 3 & -7 \\
0 & 0 & 2
\end{pmatrix}
\]

with respect to the basis \( \{e_1, e_2, e_3\} \) of \( V \). Show that \( V \) is completely reducible. Describe \( V \) as a direct sum of simple modules.

(b) \[
\begin{pmatrix}
1 & 2 & -4 & 4 \\
2 & -1 & 4 & -8 \\
1 & 0 & 1 & -2 \\
0 & 1 & -2 & 3
\end{pmatrix}
\]

with respect to the basis \( \{e_1, e_2, e_3, e_4\} \) of \( V \). Show that \( V \) is not completely reducible. Describe \( V \) as the direct sum of two indecomposable submodules and give a composition series for \( V \).

(a) We have that the Jordan form of \( T \) is \( J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \) with \( S = \begin{pmatrix} 2 & -2 & 0 \\ 7 & 1 & 7 \\ 1 & 0 & 1 \end{pmatrix} \) such that \( J = S^{-1}TS \). Then with respect to the basis

\[
\left\{ f_1 = 2e_1 + 7e_2 + e_3, f_2 = -2e_1 + e_2, f_3 = 7e_2 + e_3 \right\}
\]

of \( V \), \( T \) has matrix \( J \). This means that \( \langle f_1 \rangle, \langle f_2 \rangle, \langle f_3 \rangle \) are \( A \)-submodules of \( V \) since these are the \( J \)-stable \( \mathbb{Q} \)-subspaces of \( V \). Also,

\[
V = \langle f_1 \rangle \oplus \langle f_2 \rangle \oplus \langle f_3 \rangle
\]

since the \( f_j \) are the columns of \( S \), which is invertible.

(b) The Jordan form of \( T \) is \( R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \). Letting \( \langle f_1, f_2, f_3, f_4 \rangle \) be the columns of the conjugating matrix as before, we have that \( V_1 = \langle f_1, f_2 \rangle \) and \( V_2 = \langle f_3, f_4 \rangle \) are \( A \)-submodules of \( V \) which have irreducible sub-submodules \( \langle f_1 \rangle \) and \( \langle f_3 \rangle \) respectively. Then \( V_1 \) and \( V_2 \) are indecomposable (by the general arguments given above, taking \( p(x) = x - 1 \) and noting that \( V_1 \) and \( V_2 \) are isomorphic to \( \mathbb{Q}[x]/(x - 1)^2 \) and are cyclic) and \( V = V_1 \oplus V_2 \).

A composition series for \( V \) is

\[
V = \langle f_1, f_2, f_3, f_4 \rangle \geq \langle f_1, f_2, f_3 \rangle \geq \langle f_1, f_2 \rangle \geq \langle f_1 \rangle \geq 0.
\]

2. Let \( G = \langle a : a^n = 1 \rangle \) be the cyclic group of order \( n \) and let \( R = \mathbb{Q}G \). Describe the Wedderburn decomposition of \( R \) and find the number and the degrees of the irreducible representations of \( G \) over \( \mathbb{Q} \). In particular, show that if \( n = p \) is a prime, then \( G \) has exactly one nontrivial irreducible representation over \( \mathbb{Q} \) and this representation has degree \( p - 1 \).
The map \( \mathbb{Q}[x] \to \mathbb{Q}G \) defined \( x^j \to a^j \) is a surjective ring homomorphism with kernel 
\((1 - x^n)\) so that
\[
\mathbb{Q}[x]/(1 - x^n) \cong \mathbb{Q}G.
\]
Next, the polynomial \( 1 - x^n \in \mathbb{Q}[x] \) factors as
\[
1 - x^n = \prod_{d|n} \Phi_d(x)
\]
where \( \Phi_d(x) \) is the \( d \)th cyclotomic polynomial. Then by the Chinese Remainder Theorem, since the \( (\Phi_d(x)) \) are comaximal, \( \Phi_d(x) \) being irreducible, we have
\[
\mathbb{Q}G \cong \mathbb{Q}[x]/(1 - x^n) \cong \prod_{d|n} \mathbb{Q}[x]/(\Phi_d(x))
\]
as rings. Note that as a \( \mathbb{Q}[x] \) module, \( \mathbb{Q}[x]/(\Phi_d(x)) \) is irreducible, again by the general arguments. Note also that \( \mathbb{Q}[x]/(\Phi_d(x)) \) has dimension \( \varphi(d) \). Therefore,
\[
\mathbb{Q}G \cong \bigoplus_{d|n} \mathbb{Q}[x]/(\Phi_d(x))
\]
as modules so that
\[
\mathbb{Q}G \cong \bigoplus_{d|n} M_1(D_d)
\]
where \( D_d = \mathbb{Q}(\zeta) \) where \( \zeta \) is a primitive \( d \)-th root of unity.

**Homework 3 Solutions, added to Marcus’**

*Let \( G = S_3 \) and \( A = \mathbb{Q}G \).*

1. **Find the central idempotents** \( e_1, e_2 \) and \( e_3 \) such that \( A = Ae_1 \oplus Ae_2 \oplus Ae_3 \) where
   the \( Ae_j \) are ideals which are simple algebras with \( \dim Ae_1 = \dim Ae_2 = 1 \) and
   \( \dim Ae_3 = 4 \).

2. **Find orthogonal idempotents (not necessarily central)** \( f_1, f_2, f_3 \) and \( f_4 \) such that
   \( A = \bigoplus_{j=1}^4 Af_j \) where the \( Af_j \) are minimal left ideals (simple \( A \)-modules).

3. **Let \( p = 2 \).** We consider two modular representations of \( G \). Let \( R \subseteq \mathbb{Q} \) be given
   by \( R = \{ a/b : a, b \in \mathbb{Z}, 2 \nmid b \} \). Then \( R \) is a local ring and \( R/(2) \cong \mathbb{F}_2 \). Let
   \( \bar{A} = RG \) and \( \bar{A} = \mathbb{F}_2G \). Using the decomposition in (1), write \( \bar{A} = \bar{A}E_1 \oplus \bar{A}E_2 \) where \( E_1 \)
   and \( E_2 \) are central idempotents. It will not be possible to reduce \( e_1, e_2, \) and \( e_3 \mod 2 \)
   as some of them will not lie in \( R \). So you will have to combine them into two
   idempotents. Finally, write \( \bar{A} = \bar{A}E_1 \oplus \bar{A}E_2 \) after reducing mod 2.

4. **Try doing the same procedure as in (3) for \( p = 3 \).** Here \( R = \{ a/b : a, b \in \mathbb{Z}, 3 \nmid b \} \).
   Can you decompose \( \bar{A} = RG \)?
1. Let $e_1 = \frac{1}{6} \sum g \in G g$. Then

$$e_1^2 = \left( \frac{1}{6} \sum g \right) \left( \frac{1}{6} \sum g \right) = \frac{1}{36} \sum_g \left( g \sum g \right) = \frac{1}{36} \sum_g \left( \sum g \right) = \frac{1}{6} \left( \sum g \right)$$

so that $e_1$ is idempotent. Also,

$$\left( \sum g \alpha g \right) \left( \frac{1}{6} \sum g \right) = \frac{1}{6} \sum_g \alpha_g \left( \sum g \right) = \frac{1}{6} \sum_g \alpha_g \left( \sum g \right)$$

so that $e_1$ is central. Let $e_2 = \frac{1}{6} \sum (-1)^{\text{sgn} g} g$. We similarly compute that $e_2$ is a central idempotent. We define $e_3 = 1 - e_1 - e_2 = e_3$ and confirm directly that $e_3$ is central.

$Ae_1$ and $Ae_2$ are 1-dimensional, since, as suggested in (1), $ae_1 = \alpha e_1$ for some $\alpha \in \mathbb{Q}$ and similarly for $e_2$. We confirm that $Ae_3$ is 4-dimensional by explicitly computing its elements and confirming independence of four of them. Alternately, GAP verifies this claim and everything stated above as follows.

```gap
G:=SymmetricGroup(3);A:=GroupRing(Rationals,G);
Sym( [ 1 .. 3 ] )
gap> e:=CentralIdempotentsOfAlgebra(A);
[ (1/6)*()+(-1/6)*(2,3)+(-1/6)*(1,2)+(1/6)*(1,2,3)+(1/6)*(1,3,2)+(-1/6)*(1,3),
  (1/6)*()+(1/6)*(2,3)+(1/6)*(1,2)+(1/6)*(1,2,3)+(1/6)*(1,3,2)+(1/6)*(1,3),
  (2/3)*()+(-1/3)*(1,2,3)+(-1/3)*(1,3,2) ]
1
1
4
```

2. To find $f_3$ and $f_4$, we consider the representation $\rho$ of $\mathbb{Q}G$ given by

$$(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (123) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

and take the inverse images of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We begin with the following. The newlines were added to this document for legibility. A.1 and A.2 give the first and second generators of $A$ which are then printed to the standard output.

```gap
s:=A.1;t:=A.2;
(1)*(1,2,3)
(1)*(1,2)
gap> S:=[[0,-1],[1,-1]];T:=[[1,-1],[0,1]];
[ [ 0 , -1 ] , [ 1 , -1 ] ]
[ [ -1 , 1 ] , [ 0 , 1 ] ]
```

5
The next command defines \( f \) to be the algebra homomorphism given by the images of the generators in 2.

\[
\text{gap> } f:=\text{AlgebraHomomorphismByImages(}
\quad A,\text{FullMatrixAlgebra(Rationals,2)},{[t,s]},{[T,S]});\]
\[
\begin{bmatrix}
(1)*(1,2), (1)*(1,2,3) \rightarrow \\
[ [ [ -1, 1 ], [ 0, 1 ] ], [ [ 0, -1 ], [ 1, -1 ] ] ]
\end{bmatrix}
\]
\[
v:=\text{PreImagesRepresentative}(f,[[1,0],[0,0]]);
\]
\[
(1/3)*()+(1/3)*(2,3)+(-2/3)*(1,2)+(-1/3)*(1,2,3)
\]
\[
\text{gap> } \text{Dimension}(A*v);
\]
\[
4
\]

It is somewhat disturbing that GAP’s preimage generates a subalgebra of dimension 4, but we consider the possibility that the GAP has returned the sum of the preimage of \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and something in the kernel of \( \rho \) (note that \( e_1, e_2 \in \ker \rho \)). We experiment with \( e_1 \) and \( e_2 \) to find \( f_3 \) and \( f_4 \) as follows.

\[
\text{gap> } f3:=v-e[1]+e[2];
\]
\[
(1/3)*()+(1/3)*(1,2)+(-1/3)*(1,2,3)+(1/3)*(1,3)
\]
\[
\text{gap> } \text{Dimension}(A*f3);
\]
\[
2
\]
\[
\text{gap> } f3^2=f3;
\]
\[
\text{true}
\]
\[
\text{gap> } f4:=e[3]-f3;
\]
\[
(1/3)*()+(1/3)*(1,2)+(-1/3)*(1,3,2)+(-1/3)*(1,3)
\]
\[
\text{gap> } \text{Dimension}(A*f4);
\]
\[
2
\]
\[
\text{gap> } f4^2=f4;
\]
\[
\text{true}
\]

Addendum (B.S):

Thus, one solution for \( f_3 \) and \( f_4 \) is given by:

\[
f_3 = (1/3)[1 - (1,2) + (1,2,3) - (1,3)], \quad f_4 = e_3 - f_3.
\]

Other solutions for \( f_3 \) might be obtained by observing that the representation \( \rho \) of \( \mathbb{Q}G \) is given by:

\[
(12) \rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, (123) \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, (13) \rightarrow \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, (132) \rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix},
\]

\[
(23) \rightarrow \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.
\]

Thus by inspection we see easily that \( -(13) - (123) - (132) \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \), and so we could take \( f_3 = -1/2[(13) + (123) + (132)] \).

3. We have

\[
E_2 := e_3 = 1 - e_1 - e_2 = \frac{2}{3} - \frac{1}{3} (123) - \frac{1}{3} (132) \in R
\]
and observe that
\[ E_1 = e_1 + e_2 = \frac{1}{3} + \frac{1}{3} (123) + \frac{1}{3} (132) \in R \]
and that \( E_1 \) is idempotent by FOIL and orthogonality. As in (1), \( \tilde{A}E_2 \) is irreducible and of dimension 4, the only difference being that \( R \) is a subring of \( \mathbb{Q} \).

Then \( \tilde{A} = \tilde{A}E_1 \oplus \tilde{A}E_2 \). Reducing both sides by the ideal generated by \( 2 \cdot 1 \), which has the effect of reducing all the coefficients modulo the ideal (2), we have 
\[ A = AE_1 \oplus AE_2. \]

However, if we replace 2 by 3 we find that the \( R \)-algebra \( A \) is not decomposable into a sum of proper two-sided ideals.

Homework 4

Let \( G = SL(2,3) \). You are given that the representatives of the conjugacy classes of \( G \) are \( \{g_j : 1 \leq j \leq 7\} \) where 
\[ g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ g_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ g_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ g_5 = g_4^{-1}, \ g_6 = g_2 g_4, \text{ and } g_7 = g_2 g_5. \]
Fill in the character table by the following steps.

1. (Projective Geometry) \( G/Z = PSL(2,3) \) permutes the 4 points of the projective line over \( \mathbb{F}_3 \) and is isomorphic to \( A_4 \). Use this to fill in \( \chi_2, \chi_3, \text{ and } \chi_4 \).

2. Find the other character degrees and the values on \( Z \).

3. Of the three characters \( \chi_5, \chi_6, \text{ and } \chi_7 \), at least one has values in \( \mathbb{R} \) since complex conjugate characters come in pairs. Suppose this is \( \chi_5 \) with corresponding representation \( \rho_5 \). Consider the eigenvalues of \( \rho_5(g_i) \) and use \( \chi_5(g_4) \in \mathbb{R} \) to find \( \chi_5(g_4) \). Then find \( \chi_5(g_5), \chi_5(g_6) \) and \( \chi_5(g_7) \).

4. Tensoring with suitable linear characters gives \( \chi_6 \text{ and } \chi_7 \) on \( g_i \) for \( 4 \leq i \leq 7 \).

5. Finally, find \( \chi_5(g_3), \chi_6(g_3) \) and \( \chi_7(g_3) \) by orthogonality.

1. Given that 
\[ \varphi : GL(2,3)/Z = PSL(2,3) \xrightarrow{\sim} A_4 \]
is an isomorphism for some map \( \varphi \), we have that \( |Z| = |GL(2,3)|/|A_4| = 2 \) so that \( Z = \{g_1, g_2\} \) since \( g_1 \) and \( g_2 \) must be central in \( SL(2,3) \), being central in all of \( M_2(\mathbb{F}_3) \).

This means that elements are in the same coset of the quotient if they differ by \( g_2 \). Therefore, the conjugacy classes represented by \( g_4 Z \) and \( g_6 Z = g_2 g_4 Z \) must collapse under \( \varphi \) to the same conjugacy class and this class must either be (123) or (124) since \( |\varphi(g_4 Z)| \) must divide \( |g_4 Z| = 3 \) and isn’t 1 because \( \varphi \) is injective. Similarly, \( g_5 Z \) and \( g_7 Z \) must collapse to the conjugacy class represented by the other three cycle. By elimination, we have that the conjugacy class of \( g_3 Z \) corresponds with
the conjugacy class of (12) (34) and observe that these classes are in bijection unlike the others.

We therefore lift the characters of $A_4$ to characters of $G$ as follows

$$
\begin{array}{c|cccccc}
SL(2,3) & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \\
\hline
\chi_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & 1 & \omega & \overline{\omega} & \omega & \overline{\omega} \\
\chi_3 & 1 & 1 & 1 & \overline{\omega} & \omega & \omega & \overline{\omega} \\
\chi_4 & 3 & 3 & -1 & 0 & 0 & 0 & 0 \\
\end{array}
$$

2. Let $n_j = \dim \chi_j$. Then since $\sum_{j=1}^7 n_j^2 = 24$, we have $\sum_{j=5}^7 n_j^2 = 12$. The only squares $\leq 12$ are 1, 2, and 9. If any $n_j^2 = 9$, then the sum of the squares of the other two must be 4, but 4 is not the sum of two squares. Similarly $n_j^2 \neq 1$ for any $j$ so that $4 = n_5^2 = n_6^2 = n_7^2 = \chi_5 (g_1) = \chi_6 (g_1) = \chi_7 (g_1)$.

Since $g_2 \in Z$, we have that $\rho_j (g_2)$ must be a diagonal matrix for $5 \leq j \leq 7$. The diagonal entries of $\rho_j (g_2)$ are $\pm 1$. The trace of such a matrix is therefore $-2, 0,$ or $2$.

However, we must have $\chi_5 (g_2) = \chi_6 (g_2) = \chi_7 (g_2) = -2$ since this is the only way that

$$0 = \langle g_1, g_2 \rangle = 1^2 + 1^2 + 1^2 + 3^2 + 2\chi_5 (g_2) + 2\chi_6 (g_2) + 2\chi_7 (g_2)$$

will be satisfied. We therefore have the following situation.

$$
\begin{array}{c|cccccc}
SL(2,3) & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \\
\hline
\chi_5 & 2 & -2 & & & & & \\
\chi_6 & 2 & -2 & & & & & \\
\chi_7 & 2 & -2 & & & & & \\
\end{array}
$$

3. Assume $\chi_5$ is an irreducible real-valued character. We have $|\rho_5 (g_4)|$ divides $|g_4| = 3$ so that $|\rho_5 (g_4)| = 1$ or 3. If $|\rho_5 (g_4)| = 1$, then $g_4 \in \ker \rho_5 < G$, but the only normal subgroup of $A_4$ has order 4. This subgroup pulls back under $\pi \circ \varphi$ to a normal subgroup of order 8, but $g_4$ cannot be in this subgroup, having order 3.

Therefore $|\rho_5 (g_4)| = 3$. Let $\varepsilon_1$ and $\varepsilon_2$ be its eigenvalues with eigenvectors $v_1$ and $v_2$. Then $v_j = (\rho_5 (g_4))^3 v_j = \varepsilon_j^3 v_j$ so that $\varepsilon_j^3 = 1$. We have agreed that $\chi_5 (g_4) = \varepsilon_1 + \varepsilon_2 \in \mathbb{R}$. But for $\{\varepsilon_1, \varepsilon_2\} \subset \{1, \omega, \overline{\omega}\}$, and $\varepsilon_1 + \varepsilon_2 \in \mathbb{R}$, we must have either have $\varepsilon_1 = \varepsilon_2 = 1$ or $\{\varepsilon_1, \varepsilon_2\} = \{\omega, \overline{\omega}\}$. But the first case is impossible since we have seen that $\rho_5 (g_4) \neq I$. This means we have the latter case and $\chi_5 (g_4) = -1$. Similarly, since $|\rho_5 (g_5)| = 3$, we have $\chi_5 (g_5) = -1$.

We have observed above that $\rho_5 (g_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (with respect to any basis). This means that $\rho_5 (g_6) v_j = \rho_5 (g_2) \rho_5 (g_4) v_j = \rho_5 (g_2) \varepsilon_j v_j = -\varepsilon_j v_j$ so that $\chi_5 (g_6) = 1$ and similarly for $g_7$.

$$
\begin{array}{c|cccccc}
SL(2,3) & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \\
\hline
\chi_5 & 2 & -2 & -1 & -1 & 1 & 1 & \\
\end{array}
$$
4. We provisionally put $\chi_6 = \chi_5 \otimes \chi_2$ and $\chi_7 = \chi_5 \otimes \chi_3$.

\[
\begin{array}{cccccccc}
SL(2, 3) & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \\
\hline
\chi_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & 1 & 1 & \omega & \omega & \omega \\
\chi_3 & 1 & 1 & 1 & \omega & \omega & \omega & \omega \\
\chi_4 & 3 & 3 & -1 & 0 & 0 & 0 & 0 \\
\chi_5 & 2 & -2 & -1 & -1 & 1 & 1 & \\
\chi_6 & 2 & -2 & -\omega & -\omega & \omega & \omega & \\
\chi_7 & 2 & -2 & -\omega & -\omega & \omega & \omega & \\
\end{array}
\]

5. To use the first orthogonality relation, we need to compute the class sizes. We have $|C_1| = |C_2| = 1$ since $g_1$ and $g_2$ are central. We saw that $C_3$ corresponds under the isomorphism $\varphi$ to the conjugacy class of $A_4$ containing $(12)(34)$ which has 3 members. Therefore, $|C_3| = 6$. Finally, we have $|C_j| = 4$ for $4 \leq j \leq 7$ by the second orthogonality relation. For example,

$$|C_5| = 1 + 1 + 1 + 0 + 1 - 1 - 1 = 4.$$ 

Since we have assumed $\chi_5$ to be an irreducible character, we can use the first orthogonality relation to compute its remaining element

$$4 + 4 + 6\chi_5(g_3)\chi_5(g_3) + 4 + 4 + 4 = |G| = 24$$

giving $\chi_5(g_3) = 0$. Finally, $\chi_6(g_3) = \chi_5(g_2)\chi_2(g_2) = 0$ and similarly for $\chi_7(g_3)$ giving the final character table.

\[
\begin{array}{cccccccc}
SL(2, 3) & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \\
\hline
\chi_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & 1 & \omega & \omega & \omega & \omega \\
\chi_3 & 1 & 1 & 1 & \omega & \omega & \omega & \omega \\
\chi_4 & 3 & 3 & -1 & 0 & 0 & 0 & 0 \\
\chi_5 & 2 & -2 & 0 & -1 & -1 & 1 & 1 \\
\chi_6 & 2 & -2 & 0 & -\omega & -\omega & \omega & \omega \\
\chi_7 & 2 & -2 & 0 & -\omega & -\omega & \omega & \omega \\
\end{array}
\]

It remains to confirm that $\chi_6$ and $\chi_7$ are irreducible by checking that $\langle \chi_6, \chi_6 \rangle = \langle \chi_7, \chi_7 \rangle = 1$.

**Homework 5**
0.1 Introduction

We compute the conjugacy classes of $G$ by enumerating all the permissible rational canonical forms, we compute the orders of the representatives, and we match the representatives with the columns of the given table by order. We have the following table.

<table>
<thead>
<tr>
<th>Pseudonym</th>
<th>$g_1$</th>
<th>$g_2$</th>
<th>$g_3$</th>
<th>$g_4$</th>
<th>$g_5$</th>
<th>$g_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order of element</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Class size</td>
<td>1</td>
<td>21</td>
<td>56</td>
<td>42</td>
<td>24</td>
<td>24</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Character</th>
<th>$\chi_1$</th>
<th>$\chi_2$</th>
<th>$\chi_3$</th>
<th>$\chi_4$</th>
<th>$\chi_5$</th>
<th>$\chi_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>7</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>8</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>$-1+i\sqrt{7}$</td>
<td>$-1-i\sqrt{7}$</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>$-1-i\sqrt{7}$</td>
<td>$-1+i\sqrt{7}$</td>
</tr>
</tbody>
</table>

0.2 Stabilizer of Lines

Let $V$ be a vector space over $\mathbb{F}_2$ with basis $\{v_1, v_2, v_3\}$. Then $G$ acts transitively on the set of one-dimensional subspaces of $V$. We count that $V$ has 7 one-dimensional subspaces $\{\langle \alpha v_1 + \beta v_2 + \gamma v_3 \rangle : \alpha, \beta, \gamma \in \mathbb{F}_2\}$.

With respect to this basis, the subgroup $P = \begin{pmatrix} ♣ & ♣ & ♣ \\ 0 & ♣ & ♣ \\ ♣ & 0 & ♣ \end{pmatrix} \leq G$ is the stabilizer of the line $\langle v_1 \rangle$. Therefore, elements in the same coset of $G/P$ act equivalently on $\langle v_1 \rangle$. Since $G$ acts transitively on the set of lines, there should be one coset for each line. $|G|/|P| = 7$ confirms this claim.

Let $\{x_j : 1 \leq j \leq 7\}$ be coset representatives of $G/P$ and let 1 denote the trivial character of $G/P$. We will use the following observation to compute

$$\text{Ind}_{G/P}^G (1) = \sum_{j=1}^{7} 1(x_j^{-1}gx_j).$$

We have $x_j^{-1}gx_j \in P$ iff $gx_j \in x_jP$ iff $gx_jP = x_jP$. Therefore, $\text{Ind}_{G/P}^G (1)$ counts 1 for each coset $x_jP$ that $g$ fixes. But since the elements of the coset $x_jP$ act equivalently on $\langle v_1 \rangle$, we can regard $x_jP$ as the well defined line $x_j \langle v_1 \rangle$ so that $\text{Ind}_{G/P}^G (1)$ is the number of lines $g$ fixes.

We directly compute that $g_2$ fixes the lines $\langle v_1 \rangle$, $\langle v_2 + v_3 \rangle$, and $\langle v_1 + v_2 + v_3 \rangle$, $g_3$ fixes the line $\langle v_1 + v_2 + v_3 \rangle$, and $g_4$ fixes the line $\langle v_1 + v_3 \rangle$. We therefore have the following character.

<table>
<thead>
<tr>
<th>Ind$_{G/P}^G (1)$</th>
<th>\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 0 \end{pmatrix}</th>
<th>\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}</th>
<th>\begin{pmatrix} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 1 \end{pmatrix}</th>
<th>\begin{pmatrix} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 1 \end{pmatrix}</th>
<th>\begin{pmatrix} 0 &amp; 0 &amp; 1 \ 1 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Taking inner products, we have

$$\text{Ind}_{G/P}^G (1) = \chi_1 + \chi_2$$

giving us $\chi_2$. 

10
0.3 Stabilizer of Flags

$G$ also acts transitively on the set of complete flags. Let $B = \left\{ \begin{pmatrix}
\ast & \ast & \ast \\
0 & 0 & 1
\end{pmatrix} \right\}$ be the stabilizer of the complete flag $\langle v_1 \rangle \leq \langle v_1, v_2 \rangle$. Then by the same argument used in section 0.2, $\text{Ind}_B^G(1)(g)$ is the number of flags fixed by $g$.

For each one-dimensional subspace of $V$ enumerated in section 0.2, there are three two-dimensional subspaces of $V$ containing it, making a total of 21 complete flags. Thus, $\text{Ind}_B^G(1)(g_1) = 21$.

To count flags fixed by $g_2 = \begin{pmatrix} 0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}$, we recall from section 0.2 that the only one-dimensional subspaces of $V$ fixed by $g_2$ are $\langle v_1 \rangle$, $\langle v_2 + v_3 \rangle$, and $\langle v_1 + v_2 + v_3 \rangle$. By enumerating the two-dimensional subspaces containing these, we see that $g_2$ fixes the following flags

$\langle v_1 \rangle \leq \langle v_1, v_2 + v_3 \rangle$

$\langle v_2 + v_3 \rangle \leq \langle v_2, v_3 \rangle$

$\langle v_2 + v_3 \rangle \leq \langle v_1, v_2 + v_3 \rangle$

$\langle v_1 + v_2 + v_3 \rangle \leq \langle v_1, v_1 + v_2 + v_3 \rangle$

and $g_4$ fixes the flag $\langle v_1 + v_3 \rangle \leq \langle v_1 + v_3, v_2 + v_3 \rangle$ so that we have the following character.

| $\text{Ind}_B^G(1)$ | $\begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}$ |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Taking inner products, we see that $\text{Ind}_B^G(1) = \chi_1 + 2\chi_2 + \chi_4$ giving us $\chi_4$.

0.4 The Parabolic Subgroup

Consider the subgroup

$P = \left\{ \begin{pmatrix}
\ast & \ast & \ast \\
0 & 0 & 1
\end{pmatrix} \right\} \supset \left\{ \begin{pmatrix}
\ast & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \right\} := L \cong GL_2(F_2) \times GL_1(F_2) \cong S_3$.

Since $P = UL$ for $U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \right\} \triangleleft P$, we will lift a character from $L \cong P/U$ up to $P$ and induce from this a character of $G$.

Again, using the rational form, we compute representatives of the conjugacy classes of $GL_2(F_2)$ to be $\begin{pmatrix} 1 & 0 \\
1 & 0
\end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\
0 & 0
\end{pmatrix}$, and $\begin{pmatrix} 1 & 1 \\
0 & 1
\end{pmatrix}$ so that representatives of the conjugacy classes of $L$ are $\begin{pmatrix} 1 & 0 \\
0 & 1
\end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\
0 & 0
\end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\
0 & 1
\end{pmatrix}$. The classes in $L$ represented by these three elements do not merge in $P$ since these elements have different orders.
We take the following character from the character table for $S_3$.

$$
\begin{array}{ccc}
L & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\varepsilon & 1 & -1 & 1 \\
\end{array}
$$

We will compute

$$\text{Ind}^G_P \varepsilon (g_2) = |C_G(g_2)| \left( \sum_{j=1}^{s} \frac{1}{|C_P(h_j)|} \varepsilon(h_j) \right)$$

where $h_j$ are the representatives of the conjugacy classes in $P$ whose union is the conjugacy class of $g_2$ in $G$.

The GAP program given in the appendix gives the following conjugacy classes of $P$.

```plaintext
gap> CCC;
[[[0, 1, 0], [1, 0, 0], [0, 0, 1]],
 [[0, 1, 1], [1, 1, 0], [0, 0, 1]],
 [[1, 0, 0], [1, 1, 0], [0, 0, 1]],
 [[1, 1, 0], [1, 0, 0], [0, 0, 1]],
 [[1, 1, 0], [1, 0, 1], [0, 0, 1]],
 [[1, 1, 1], [1, 0, 0], [0, 0, 1]],
 [[1, 1, 1], [1, 0, 1], [0, 0, 1]],
 [[1, 0, 0], [0, 1, 0], [0, 0, 1]],
 [[1, 0, 1], [0, 1, 0], [0, 0, 1]],
 [[1, 0, 1], [0, 1, 1], [0, 0, 1]],
 [[0, 1, 0], [1, 1, 0], [0, 0, 1]],
 [[0, 1, 1], [1, 1, 1], [0, 0, 1]],
 [[0, 1, 1], [1, 1, 1], [0, 0, 1]],
 [[0, 1, 0], [1, 1, 0], [0, 0, 1]],
 [[0, 1, 1], [1, 1, 1], [0, 0, 1]],
 [[0, 1, 1], [1, 1, 1], [0, 0, 1]],
 [[1, 0, 0], [1, 1, 0], [0, 0, 1]],
 [[1, 0, 1], [1, 1, 0], [0, 0, 1]],
 [[1, 0, 1], [1, 1, 1], [0, 0, 1]],
 [[1, 0, 1], [1, 1, 1], [0, 0, 1]],
 [[1, 0, 0], [1, 0, 0], [0, 0, 1]],
 [[1, 0, 1], [1, 0, 1], [0, 0, 1]],
 [[1, 0, 1], [1, 0, 1], [0, 0, 1]],
 [[1, 0, 0], [1, 0, 1], [0, 0, 1]],
 [[1, 0, 1], [1, 0, 1], [0, 0, 1]],
]
```

Since the factorization $\begin{pmatrix} a & b \\ d & e \\ f \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} a & b \\ d & e \\ f \end{pmatrix} \in UL$ is transparent, we can easily see which elements are in which coset of $G/P$.

We observe that the first and the last conjugacy class listed contain elements of order 2. Since only one class of $G$ contains elements of order 2, namely the class containing $g_2$ at which we’re trying to compute the character, we must have that the class of $g_2$ splits in $P$ into the first and last classes above. The first class contains a transposition, so $\varepsilon$ must be $-1$ on that class, and the last class is the identity coset, so $\varepsilon$ must be 1 on that class. Therefore,

$$\text{Ind}^G_P \varepsilon (g_2) = \frac{1}{8} \left( \frac{1}{4}(-1) + \frac{1}{8}(1) \right) = -1.$$
Similarly for the other values giving the following character.

<table>
<thead>
<tr>
<th>[\begin{array}{ccc} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{array} ]</th>
<th>[\begin{array}{ccc} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{array} ]</th>
<th>[\begin{array}{ccc} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 1 \end{array} ]</th>
<th>[\begin{array}{ccc} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 0 \end{array} ]</th>
<th>[\begin{array}{ccc} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 1 \end{array} ]</th>
<th>[\begin{array}{ccc} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 0 \end{array} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Ind}_{\psi_i}^G(1))</td>
<td>7</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

We confirm that this character is irreducible by taking inner products, or alternately, we notice that this character is \(\chi_3\) from the given table.

0.5 The 7-Subgroup

Next, consider the Sylow 7-subgroup \(S = \langle g_5 \rangle\). We will induce the following characters.

<table>
<thead>
<tr>
<th>(S)</th>
<th>(g_1)</th>
<th>(g_5)</th>
<th>(g_5^2)</th>
<th>(g_5^3)</th>
<th>(g_5^4)</th>
<th>(g_5^6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\psi_1)</td>
<td>1</td>
<td>(\omega)</td>
<td>(\omega^2)</td>
<td>(\omega^3)</td>
<td>(\omega^4)</td>
<td>(\omega^5)</td>
</tr>
<tr>
<td>(\psi_2)</td>
<td>1</td>
<td>(\omega^3)</td>
<td>(\omega^6)</td>
<td>(\omega^2)</td>
<td>(\omega^5)</td>
<td>(\omega)</td>
</tr>
</tbody>
</table>

We determine which conjugacy classes of the powers of \(g_5\) lie in by computing their rational forms. Gap gives the characteristic polynomials for the powers of \(g_5\) below.

\[
\text{gap> g5} := \text{[[0,0,1],[1,0,0],[0,1,1]]} \\
\text{[ [ 0, 0, 1 ], [ 1, 0, 0 ], [ 0, 1, 1 ] ]} \\
\text{gap> for j in [0..6] do Print(j,": \text{CharacteristicPolynomial(g5^j) mod 2,\"\n")$}
\]

\[
0: 1+x_1+x_1^2+x_1^3 \\
1: 1+x_1^2+x_1^3 \\
2: 1+x_1^2+x_1^3 \\
3: 1+x_1^2+x_1^3 \\
4: 1+x_1^2+x_1^3 \\
5: 1+x_1^2+x_1^3 \\
6: 1+x_1^2+x_1^3 \\
\]

We check that both \(1 + x^2 + x^3\) and \(1 + x + x^3\) are irreducible over \(\mathbb{F}_2\) so that the rational forms of \(A^j\) have one block containing the companion matrices \(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \) and \(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \) respectively. We conclude that

\[
g_5^3 \sim g_5^5 \sim g_5^6 \sim g_6
\]

and

\[
g_5 \sim g_5^2 \sim g_5^4.
\]

in \(G\). Of course, in \(S\), none of these elements are conjugate. This means that the conjugacy class of \(g_5\) in \(G\) splits into three conjugacy classes of \(H\) and similarly for \(g_6\).

We compute

\[
\text{\text{Ind}}_S^G \psi_1(g_5) = 7 \cdot \frac{1}{7} (\omega + \omega^2 + \omega^4)
\]

and similarly for the remaining values giving the following table.

<table>
<thead>
<tr>
<th>(\text{Ind}_{\psi_1}^G)</th>
<th>(g_1)</th>
<th>(g_2)</th>
<th>(g_3)</th>
<th>(g_4)</th>
<th>(g_5)</th>
<th>(g_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Ind}_{\psi_1}^G)</td>
<td>24</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\omega + \omega^2 + \omega^4)</td>
<td>(\omega^3 + \omega^5 + \omega^6)</td>
</tr>
<tr>
<td>(\text{Ind}_{\psi_2}^G)</td>
<td>24</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\omega^3 + \omega^5 + \omega^6)</td>
<td>(\omega + \omega^2 + \omega^4)</td>
</tr>
</tbody>
</table>
Taking inner products, we have

\[ \text{Ind}_G^S \psi_1 = \chi_2 + \chi_3 + \chi_4 + \chi_6 \]

\[ \text{Ind}_G^S \psi_2 = \chi_2 + \chi_3 + \chi_4 + \chi_5 \]

giving us \( \chi_5 \) and \( \chi_6 \).

### 0.6 Appendix: The Conjugacy Classes of \( P \)

The following GAP program was used to compute the conjugacy classes of \( P \). Our method is extremely simple minded. We simply compute all the conjugates of all the elements of \( P \) by all the elements of \( P \). While this may not be such a good algorithm for larger groups, it works quite well for \( P \). Since \( P = UL \), we produce \( P \) by first producing the sets \( U \) and \( L \).

\[ \text{UU}:=[]; \text{LL}:=[]; \]

for \( i \) in [0..1] do for \( j \) in [0..1] do

for \( k \) in [0..1] do for \( l \) in [0..1] do

\[ B:=[[i,j,0],[k,l,0],[0,0,1]]; \]

if not Determinant(B) = 0 then

UniteSet(LL,[B]);

fi;od;od;od;od;

for \( i \) in [0..1] do for \( j \) in [0..1] do

\[ B:=[[1,0,i],[0,1,j],[0,0,1]]; \]

UniteSet(UU,[B]);

od;od;

Then we take all conjugates. The following routine makes use of the idempotency of GAP’s UniteSet function.

\[ \text{CCC}:=[]; \]

for \( i \) in [1..4] do for \( j \) in [1..6] do

\[ \text{DD}:=[]; \]

for \( U \) in UU do for \( L \) in LL do

\[ B:=(U*L)^(-1)*\text{UU}[i]*\text{LL}[j]*U*L \mod 2; \]

UniteSet(DD,[B]);

od;od;

UniteSet(CCC,[DD]);

od;od;

Then the set \( \text{CCC} \) contains all the conjugacy classes of \( P \).

---

**Homework 6**
0.7 The Sylow 7-Subgroup

We recall that $G$ has the following character table.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$g_1$</th>
<th>$g_2$</th>
<th>$g_3$</th>
<th>$g_4$</th>
<th>$g_5$</th>
<th>$g_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>order of element</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>class size</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>21</td>
<td>56</td>
<td>42</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>7</td>
<td>$-1$</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>8</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>3</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
<td>$-\frac{1+i\sqrt{7}}{2}$</td>
<td>$-\frac{1-i\sqrt{7}}{2}$</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>3</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
<td>$-\frac{1-i\sqrt{7}}{2}$</td>
<td>$-\frac{1+i\sqrt{7}}{2}$</td>
</tr>
</tbody>
</table>

Let $P_7 \in \text{Syl}_7(G)$. We compute the conjugacy classes of $N_G(P_7)$ and its character table by restricting characters of $G$.

We have from DF that $|N_G(P_7)| = 21$. Since $P_7 \trianglelefteq N_G(P_7)$, we have that $N_G(P_7)$ has one Sylow 7-subgroup so that $N_G(P_7)$ has 6 elements of order 7. By our computations in Homework 5, we have that these 6 elements form two distinct conjugacy classes. The remaining elements of $N_G(P_7)$ must be the identity and 14 elements of order 3. The elements of order 3 lie in two conjugacy classes of $N_G(P_7)$ as follows.

Let $Q \in \text{Syl}_3(N_G(P_7))$. There are 7 Sylow 3-subgroups of $N_{P_7}(P_3)$ since this is the only way we can account for the the 14 elements of order 3. Therefore,

$$[N_G(P_7) : N_{N_G(P_7)}(Q)] = 7.$$  

Then $|N_{N_G(P_7)}(Q)| = 3$ so that $N_{N_G(P_7)}(Q) = Q$. Let $q \in Q$. Then since $Q$ is cyclic, we have $C_{N_G(P_7)}(q) = C_{N_G(P_7)}(Q)$ and this set at least contains $Q$. But

$$Q \leq C_{N_G(P_7)}(q) = C_{N_G(P_7)}(Q) \leq N_{N_G(P_7)}(Q) = Q$$

so that $|C_{N_G(P_7)}(q)| = 3$. Therefore the size of the conjugacy class of $q$ in $N_G(P_7)$ is 7. This means that $N_G(P_7)$ has two conjugacy classes of elements of order 3.

We begin with the restrictions of $\chi_1$, $\chi_5$ and $\chi_6$ to $N_G(P_7)$ and the characters induced from the nontrivial characters $\varepsilon_2$ and $\varepsilon_3$ of $Q \leq N_G(P_7)$.

<table>
<thead>
<tr>
<th>order of element</th>
<th>1</th>
<th>3</th>
<th>3</th>
<th>7</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>class size</td>
<td>1</td>
<td>7</td>
<td>7</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$\omega + \omega^2 + \omega^4$</td>
<td>$\omega^3 + \omega^5 + \omega^6$</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$\omega^3 + \omega^5 + \omega^6$</td>
<td>$\omega + \omega^3 + \omega^6$</td>
</tr>
<tr>
<td>$\varepsilon_2$</td>
<td>7</td>
<td>$\zeta$</td>
<td>$\zeta^2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\varepsilon_3$</td>
<td>7</td>
<td>$\zeta^2$</td>
<td>$\zeta$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where $\omega$ is a primitive 7th root of unity and $\zeta$ is a primitive 3rd root of unity. We find $\psi_1$, $\psi_5$ and $\psi_6$ to be irreducible by taking inner products, we compute

$$\langle \varepsilon_2, \chi_5 \rangle = \langle \varepsilon_3, \chi_5 \rangle = \langle \varepsilon_2, \chi_6 \rangle = \langle \varepsilon_3, \chi_6 \rangle = 1$$

and we find $\varepsilon_2 - \chi_5 - \chi_6$ and $\varepsilon_3 - \chi_5 - \chi_6$ to be irreducible giving the following character table.
0.8 The Sylow 2- and 3-Subgroups

Next, let $P_2 \in \text{Syl}_2(G)$ and $P_3 \in \text{Syl}_3(G)$ We have from DF that the number of Sylow 2- and 3-subgroups of $G$ are 21 and 28\(^1\) so that $|N_G(P_2)| = 8$ and $|N_G(P_3)| = 6$. We therefore have $N_G(P_2) = P_2 \cong D_8$ by DF and $N_G(P_3) \cong S_3$ by elimination.

To compute the character table for $D_8 = \langle r, s : r^4 = s^2 = 1, rs = sr^3 \rangle$, we lift the three nontrivial characters from the quotient $D_8/\langle r^2 \rangle$ giving $\chi_2$, $\chi_3$, and $\chi_4$, and induce a nontrivial character from $\langle r \rangle$ giving $\chi_5$. We thus have the following table.

<table>
<thead>
<tr>
<th>$D_8$</th>
<th>1</th>
<th>$r$</th>
<th>$s$</th>
<th>$sr$</th>
<th>$r^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

The character table for $S_3$ was computed previously and is reprinted below.

<table>
<thead>
<tr>
<th>$S_3$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>2</td>
</tr>
</tbody>
</table>

0.9 The Isaacs-Navarro Conjecture for $G$

When $p = 2$, we have

$$M_1(G) = 4 = M_1(N(P_2)),$$

for $p = 3$ we have

$$M_1(G) = 3 = M_1(N(P_3)),$$

and for $p = 7$ we have

$$M_1(G) = 3 = M_1(N(P_3)),$$

$$M_2(G) = 0 = M_2(N(P_3)),$$

$$M_3(G) = 2 = M_3(N(P_3))$$

\(^1\)There is a typographic error in the statement of Proposition 14 in DF, which has $n_3 = 7$. The proof, however, has $n_3 = 28$. 

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satisfying the Issaacs-Navarro Conjecture.

0.10 The Issaacs-Navarro Conjecture for $A_5$

We compute the character degrees with GAP as follows.

```gap
G := AlternatingGroup(5);  # Computes the group A_5

P := Normalizer(G, SylowSubgroup(G, 2));  # Computes the normalizer of the Sylow 2-subgroup
P := Group([ (1,3)(2,4), (1,2)(3,4), (2,3,4) ]);  # Represents the Sylow 2-subgroup

Q := Normalizer(G, SylowSubgroup(G, 3));  # Computes the normalizer of the Sylow 3-subgroup
Q := Group([ (1,2,3), (2,3)(4,5) ]);  # Represents the Sylow 3-subgroup

R := Normalizer(G, SylowSubgroup(G, 5));  # Computes the normalizer of the Sylow 5-subgroup
R := Group([ (1,2,3,4,5), (2,5)(3,4) ]);  # Represents the Sylow 5-subgroup

CharacterDegrees(G);  # Computes the character degrees of G
CharacterDegrees(P);  # Computes the character degrees of P
CharacterDegrees(Q);  # Computes the character degrees of Q
CharacterDegrees(R);  # Computes the character degrees of R
```

For example, the output on line 12 tells us that the Sylow 2-subgroup has three characters of degree 1 and one character of degree 3. For $p = 2$, we have

$$M_1(G) = 4 = M_1(P),$$

for $p = 3$ we have

$$M_1(G) = 3 = M_1(Q),$$

and for $p = 5$ we have

$$M_1(G) = 2 = M_1(Q),$$

$$M_2(G) = 2 = M_2(Q)$$

satisfying the Issaacs-Navarro Conjecture.

Homework 8

1. Find the $p$-blocks of $G$ for $p = 2, 3, 5$.

2. Construct the isotypies between $B_3(G)$ and $B_3(N_G(P_3))$ and between $B_7(G)$ and $B_7(N_G(P_7))$, where $P_3$ and $P_7$ are Sylow 3- and 7-subgroups.
3. In constructing isptypies, a sign is attached to an irreducible character in $B_p(G)$. Is there any connection with the sign which appears in Isaacs-Navarro?

4. Let $p = 3$. Find
   
   (a) the Brauer characters
   (b) the decomposition numbers
   (c) the Cartan invariants
   (d) the dimension of the PIMs.

   and repeat for $p = 7$.

We have the following table for $GL(3, 2)$.

<table>
<thead>
<tr>
<th>order of element</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>7</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>class size</td>
<td>1</td>
<td>21</td>
<td>56</td>
<td>42</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>−1</td>
<td>−1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>7</td>
<td>−1</td>
<td>1</td>
<td>−1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>8</td>
<td>0</td>
<td>−1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>3</td>
<td>−1</td>
<td>0</td>
<td>1</td>
<td>$\frac{1+i\sqrt{7}}{2}$</td>
<td>$\frac{-1-i\sqrt{7}}{2}$</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>3</td>
<td>−1</td>
<td>0</td>
<td>1</td>
<td>$\frac{-1-i\sqrt{7}}{2}$</td>
<td>$\frac{1+i\sqrt{7}}{2}$</td>
</tr>
</tbody>
</table>

We therefore have the following table for $Z(\mathbb{A})$.

<table>
<thead>
<tr>
<th>class size</th>
<th>1</th>
<th>21</th>
<th>56</th>
<th>42</th>
<th>24</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>1</td>
<td>21</td>
<td>56</td>
<td>42</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>1</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>−4</td>
<td>−4</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>1</td>
<td>−3</td>
<td>8</td>
<td>−6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>1</td>
<td>0</td>
<td>−7</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>1</td>
<td>−7</td>
<td>0</td>
<td>14</td>
<td>$-4+4i\sqrt{7}$</td>
<td>$-4-4i\sqrt{7}$</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>1</td>
<td>−7</td>
<td>0</td>
<td>14</td>
<td>$-4-4i\sqrt{7}$</td>
<td>$-4+4i\sqrt{7}$</td>
</tr>
</tbody>
</table>
Reducing this mod $2R$, $3R$, and $7R$, we have the following.

<table>
<thead>
<tr>
<th>class size</th>
<th>1</th>
<th>21</th>
<th>56</th>
<th>42</th>
<th>24</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>♣</td>
<td>♣</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>♣</td>
<td>♣</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>♣</td>
<td>♣</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>♣</td>
<td>♣</td>
</tr>
</tbody>
</table>

From the table mod 2, we see that the 2-blocks of $G$ are $\{\chi_1, \chi_2, \chi_3, \chi_5, \chi_6\}$ and $\{\chi_4\}$. Some of the elements of the table mod 3 have not been reduced. From the other elements, however, we can see that $\{\chi_1, \chi_3, \chi_4\}$ and $\{\chi_2\}$ form 3-blocks. Since 3 divides $\chi_5(1) = \chi_6(1) = 3$, we have that $\chi_5$ and $\chi_6$ mod 3 are the only irreducible Brauer characters in their blocks. This means that $\{\chi_5\}$ and $\{\chi_6\}$ are the remaining 3-blocks.

Next, we compute the 7-blocks. Since $7 \in p$ by assumption, we have that

$$-7 = \sqrt{-7} \sqrt{-7} \in p,$$

but since $p$ is prime, this means that $\sqrt{-7} \in p$. Thus,

$$24 - (-4 + 4 \sqrt{-7}) = 28 - 4 \sqrt{-7} \in p$$

which shows that $\chi_1$ and $\chi_5$ are in the same 7-block. Repeating this argument, we have that $\{\chi_1, \chi_2, \chi_4, \chi_6, \chi_7\}$ is a 7-block. This means that $\{\chi_3\}$ is the remaining 7-block.

We next look at the character tables for $N(P_3)$ and $N(P_7)$ where $P_3$ and $P_7$ are Sylow 3- and 7- subgroups.

<table>
<thead>
<tr>
<th>$N(P_3) = S_3$</th>
<th>Table</th>
<th>$\omega_i(C_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>order of element</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>class size</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_2$</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\psi_3$</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

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where \( \omega \) is a primitive 7th root of unity and \( \zeta \) is a primitive 3rd root of unity.

Reducing \( \omega_i (C_j) \) from the first table mod 3, we see that
\[ B_3 (N_G (P_3)) = \{ \psi_1, \psi_2, \psi_3 \} \]
For \( \psi_i (C_j) \) from the second table, we observe that
\[
\frac{\sqrt{-7}}{2} \cdot 2 = \sqrt{-7} \in \mathfrak{p},
\]
but since \( 2 \not\in \mathfrak{p} \) else \( 7 - 2 - 2 - 2 = 1 \in \mathfrak{p} \), we must have that \( \frac{\sqrt{-7}}{2} \in \mathfrak{p} \) and \( \frac{-1 + \sqrt{-7}}{2} \in \mathfrak{p} \). Thus,
\[
\frac{-1 + \sqrt{-7}}{2} - 3 = \frac{-7 + \sqrt{-7}}{2} \in \mathfrak{p}
\]
which shows that \( \chi_1 \) and \( \chi_4 \) are in the same 7-block. Repeating this argument, we have
\[ B_7 (N_G (P_7)) = \{ \psi_1, \psi_2, \psi_3, \psi_4, \psi_5 \} \]

Next, we see that the maps
\[
I_3 : \begin{cases} \chi_1 \\ \chi_3 \\ \chi_4 \end{cases} \rightarrow \begin{cases} \psi_1 \\ \psi_2 \\ \psi_3 \end{cases}
\]
and
\[
I_7 : \begin{cases} \chi_1 \\ -\chi_2 \\ \chi_4 \\ \chi_5 \\ \chi_6 \end{cases} \rightarrow \begin{cases} \psi_1 \\ \psi_2 \\ \psi_3 \\ \chi_4 \\ \chi_5 \end{cases}
\]
preserve the character degrees mod 3 and 7 and preserve the values of the characters on the 3- and 7-elements. Then \( I_3 \) and \( I_7 \) are the isotypies.

For \( p = 3 \), we have that the trivial character \( \chi_1 \) is a Brauer character and that \( \chi_2, \chi_5 \) and \( \chi_6 \) are also Brauer characters since 3 divides \( \chi_2 (1) \), \( \chi_5 (1) \), and \( \chi_6 (1) \). We then have the decomposition matrix
\[
\begin{array}{|c|ccccc|}
\hline
& \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 \\
\hline
\chi_1 & 1 & - & - & - & - \\
\chi_3 & 7 & m & n & - & - \\
\chi_4 & 8 & m + 1 & n & - & - \\
\chi_2 & 6 & - & - & 1 & - \\
\chi_5 & 3 & - & - & - & 1 \\
\chi_6 & 3 & - & - & - & 1 \\
\hline
\end{array}
\]
Let $m$ and $n$ be the multiplicities of $\varphi_1$ and $\varphi_2$ in $\chi_3$ as in the table. Then
\[
0 = \chi_3(g) = m\varphi_1(g) + n\varphi_2(g) = m + n\varphi_2(g) = \Phi(\omega)
\] (3)
for $g \in G$ an element of order 7. The right hand side of (3) is a sum of 7th roots of unity and is therefore some polynomial $\Phi$ in $\omega$. Since the minimal polynomial for $\omega$ is
\[
1 + x + x^2 + x^3 + x^4 + x^5 + x^6,
\]
we must have either $\Phi(x) = 0$ or $\Phi(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$. Thus, we either have that $m = 0$, in which case $\chi_3$ is an irreducible Brauer character, or else $m = 1$, (1 is the only integer in the sum) in which case $\chi_3 = \varphi_1 + \varphi_2$.

In the latter case,
\[
C = \begin{pmatrix}
2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

Final Thoughts on $GL(3,2)$

Let $G = GL(3,2)$, the simple group of order 168. We begin by computing the conjugacy classes of $G$. We do this by enumerating all the possible rational forms. There are 6 conjugacy classes.

Next, we construct the characters of $G$. We begin with $\chi_1$, the trivial character. Next, we define the subgroup $P = \{(\begin{smallmatrix}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
0 & 0 & \bullet
\end{smallmatrix}) \} \leq G$ and compute $\text{Ind}_P^G(1)$ where 1 is the trivial character of $P$. To do this, we observe that $P$ is the stabilizer of the line $\langle v_1 \rangle$ in the vector space $V = \langle v_1, v_2, v_3 \rangle$ over $F_2$ so that $\text{Ind}_P^G(1)(g)$ is the number of lines fixed by $g \in G$. This gives us $\chi_2$.

Similarly, $B = \{(\begin{smallmatrix}
\bullet & \bullet & \bullet \\
0 & \bullet & \bullet \\
0 & 0 & \bullet
\end{smallmatrix}) \} \leq G$ is the stabilizer of the flag $\langle v_1 \rangle \leq \langle v_1, v_2 \rangle$ in $V$ so that $\text{Ind}_B^G(1)(g)$ is number of flags fixed by $g$. This gives us $\chi_4$.

We next construct the character of degree 7 using Harish-Chandra induction. We take
\[
P = \{(\begin{smallmatrix}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
0 & 0 & \bullet
\end{smallmatrix}) \} \supset L = \{(\begin{smallmatrix}
\bullet & \bullet & \bullet \\
0 & 0 & \bullet
\end{smallmatrix}) \} \cong S_3
\]

We lift the sign character of $S_3 \cong L$ to a character of $P$ and induce from this a character of $G$. This is not extremely easy, most of the complication arising from determining how the conjugacy classes of $G$ split in $P$, a computation required in the induced character formula. This can at least be accomplished by explicit computation, which I'm sorry to say is what I did. This gives us an irreducible character $\chi_3$ of $G$.

We compute the remaining two characters for $G$ by inducing two characters from a 7-Sylow subgroup $S \leq G$. Again, the difficulty is in determining how the conjugacy classes of $G$ split in $S$, but again, this can at least be accomplished with an explicit computation. This produces the characters $\chi_5$ and $\chi_6$ of $G$.

The characters computed in the discussion are presented in the character table at the head of this document.

Next, we determine the Sylow 2-, 3-, and 7-subgroups $P_2$, $P_3$, and $P_7$ of $G$ and their normalizers. By order considerations and other algebraic manipulations, we find that
$P_2 = N_G(P_2) \cong B$ and $P_3 \triangleleft N_G(P_3) \cong L \cong S_5$. For $P_7$, we use the construction of the characters for groups of order $p^g$ to compute the character table for $N_G(P_7)$ which has order $3 \cdot 7$. The character tables for $N_G(P_3)$ and $N_G(P_7)$ are presented on pages 19 and 20 of this document.

Having the character tables for $N_G(P_2)$, $N_G(P_3)$, and $N_G(P_7)$, we were able to verify the Issaacs-Navarro conjecture for $G$.

Finally, in this document, we computed various $p$-blocks, isotypies, Brauer characters and Cartan invariants for $G$. 