Homework 1, Solutions

1. *Let V be a three-dimensional vector space over $\mathbb{C}$ with basis $\{v_1, v_2, v_3\}$ and let $G = S_3$ act on V by permuting the indicies. Let $V_1 = \langle v_1 + v_2 + v_3 \rangle$ and*

$$V_2 = \{\alpha v_1 + \beta v_2 + \gamma v_3 : \alpha + \beta + \gamma = 0\} = \langle v_1 - v_2, v_2 - v_3 \rangle$$

be $\mathbb{C}G$-submodules.

   (a) Show that $V = V_1 \oplus V_2$.

   (b) Show that V_2 is simple.

   (c) Let $H = \langle (123) \rangle \leq G$ and let G act on G/H by multiplication. Show that $\mathbb{C}G/H$ is reducible as a $\mathbb{C}G$-module.

   (d) Let F be a field of characteristic 3 and let V_1 and V_2 be the FG-submodules of V defined above. Do we still have $V = V_1 \oplus V_2$?

   (a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ is invertible iff $\{v_1 + v_2 + v_3, v_1 - v_2, v_2 - v_3\}$ is a basis for M iff every element of V can be expressed uniquely as a linear combination of $v_1 + v_2 + v_3, v_1 - v_2$, and $, v_2 - v_3$ iff every element of V can be expressed uniquely as $u_1 + u_2$ for $u_1 \in V_1 = \langle v_1 + v_2 + v_3 \rangle$ and $u_2 \in V_2 = \langle v_1 - v_2, v_2 - v_3 \rangle$ iff $V = V_1 \oplus V_2$.

   (b) If W is a non-zero submodule of V_2, then there must be some $w = \alpha v_1 + \beta v_2 + \gamma v_3 \in W$ with $w \neq 0$. Also, $\alpha + \beta + \gamma = 0$ as $W \subset V_2$. Now since W is a $\mathbb{C}S_3$-submodule, we have that

$$\begin{aligned} w - (12) \cdot w &= (\alpha - \beta)(v_1 - v_2) \\ w - (23) \cdot w &= (\beta - \gamma)(v_2 - v_3) \\ w - (13) \cdot w &= (\alpha - \gamma)(v_1 - v_3) \end{aligned}$$

   are all in W. If any two of the scalars $\alpha - \beta$, $\beta - \gamma$, and $\alpha - \gamma$ are both zero, then $\alpha = \beta = \gamma$ so that $w \notin V_2$, a contradiction. We therefore have that two of the above scalars, say $\alpha - \beta$ and $\beta - \gamma$ nonzero. But this means that both $v_1 - v_2$ and $v_2 - v_3$ are in W so that $W = V_2$. This shows that V_2 is irreducible.

   (c) We have that $\langle H + (12)H \rangle$ is a $\mathbb{C}G$-submodule of $\mathbb{C}G/H$ as $g \cdot (H + (12)H) = H + (12)H$ for all $g \in G$. Since this submodule has dimension 1, $\mathbb{C}G/H$ is reducible.

   (d) If F is a field of characteristic 3, then $V_1 = \langle v_1 + v_2 + v_3 \rangle$ is a submodule of $V_2 = \{\alpha v_1 + \beta v_2 + \gamma v_3 : \alpha + \beta + \gamma = 0\}$ so that $V \neq V_1 \oplus V_2$. Also note that the matrix given above has determinant 3.

2. *Let D be a division ring and let $\mathcal{A}$ be the submodule of $M_n(D)$ of upper triangular matrices.*

   (a) Describe $J(\mathcal{A})$.

(b) Describe the decomposition of $A/J(\mathcal{A})$ as a direct sum of simple modules.

(a) Define $L_k = \{A = (a_{i,j}) \in \mathcal{A} : a_{k,k} = 0\}$. To show that L_k is an ideal, it suffices to show that $E_{i,j}A \in L_k$ for all $i \le j$. But left multiplying A by $E_{i,j}$ results in a matrix with the jth row of A in the ith row and zeros elsewhere. Also, since $i < j$, we have, that this operation has the effect of moving rows upward and clobbering the rest so that the k,k entry of $E_{i,j}A$ is 0.

To show that L_k is a maximal ideal, let $B \in \mathcal{A} \backslash L_k$. Then $(b_{k,k}) \neq 0$. But then $E_{k,k} b_{k,k}^{-1} B = E_{k,k} \in L_k$ so that $(L_k, B) = \mathcal{A}$.

We therefore have that

$$J(\mathcal{A}) \subset \bigcap_{k=1}^{n} L_k = \left\{ A = (a_{i,j}) \in \mathcal{A} : a_{k,k} = 0, 1 \le k \le n \right\}.$$

However, since the righthand side is a nilpotent ideal, we have the reverse inclusion since $J(\mathcal{A})$ contains all nilpotent ideals.

Remark: The final answer is that J is precisely the set of upper triangular matrices with 0 along the main diagonal. This is the right hand side above. Check that this is a nilpotent ideal, by showing that the k-th power of a matrix of that form has the k-th diagonal (i.e. the $k-1$-st diagonal to the right of the main diagonal) equal to zero.

(b) We have that A_1 and A_2 will be in the same coset of $J(\mathcal{A})$ if and only if they have the same diagonal elements. Therefore, if we define

$$M_j = \{\alpha_j E_{j,j} + J(\mathcal{A}) : \alpha_j \in D\},$$

then M_j is irreducible as D is a division ring and

$$\mathcal{A}/(J\mathcal{A}) = \bigoplus_{j=1}^{n} M_j.$$

Homework 2 Solutions

Added to Marcus' Homework

Here are some general comments on the decomposition of $F[x]$-modules, F a field. Let V be a cyclic $F[x]$-module, i.e. V has a basis $\{v, xv, \dots x^{n-1}v\}$ and the minimal polynomial $f(x)$ of x (as a linear transformation of V) has degree equal to the dimension n of V. Then in fact V is isomorphic to $F[x]/(f(x))$ and the latter can be written as $\bigoplus_1^k F[x]/(p_i(x)^{r_i})$, where each factor $F[x]/(p_i(x)^{r_i})$ is cyclic and the $p_i(x)$ are distinct and irreducible. (Note that in general, if we have a decomposition of an $F[x]$-module according to the "rational form" as in DF, p.455, we have to apply the above to each cyclic factor.) Now consider a cyclic module $V = F[x]/(p(x)^r)$. Then V is indecomposable as $F[x]$-module and is irreducible if $r = 1$. To see that such a V is indecomposable, suppose $V = V_1 \oplus V_2$. Then on V_1 (resp. V_2), the minimal polynomial of x would be of the form $p(x)^k$ (resp. $p(x)^l$) where $k, l < r$ and then the minimal polynomial on V would have degree less than $deg(p).r$. Similarly if $r = 1$ and V had a proper submodule V_1 (not zero) then the minimal polynomial of x on V_1 would be a factor of $p(x)$ of smaller degree, but $p(x)$ is irreducible. We can apply these arguments to the problems below. In Problem 1 the modules are not cyclic, but we apply the above criteria to each cyclic factor. In Problem 2 the module $Q[x]/(x^n - 1)$ is cyclic.

1. *Let $A = \mathbb{Q}[x]$, V an A-module where $xv = Tv$ for $v \in V$ with T the linear transformation given by*

   (a) $\begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}$ *with respect to the basis $\{e_1, e_2, e_3\}$ of V. Show that V is completely reducible. Describe V as a direct sum of simple modules.*

   (b) $\begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}$ *with respect to the basis $\{e_1, e_2, e_3, e_4\}$ of V. Show that V is* not *completely reducible. Describe V as the direct sum of two indecomposable submodules and give a composition series for V*

   (a) We have that the Jordan form of T is $J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ with $S = \begin{pmatrix} 2 & -2 & 0 \\ 7 & 1 & 7 \\ 1 & 0 & 1 \end{pmatrix}$ such that $J = S^{-1}TS$. Then with respect to the basis

   $$\left\{ f_1 = 2e_1 + 7e_2 + e_3, f_2 = -2e_1 + e_2, f_3 = 7e_2 + e_3 \right\}$$

   of V, T has matrix J. This means that $\langle f_1 \rangle, \langle f_2 \rangle, \langle f_3 \rangle$ are A-submodules of V since these are the J-stable $\mathbb{Q}$-subspaces of V. Also,

   $$V = \langle f_1 \rangle \oplus \langle f_2 \rangle \oplus \langle f_3 \rangle$$

   since the f_j are the columns of S, which is invertible.

   (b) The Jordan form of T is $R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Letting $\{f_1, f_2, f_3, f_4\}$ be the columns of the conjugating matrix as before, we have that $V_1 = \langle f_1, f_2 \rangle$ and $V_2 = \langle f_3, f_4 \rangle$ are A-submodules of V which have irreducible sub-submodules $\langle f_1 \rangle$ and $\langle f_3 \rangle$ respectively. Then V_1 and V_2 are indecomposable (by the general arguments given above, taking $p(x) = x - 1$ and noting that V_1 and V_2 are isomorphic to $Q[x]/(x-1)^2$ and are cyclic) and $V = V_1 \oplus V_2$.
   A composition series for V is

   $$V = \langle f_1, f_2, f_3, f_4 \rangle \geq \langle f_1, f_2, f_3 \rangle \geq \langle f_1, f_2 \rangle \geq \langle f_1 \rangle \geq 0.$$

2. *Let $G = \langle a : a^n = 1 \rangle$ be the cyclic group of order n and let $R = \mathbb{Q}G$. Describe the Wedderburn decomposition of R and find the number and the degrees of the irreducible representations of G over $\mathbb{Q}$. In particular, show that if $n = p$ is a prime, then G has exactly one nontrivial irreducible representation over $\mathbb{Q}$ and this representation has degree $p - 1$.*

The map $\mathbb{Q}[x] \to \mathbb{Q}G$ defined $x^j \to a^j$ is a surjective ring homomorphism with kernel $(1-x^n)$ so that

$$\mathbb{Q}[x]/\left(1-x^n\right) \cong \mathbb{Q}G.$$

Next, the polynomial $1-x^n \in \mathbb{Q}[x]$ factors as

$$1-x^n = \prod_{d|n} \Phi_d(x)$$

where $\Phi_d(x)$ is the dth cyclotomic polynomial. Then by the Chineese Remainder Theorem, since the $(\Phi_d(x))$ are comaximal, $\Phi_d(x)$ being irreducible, we have

$$\mathbb{Q}G \cong \mathbb{Q}[x]/\left(1-x^n\right) \cong \prod_{d|n} \mathbb{Q}[x]/\left(\Phi_d(x)\right)$$

as rings. Note that as a $\mathbb{Q}[x]$ module, $\mathbb{Q}[x]/\left(\Phi_d(x)\right)$ is irreducible, again by the general arguments. Note also that $\mathbb{Q}[x]/\left(\Phi_d(x)\right)$ has dimension $\varphi(d)$. Therefore,

$$\mathbb{Q}G \cong \bigoplus_{d|n} \mathbb{Q}[x]/\left(\Phi_d(x)\right)$$

as modules so that

$$\mathbb{Q}G \cong \bigoplus_{d|n} M_1\left(D_d\right)$$

where $D_d = \mathbb{Q}(\zeta)$ where ζ is a primitive d-th root of unity.

Homework 3 Solutions, added to Marcus'

Let $G = S_3$ and $A = \mathbb{Q}G$.

1. *Find the central idempotents e_1, e_2 and e_3 such that $A = Ae_1 \oplus Ae_2 \oplus Ae_3$ where the Ae_j are ideals which are simple algebras with $\dim Ae_1 = \dim Ae_2 = 1$ and $\dim Ae_3 = 4$.*

2. *Find orthogonal idempotents (not necessarily central) f_1, f_2 f_3 and f_4 such that ${}_AA = \bigoplus_{j=1}^4 Af_j$ where the Af_j are minimal left ideals (simple A-modules).*

3. *Let $p = 2$. We consider two modular representations of G. Let $R \subset \mathbb{Q}$ be given by $R = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, 2 \not| b\right\}$. Then R is a local ring and $R/(2) \cong \mathbb{F}_2$. Let $\widetilde{A} = RG$ and $\overline{A} = \mathbb{F}_2 G$. Using the decomposition in (1), write $\widetilde{A} = \widetilde{A}E_1 \oplus \widetilde{A}E_2$ where E_1 and E_2 are central idempotents. It will not be possible to reduce e_1, e_2, and e_3 mod 2 as some of them will not lie in R. So you will have to combine them into two idempotents. Finally, write $\overline{A} = \overline{AE_1} \oplus \overline{AE_2}$ after reducing mod 2.*

4. *Try doing the same procedure as in (3) for $p = 3$. Here $R = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, 3 \not| b\right\}$. Can you decompose $\widetilde{A} = RG$?*

1. Let $e_1 = \frac{1}{6}\sum_{g\in G} g$. Then

$$e_1^2 = \left(\frac{1}{6}\sum_g g\right)\left(\frac{1}{6}\sum_g g\right) = \frac{1}{36}\sum_g\left(g\sum_g g\right) = \frac{1}{36}\sum_g\left(\sum_g g\right) = \frac{1}{6}\left(\sum_g g\right)$$

so that e_1 is idempotent. Also,

$$\left(\sum_g \alpha_g g\right)\left(\frac{1}{6}\sum_g g\right) = \frac{1}{6}\sum_g \alpha_g g\left(\sum_g g\right) = \frac{1}{6}\sum_g \alpha_g\left(\sum_g g\right) \tag{1}$$

$$= \frac{1}{6}\sum_g\left(\sum_g g\right)\alpha_g = \frac{1}{6}\sum_g\left(\sum_g g\right)\alpha_g g = \frac{1}{6}\left(\sum_g g\right)\left(\sum_g \alpha_g g\right)$$

so that e_1 is central. Let $e_2 = \frac{1}{6}\sum_g (-1)^{\mathrm{sgn}\ g} g$. We similarly compute that e_2 is a central idempotent. We define $e_3 = 1 - e_1 - e_2 = e_3$ and confirm directly that e_3 is central.

Ae_1 and Ae_2 are 1-dimensional, since, as suggested in (1), $ae_1 = \alpha e_1$ for some $\alpha \in \mathbb{Q}$ and similarly for e_2. We confirm that Ae_3 is 4-dimensional by explicitly computing its elements and confirming independence of four of them. Alternately, GAP verifies this claim and everything stated above as follows.

```
gap> G:=SymmetricGroup(3);A:=GroupRing(Rationals,G);
Sym( [ 1 .. 3 ] )
<algebra-with-one over Rationals, with 2 generators>
gap> e:=CentralIdempotentsOfAlgebra(A);
[ (1/6)*()+(-1/6)*(2,3)+(-1/6)*(1,2)+(1/6)*(1,2,3)+(1/6)*(1,3,2)+(-1/6)*(1,3),
  (1/6)*()+(1/6)*(2,3)+(1/6)*(1,2)+(1/6)*(1,2,3)+(1/6)*(1,3,2)+(1/6)*(1,3),
  (2/3)*()+(-1/3)*(1,2,3)+(-1/3)*(1,3,2) ]
gap> Dimension(A*e[1]);Dimension(A*e[2]);Dimension(A*e[3]);
1
1
4
```

2. To find f_3 and f_4, we consider the representation ρ of $\mathbb{Q}G$ given by

$$(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } (123) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \tag{2}$$

and take the inverse images of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We begin with the following. The newlines were added to this document for legibility. `A.1` and `A.2` give the first and second generators of A which are then printed to the standard output.

```
gap> s:=A.1;t:=A.2;
(1)*(1,2,3)
(1)*(1,2)
gap> S:=[[0,-1],[1,-1]];T:=[[-1,1],[0,1]];
[ [ 0, -1 ], [ 1, -1 ] ]
[ [ -1, 1 ], [ 0, 1 ] ]
```

The next command defines `f` to be the algebra homomorphism given by the images of the generators in 2.

```
gap> f:=AlgebraHomomorphismByImages(
        A,FullMatrixAlgebra(Rationals,2),[t,s],[T,S]);
[ (1)*(1,2), (1)*(1,2,3) ] ->
[ [ [ -1, 1 ], [ 0, 1 ] ], [ [ 0, -1 ], [ 1, -1 ] ] ]
v:=PreImagesRepresentative(f,[[1,0],[0,0]]);
(1/3)*()+(-1/3)*(2,3)+(-2/3)*(1,2)+(-1/3)*(1,2,3)
gap> Dimension(A*v);
4
```

It is somewhat disturbing that GAP's preimage generates a subalgebra of dimension 4, but we consider the possibility that the GAP has returned the sum of the preimage of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and something in the kernel of ρ (note that $e_1, e_2 \in \ker \rho$). We experiment with e_1 and e_2 to find f_3 and f_4 as follows.

```
gap> f3:=v-e[1]+e[2];
(1/3)*()+(-1/3)*(1,2)+(-1/3)*(1,2,3)+(1/3)*(1,3)
gap> Dimension(A*f3);
2
gap> f3^2=f3;
true
gap> f4:=e[3]-f3;
(1/3)*()+(1/3)*(1,2)+(-1/3)*(1,3,2)+(-1/3)*(1,3)
gap> Dimension(A*f4);
2
gap> f4^2=f4;
true
```

Addendum (B.S) :

Thus, one solution for f_3 and f_4 is given by:
$f_3 = (1/3)[1 - (1,2) + (1,2,3) - (1,3)]$, $f_4 = e_3 - f_3$.
Other solutions for f_3 might be obtained by observing that the representation ρ of $\mathbb{Q}G$ is given by $(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, $(123) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $(13) \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $(132) \mapsto \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $(23) \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

Thus by inspection we see easily that $-(13) - (123) - 132) \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and so we could take $f_3 = -1/2[(13) + (123) + (132)]$.

3. We have

$$E_2 := e_3 = 1 - e_1 - e_2 = \frac{2}{3} - \frac{1}{3}(123) - \frac{1}{3}(132) \in R$$

and observe that

$$E_1 = e_1 + e_2 = \frac{1}{3} + \frac{1}{3}(123) + \frac{1}{3}(132) \in R$$

and that E_1 is idempotent by FOIL and orthogonality. As in (1), $\widetilde{A}E_2$ is irreducible and of dimension 4, the only difference being that R is a subring of $\mathbb{Q}$.

Then $\widetilde{A} = \widetilde{A}E_1 \oplus \widetilde{A}E_2$. Reducing both sides by the ideal generated by $2 \cdot 1$, which has the effect of reducing all the coefficients modulo the ideal (2), we have $\overline{A} = \overline{A}E_1 \oplus \overline{A}E_2$.

However, if we replace 2 by 3 we find that the R-algebra A is not decomposable into a sum of proper two-sided ideals.

Homework 4

Let $G = SL(2,3)$. You are given that the representatives of the conjugacy classes of G are $\{g_j : 1 \leq j \leq 7\}$ where $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $g_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $g_5 = g_4^{-1}$, $g_6 = g_2 g_4$, and $g_7 = g_2 g_5$. Fill in the character table by the following steps.

1. *(Projective Geometry) $G/Z = \mathrm{PSL}(2,3)$ permutes the 4 points of the projective line over $\mathbb{F}_3$ and is isomorphic to A_4. Use this to fill in χ_2, χ_3, and χ_4.*
2. *Find the other character degrees and the values on Z.*
3. *Of the three characters χ_5, χ_6, and χ_7, at least one has values in $\mathbb{R}$ since complex conjugate characters come in pairs. Suppose this is χ_5 with corresponding representation ρ_5. Consider the eigenvalues of $\rho_5(g_r)$ and use $\chi_5(g_4) \in \mathbb{R}$ to find $\chi_5(g_4)$. Then find $\chi_5(g_5)$,$\chi_5(g_6)$ and $\chi_5(g_7)$.*
4. *Tensoring with suitable linear characters gives χ_6 and χ_7 on g_i for $4 \leq i \leq 7$.*
5. *Finally, find $\chi_5(g_3)$,$\chi_6(g_3)$ and $\chi_7(g_3)(g_3)$ by orthogonality.*

1. Given that

   $$\varphi : GL(2,3)/Z = PSL(2,3) \xrightarrow{\sim} A_4$$

   is an isomorphism for some map φ, we have that $|Z| = |GL(2,3)| / |A_4| = 2$ so that $Z = \{g_1, g_2\}$ since g_1 and g_2 must be central in $SL(2,3)$, being central in all of $M_2(\mathbb{F}_3)$.

   This means that elements are in the same coset of the quotient if they differ by g_2.

   Therefore, the conjugacy classes represented by $g_4 Z$ and $g_6 Z = g_2 g_4 Z$ must collapse under φ to the same conjugacy class and this class must either be (123) or (124) since $|\varphi(g_4 Z)|$ must divide $|g_4 Z| = 3$ and isn't 1 because φ is injective. Similarly, $g_5 Z$ and $g_7 Z$ must collapse to the conjugacy class represented by the other three cycle. By elimination, we have that the conjugacy class of $g_3 Z$ corresponds with the conjugacy class of (12) (34) and observe that these classes are in bijection unlike the others.

We therefore lift the characters of A_4 to characters of G as follows

$SL(2,3)$	g_1	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	$\overline{\omega}$	ω	$\overline{\omega}$
χ_3	1	1	1	$\overline{\omega}$	ω	$\overline{\omega}$	ω
χ_4	3	3	-1	0	0	0	0

2. Let $n_j = \dim \chi_j$. Then since $\sum_{j=1}^{7} n_j^2 = 24$, we have $\sum_{j=5}^{7} n_j^2 = 12$. The only squares ≤ 12 are 1, 2, and 9. If any $n_j^2 = 9$, then the sum of the squares of the other two must be 4, but 4 is not the sum of two squares. Similarly $n_j^2 \neq 1$ for any j so that $4 = n_5^2 = n_6^2 = n_7^2 = \chi_5(g_1) = \chi_6(g_1) = \chi_7(g_1)$.

   Since $g_2 \in Z$, we have that $\rho_j(g_2)$ must be a diagonal matrix for $5 \leq j \leq 7$. The diagonal entries of $\rho_j(g_2)$ are ± 1. The trace of such a matrix is therefore -2, 0, or 2.

   However, we must have $\chi_5(g_2) = \chi_6(g_2) = \chi_7(g_2) = -2$ since this is the only way that
$$0 = \langle g_1, g_2 \rangle = 1^2 + 1^2 + 1^2 + 3^2 + 2\chi_5(g_2) + 2\chi_6(g_2) + 2\chi_7(g_2)$$
   will be satisfied. We therefore have the following situation.

$SL(2,3)$	g_1	g_2	g_3	g_4	g_5	g_6	g_7
χ_5	2	-2					
χ_6	2	-2					
χ_7	2	-2					

3. Assume χ_5 is an irreducible real-valued character. We have $|\rho_5(g_4)|$ divides $|g_4| = 3$ so that $|\rho_5(g_4)|$ is 1 or 3. If $|\rho_5(g_4)| = 1$, then $g_4 \in \ker \rho_5 \lhd G$, but the only normal subgroup of A_4 has order 4. This subgroup pulls back under $\pi \circ \varphi$ to a normal subgroup of order 8, but g_4 cannot be in this subgroup, having order 3.

   Therefore $|\rho_5(g_4)| = 3$. Let ε_1 and ε_2 be its eigenvalues with eigenvectors v_1 and v_2. Then $v_j = (\rho_5(g_4))^3 v_j = \varepsilon_j^3 v_j$ so that $\varepsilon_j^3 = 1$. We have agreed that $\chi_5(g_4) = \epsilon_1 + \epsilon_2 \in \mathbb{R}$. But for $\{\varepsilon_1, \varepsilon_2\} \subset \{1, \omega, \overline{\omega}\}$, and $\varepsilon_1 + \varepsilon_2 \in \mathbb{R}$, we must have either have $\varepsilon_1 = \varepsilon_2 = 1$ or $\{\varepsilon_1, \varepsilon_2\} = \{\omega, \overline{\omega}\}$. But the first case is impossible since we have seen that $\rho_5(g_4) \neq I$. This means we have the latter case and $\chi_5(g_4) = -1$. Similarly, since $|\rho_5(g_5)| = 3$, we have $\chi_5(g_5) = -1$.

   We have observed above that $\rho_5(g_2) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (with respect to any basis). This means that $\rho_5(g_6) v_j = \rho_5(g_2) \rho_5(g_4) v_j = \rho_5(g_2) \epsilon_j v_j = -\epsilon_j v_j$ so that $\chi_5(g_6) = 1$ and similarly for g_7.

$SL(2,3)$	g_1	g_2	g_3	g_4	g_5	g_6	g_7
χ_5	2	-2		-1	-1	1	1

4. We provisionally put $\chi_6 = \chi_5 \otimes \chi_2$ and $\chi_7 = \chi_5 \otimes \chi_3$.

$SL(2,3)$	g_1	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	$\overline{\omega}$	ω	$\overline{\omega}$
χ_3	1	1	1	$\overline{\omega}$	ω	$\overline{\omega}$	ω
χ_4	3	3	-1	0	0	0	0
χ_5	2	-2		-1	-1	1	1
χ_6	2	-2		$-\omega$	$-\overline{\omega}$	ω	$\overline{\omega}$
χ_7	2	-2		$-\overline{\omega}$	$-\omega$	$\overline{\omega}$	ω

5. To use the first orthogonality relation, we need to compute the class sizes. We have $|\mathcal{C}_1| = |\mathcal{C}_2| = 1$ since g_1 and g_2 are central. We saw that $\mathcal{C}_3$ corresponds under the isomorphism φ to the conjugacy class of A_4 containing $(12)(34)$ which has 3 members. Therefore, $|\mathcal{C}_3| = 6$. Finally, we have $|\mathcal{C}_j| = 4$ for $4 \leq j \leq 7$ by the second orthogonality relation. For example,

$$|\mathcal{C}_5| = 1 + 1 + 1 + 0 + 1 - 1 - 1 = 4.$$

Since we have assumed χ_5 to be an irreducible character, we can use the first orthogonality relation to compute its remaining element

$$4 + 4 + 6\chi_5(g_3)\overline{\chi_5(g_3)} + 4 + 4 + 4 + 4 = |G| = 24$$

giving $\chi_5(g_3) = 0$. Finally, $\chi_6(g_3) = \chi_5(g_2)\chi_2(g_2) = 0$ and similarly for $\chi_7(g_3)$ giving the final character table.

$SL(2,3)$	g_1 1	g_2 1	g_3 6	g_4 4	g_5 4	g_6 4	g_7 4
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	$\overline{\omega}$	ω	$\overline{\omega}$
χ_3	1	1	1	$\overline{\omega}$	ω	$\overline{\omega}$	ω
χ_4	3	3	-1	0	0	0	0
χ_5	2	-2	0	-1	-1	1	1
χ_6	2	-2	0	$-\omega$	$-\overline{\omega}$	ω	$\overline{\omega}$
χ_7	2	-2	0	$-\overline{\omega}$	$-\omega$	$\overline{\omega}$	ω

It remains to confirm that χ_6 and χ_7 are irreducible by checking that $\langle \chi_6, \chi_6 \rangle = \langle \chi_7, \chi_7 \rangle = 1$.

Homework 5

0.1 Introduction

We compute the conjugacy classes of G by enumerating all the permissible rational canonical forms, we compute the orders of the representatives, and we match the representatives with the columns of the given table by order. We have the following table.

	$\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&1\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&0\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&1\\0&1&0 \end{pmatrix}$
pseudonym	g_1	g_2	g_3	g_4	g_5	g_6
order of element	1	2	3	4	7	7
class size	1	21	56	42	24	24
χ_1	1	1	1	1	1	1
χ_2	6	2	0	0	-1	-1
χ_3	7	-1	1	-1	0	0
χ_4	8	0	-1	0	1	1
χ_5	3	-1	0	1	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
χ_6	3	-1	0	1	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$

0.2 Stabilizer of Lines

Let V be a vector space over $\mathbb{F}_2$ with basis $\{v_1, v_2, v_3\}$. Then G acts transitively on the set of one-dimensional subspaces of V. We count that V has 7 one-dimensional subspaces $\{\langle \alpha v_1 + \beta v_2 + \gamma v_3 \rangle : \alpha, \beta, \gamma \in \mathbb{F}_2\}$.

With respect to this basis, the subgroup $P = \begin{pmatrix} \clubsuit&\clubsuit&\clubsuit\\0&\clubsuit&\clubsuit\\0&\clubsuit&\clubsuit \end{pmatrix} \leq G$ is the stabilizer of the line $\langle v_1 \rangle$. Therefore, elements in the same coset of G/P act equivalently on $\langle v_1 \rangle$. Since G acts transitively on the set of lines, there should be one coset for each line. $\frac{|G|}{|P|} = 7$ confirms this claim.

Let $\{x_j : 1 \leq j \leq 7\}$ be coset representatives of G/P and let 1 denote the trivial character of G/P. We will use the following observation to compute

$$\mathrm{Ind}_P^G 1\,(g) = \sum_{j=1}^{7} 1\left(x_j^{-1} g x_j\right).$$

We have $x_j^{-1} g x_j \in P$ iff $g x_j \in x_j P$ iff $g x_j P = x_j P$. Therefore, $\mathrm{Ind}_P^G 1\,(g)$ counts 1 for each coset $x_j P$ that g fixes. But since the elements of the coset $x_j P$ act equivalently on $\langle v_1 \rangle$, we can regard $x_j P$ as the well defined line $x_j \langle v_1 \rangle$ so that $\mathrm{Ind}_P^G 1\,(g)$ is the number of lines g fixes.

We directly compute that g_2 fixes the lines $\langle v_1 \rangle$, $\langle v_2 + v_3 \rangle$, and $\langle v_1 + v_2 + v_3 \rangle$, g_3 fixes the line $\langle v_1 + v_2 + v_3 \rangle$, and g_4 fixes the line $\langle v_1 + v_3 \rangle$. We therefore have the following character.

	$\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&1\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&0\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&1\\0&1&0 \end{pmatrix}$
$\mathrm{Ind}_P^G(1)$	7	3	1	1	0	0

Taking inner products, we have

$$\mathrm{Ind}_P^G(1) = \chi_1 + \chi_2$$

giving us χ_2.

0.3 Stabilizer of Flags

G also acts transitively on the set of complete flags. Let $B = \left\{ \left(\begin{smallmatrix} \clubsuit & \clubsuit & \clubsuit \\ 0 & \clubsuit & \clubsuit \\ 0 & 0 & \clubsuit \end{smallmatrix} \right) \right\}$ be the stabilizer of the complete flag $\langle v_1 \rangle \leq \langle v_1, v_2 \rangle$. Then by the same argument used in section 0.2, $\mathrm{Ind}_B^G(1)(g)$ is the number of flags fixed by g.

For each one-dimensional subspace of V enumerated in section 0.2, there are three two-dimensional subspaces of V containing it, making a total of 21 complete flags. Thus, $\mathrm{Ind}_B^G(1)(g_1) = 21$.

To count flags fixed by $g_2 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix} \right)$, we recall from section 0.2 that the only one-dimensional subspaces of V fixed by g_2 are $\langle v_1 \rangle$, $\langle v_2 + v_3 \rangle$, and $\langle v_1 + v_2 + v_3 \rangle$. By enumerating the two-dimensional subspaces containing these, we see that g_2 fixes the following flags

$$\begin{aligned} \langle v_1 \rangle &\leq \langle v_1, v_2 + v_3 \rangle \\ \langle v_2 + v_3 \rangle &\leq \langle v_2, v_3 \rangle \\ \langle v_2 + v_3 \rangle &\leq \langle v_1, v_2 + v_3 \rangle \\ \langle v_2 + v_3 \rangle &\leq \langle v_1 + v_2, v_2 + v_3 \rangle \\ \langle v_1 + v_2 + v_3 \rangle &\leq \langle v_1, v_1 + v_2 + v_3 \rangle \end{aligned}$$

and g_4 fixes the flag

$$\langle v_1 + v_3 \rangle \leq \langle v_1 + v_3, v_2 + v_3 \rangle$$

so that we have the following character.

	$\left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix} \right)$	$\left(\begin{smallmatrix} 1&0&0\\0&0&1\\0&1&0 \end{smallmatrix} \right)$	$\left(\begin{smallmatrix} 0&0&1\\1&0&0\\0&1&0 \end{smallmatrix} \right)$	$\left(\begin{smallmatrix} 0&0&1\\1&0&1\\0&1&1 \end{smallmatrix} \right)$	$\left(\begin{smallmatrix} 0&0&1\\1&0&0\\0&1&1 \end{smallmatrix} \right)$	$\left(\begin{smallmatrix} 0&0&1\\1&0&1\\0&1&0 \end{smallmatrix} \right)$
$\mathrm{Ind}_B^G(1)$	21	5	0	1	0	0

Taking inner products, we see that

$$\mathrm{Ind}_B^G(1) = \chi_1 + 2\chi_2 + \chi_4$$

giving us χ_4.

0.4 The Parabolic Subgroup

Consider the subgroup

$$P = \left\{ \begin{pmatrix} \clubsuit & \clubsuit & \clubsuit \\ \clubsuit & \clubsuit & \clubsuit \\ 0 & 0 & 1 \end{pmatrix} \right\} \supset \left\{ \left(\begin{array}{cc|c} \clubsuit & \clubsuit & 0 \\ \clubsuit & \clubsuit & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \right\} := L \cong GL_2(\mathbb{F}_2) \times GL_1(\mathbb{F}_2) \cong S_3.$$

Since $P = UL$ for $U = \left\{ \left(\begin{smallmatrix} 1 & 0 & \clubsuit \\ 0 & 1 & \clubsuit \\ 0 & 0 & 1 \end{smallmatrix} \right) \right\} \lhd P$, we will lift a character from $L \cong P/U$ up to P and induce from this a character of G.

Again, using the rational form, we compute representatives of the conjugacy classes of $GL_2(\mathbb{F}_2)$ to be $\left(\begin{smallmatrix} 1&0\\0&1 \end{smallmatrix} \right)$, $\left(\begin{smallmatrix} 0&1\\1&0 \end{smallmatrix} \right)$, and $\left(\begin{smallmatrix} 0&1\\1&1 \end{smallmatrix} \right)$ so that representatives of the conjugacy classes of L are $\left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix} \right)$, $\left(\begin{smallmatrix} 0&1&0\\1&0&0\\0&0&1 \end{smallmatrix} \right)$, $\left(\begin{smallmatrix} 0&1&0\\1&1&0\\0&0&1 \end{smallmatrix} \right)$. The classes in L represented by these three elements do not merge in P since these elements have different orders.

We take the following character from the character table for S_3.

L	$\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&1&0\\1&1&0\\0&0&1\end{smallmatrix}\right)$
ε	1	-1	1

We will compute

$$\mathrm{Ind}_P^G \varepsilon\left(g_2\right) = \left|C_G\left(g_2\right)\right| \left(\sum_{j=1}^{s} \frac{1}{\left|C_P\left(h_j\right)\right|} \varepsilon\left(h_j\right)\right)$$

where h_j are the representatives of the conjugacy classes in P whose union is the conjugacy class of g_2 in G.

The GAP program given in the appendix gives the following conjugacy classes of P.

```
gap> CCC;
[ [ [ [ 0, 1, 0 ], [ 1, 0, 0 ], [ 0, 0, 1 ] ],
      [ [ 0, 1, 1 ], [ 1, 0, 1 ], [ 0, 0, 1 ] ],
      [ [ 1, 0, 0 ], [ 1, 1, 0 ], [ 0, 0, 1 ] ],
      [ [ 1, 0, 0 ], [ 1, 1, 1 ], [ 0, 0, 1 ] ],
      [ [ 1, 1, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],
      [ [ 1, 1, 1 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] ],
  [ [ [ 0, 1, 0 ], [ 1, 0, 1 ], [ 0, 0, 1 ] ],
      [ [ 0, 1, 1 ], [ 1, 0, 0 ], [ 0, 0, 1 ] ],
      [ [ 1, 0, 1 ], [ 1, 1, 0 ], [ 0, 0, 1 ] ],
      [ [ 1, 0, 1 ], [ 1, 1, 1 ], [ 0, 0, 1 ] ],
      [ [ 1, 1, 0 ], [ 0, 1, 1 ], [ 0, 0, 1 ] ],
      [ [ 1, 1, 1 ], [ 0, 1, 1 ], [ 0, 0, 1 ] ] ],
  [ [ [ 0, 1, 0 ], [ 1, 1, 0 ], [ 0, 0, 1 ] ],
      [ [ 0, 1, 0 ], [ 1, 1, 1 ], [ 0, 0, 1 ] ],
      [ [ 0, 1, 1 ], [ 1, 1, 0 ], [ 0, 0, 1 ] ],
      [ [ 0, 1, 1 ], [ 1, 1, 1 ], [ 0, 0, 1 ] ],
      [ [ 1, 1, 0 ], [ 1, 0, 0 ], [ 0, 0, 1 ] ],
      [ [ 1, 1, 0 ], [ 1, 0, 1 ], [ 0, 0, 1 ] ],
      [ [ 1, 1, 1 ], [ 1, 0, 0 ], [ 0, 0, 1 ] ],
      [ [ 1, 1, 1 ], [ 1, 0, 1 ], [ 0, 0, 1 ] ] ],
  [ [ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] ],
  [ [ [ 1, 0, 0 ], [ 0, 1, 1 ], [ 0, 0, 1 ] ],
      [ [ 1, 0, 1 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],
      [ [ 1, 0, 1 ], [ 0, 1, 1 ], [ 0, 0, 1 ] ] ] ]
```

Since the factorization $\left(\begin{smallmatrix}a&b&c\\d&e&f\\0&0&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0&c\\0&1&f\\0&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b&0\\d&e&0\\0&0&1\end{smallmatrix}\right) \in UL$ is transparent, we can easily see which elements are in which coset of G/P.

We observe that the first and the last conjugacy class listed contain elements of order 2. Since only one class of G contains elements of order 2, namely the class containing g_2 at which we're trying to compute the character, we must have that the class of g_2 splits in P into the first and last classes above. The first class contains a transposition, so ε must be -1 on that class, and the last class is the identity coset, so ε must be 1 on that class. Therefore,

$$\mathrm{Ind}_P^G \varepsilon\left(g_2\right) = \frac{1}{8}\left(\frac{1}{4}(-1) + \frac{1}{8}(1)\right) = -1.$$

Similarly for the other values giving the following character.

	$\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&1\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&0\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\1&0&1\\0&1&0 \end{pmatrix}$
$\mathrm{Ind}_P^G(1)$	7	-1	1	-1	0	0

We confirm that this character is irreducible by taking inner products, or alternately, we notice that this character is χ_3 from the given table.

0.5 The 7-Subgroup

Next, consider the Sylow 7-subgroup $S = \langle g_5 \rangle$. We will induce the following characters.

S	g_1	g_5	g_5^2	g_5^3	g_5^4	g_5^5	g_5^6
ψ_1	1	ω	ω^2	ω^3	ω^4	ω^5	ω^6
ψ_2	1	ω^3	ω^6	ω^2	ω^5	ω	ω^4

We determine which conjugacy classes of the powers of g_5 lie in by computing their rational forms. Gap gives the characteristic polynomials for the powers of g_5 below.

```
gap> g5:=[[0,0,1],[1,0,0],[0,1,1]];
[ [ 0, 0, 1 ], [ 1, 0, 0 ], [ 0, 1, 1 ] ]
gap> for j in [0..6] do
Print(j,": ",CharacteristicPolynomial(g5^j) mod 2,"\n")$
0: 1+x_1+x_1^2+x_1^3
1: 1+x_1^2+x_1^3
2: 1+x_1^2+x_1^3
3: 1+x_1+x_1^3
4: 1+x_1^2+x_1^3
5: 1+x_1+x_1^3
6: 1+x_1+x_1^3
```

We check that both $1 + x^2 + x^3$ and $1 + x + x^3$ are irreducible over $\mathbb{F}_2$ so that the rational forms of A^j have one block containing the companion matrices $\begin{pmatrix} 0&0&1\\1&0&0\\0&1&1 \end{pmatrix}$ and $\begin{pmatrix} 0&0&1\\1&0&1\\0&1&0 \end{pmatrix}$ respectively. We conclude that

$$g_5^3 \sim g_5^5 \sim g_5^6 \sim g_6$$

and

$$g_5 \sim g_5^2 \sim g_5^4.$$

in G. Of course, in S, none of these elements are conjugate. This means that the conjugacy class of g_5 in G splits into three conjugacy classes of H and similarly for g_6. We compute

$$\mathrm{Ind}_S^G \psi_1(g_5) = 7 \cdot \frac{1}{7}\left(\omega + \omega^2 + \omega^4\right)$$

and similarly for the remaining values giving the following table.

	g_1	g_2	g_3	g_4	g_5	g_6
$\mathrm{Ind}_S^G(\psi_1)$	24	0	0	0	$\omega + \omega^2 + \omega^4$	$\omega^3 + \omega^5 + \omega^6$
$\mathrm{Ind}_S^G(\psi_2)$	24	0	0	0	$\omega^3 + \omega^5 + \omega^6$	$\omega + \omega^2 + \omega^4$

Taking inner products, we have

$$\mathrm{Ind}_S^G \psi_1 = \chi_2 + \chi_3 + \chi_4 + \chi_6$$

$$\mathrm{Ind}_S^G \psi_2 = \chi_2 + \chi_3 + \chi_4 + \chi_5$$

giving us χ_5 and χ_6.

0.6 Appendix: The Conjugacy Classes of P

The following GAP program was used to compute the conjugacy classes of P. Our method is extremely simple minded. We simply compute all the conjugates of all the elements of P by all the elements of P. While this may not be such a good algorithm for larger groups, it works quite well for P. Since $P = UL$, we produce P by first producing the sets U and L.

```
UU:=[];LL:=[];
for i in [0..1] do for j in [0..1] do
for k in [0..1] do for l in [0..1] do
B:=[[i,j,0],[k,l,0],[0,0,1]];
if not Determinant(B) = 0 then
UniteSet(LL,[B]);
fi;od;od;od;od;
for i in [0..1] do for j in [0..1] do
B:=[[1,0,i],[0,1,j],[0,0,1]];
UniteSet(UU,[B]);
od;od;
```

Then we take all conjugates. The following routine makes use of the idempotency of GAP's `UniteSet` function.

```
CCC:=[];
for i in [1..4] do for j in [1..6] do
DD:=[];
for U in UU do for L in LL do
B:=(U*L)^(-1)*UU[i]*LL[j]*U*L mod 2;
UniteSet(DD,[B]);
od;od;
UniteSet(CCC,[DD]);
od;od;
```

Then the set `CCC` contains all the conjugacy classes of P.

Homework 6

0.7 The Sylow 7-Subgroup

We recall that G has the following character table.

G	g_1	g_2	g_3	g_4	g_5	g_6
order of element	1	2	3	4	7	7
class size	1	21	56	42	24	24
χ_1	1	1	1	1	1	1
χ_2	6	2	0	0	-1	-1
χ_3	7	-1	1	-1	0	0
χ_4	8	0	-1	0	1	1
χ_5	3	-1	0	1	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
χ_6	3	-1	0	1	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$

Let $P_7 \in \mathrm{Syl}_7(G)$. We compute the conjugacy classes of $N_G(P_7)$ and its character table by restricting characters of G.

We have from DF that $|N_G(P_7)| = 21$. Since $P_7 \lhd N_G(P_7)$, we have that $N_G(P_7)$ has one Sylow 7-subgroup so that $N_G(P_7)$ has 6 elements of order 7. By our computations in Homework 5, we have that these 6 elements form two distinct conjugacy classes. The remaining elements of $N_G(P_7)$ must be the identity and 14 elements of order 3. The elements of order 3 lie in two conjugacy classes of $N_G(P_7)$ as follows.

Let $Q \in \mathrm{Syl}_3(N_G(P_7))$. There are 7 Sylow 3-subgroups of $N_{P_7}(P_3)$ since this is the only way we can account for the the 14 elements of order 3. Therefore,

$$\left[N_G(P_7) : N_{N_G(P_7)}(Q)\right] = 7.$$

Then $\left|N_{N_G(P_7)}(Q)\right| = 3$ so that $N_{N_G(P_7)}(Q) = Q$. Let $q \in Q$. Then since Q is cyclic, we have $C_{N_G(P_7)}(q) = C_{N_G(P_7)}(Q)$ and this set at least contains Q. But

$$Q \leq C_{N_G(P_7)}(q) = C_{N_G(P_7)}(Q) \leq N_{N_G(P_7)}(Q) = Q$$

so that $\left|C_{N_G(P_7)}(q)\right| = 3$. Therefore the size of the conjugacy class of q in $N_G(P_7)$ is 7. This means that $N_G(P_7)$ has two conjugacy classes of elements of order 3.

We begin with the restrictions of χ_1, χ_5 and χ_6 to $N_G(P_7)$ and the characters induced from the nontrivial characters ε_2 and ε_3 of $Q \leq N_G(P_7)$.

order of element	1	3	3	7	7
class size	1	7	7	3	3
χ_1	1	1	1	1	1
χ_5	3	0	0	$\omega+\omega^2+\omega^4$	$\omega^3+\omega^5+\omega^6$
χ_6	3	0	0	$\omega^3+\omega^5+\omega^6$	$\omega+\omega^5+\omega^6$
ε_2	7	ζ	ζ^2	0	0
ε_3	7	ζ^2	ζ	0	0

where ω is a primitive 7th root of unity and ζ is a primitive 3rd root of unity. We find ψ_1, ψ_5 and ψ_6 to be irreducible by taking inner products, we compute

$$\langle \varepsilon_2, \chi_5 \rangle = \langle \varepsilon_3, \chi_5 \rangle = \langle \varepsilon_2, \chi_6 \rangle = \langle \varepsilon_3, \chi_6 \rangle = 1$$

and we find $\varepsilon_2 - \chi_5 - \chi_6$ and $\varepsilon_3 - \chi_5 - \chi_6$ to be irreducible giving the following character table.

order of element	1	3	3	7	7
class size	1	7	7	3	3
ψ_1	1	1	1	1	1
ψ_2	1	ζ	ζ^2	1	1
ψ_3	1	ζ^2	ζ	1	1
ψ_5	3	0	0	$\omega+\omega^2+\omega^4$	$\omega^3+\omega^5+\omega^6$
ψ_6	3	0	0	$\omega^3+\omega^5+\omega^6$	$\omega+\omega^5+\omega^6$

0.8 The Sylow 2- and 3-Subgroups

Next, let $P_2 \in \mathrm{Syl}_2(G)$ and $P_3 \in \mathrm{Syl}_3(G)$ We have from DF that the number of Sylow 2- and 3-subgroups of G are 21 and 28[1] so that $|N_G(P_2)| = 8$ and $|N_G(P_3)| = 6$. We therefore have $N_G(P_2) = P_2 \cong D_8$ by DF and $N_G(P_3) \cong S_3$ by elimination.

To compute the character table for $D_8 = \langle r, s : r^4 = s^2 = 1, rs = sr^3 \rangle$, we lift the three nontrivial characters from the quotient $D_8/\langle r^2 \rangle$ giving χ_2, χ_3, and χ_4, and induce a nontrivial character from $\langle r \rangle$ giving χ_5. We thus have the following table.

D_8	1	r	s	sr	r^2
χ_1	1	1	1	1	1
χ_2	1	-1	-1	1	1
χ_3	1	-1	1	-1	1
χ_4	1	1	-1	-1	1
χ_5	2	0	0	0	-2

The character table for S_3 was computed previously and is reprinted below.

S_3			
χ_1	1	1	1
χ_2	1	-1	1
χ_1	2	0	1

0.9 The Issaacs-Navarro Conjecture for G

When $p = 2$, we have

$$M_1(G) = 4 = M_1(N(P_2)),$$

for $p = 3$ we have

$$M_1(G) = 3 = M_1(N(P_3)),$$

and for $p = 7$ we have

$$\begin{aligned} M_1(G) &= 3 = M_1(N(P_5)), \\ M_2(G) &= 0 = M_2(N(P_5)), \\ M_3(G) &= 2 = M_3(N(P_5)) \end{aligned}$$

[1] There is a typographic error in the statement of Proposition 14 in DF, which has $n_3 = 7$. The proof, however, has $n_3 = 28$.

satisfying the Issaacs-Navarro Conjecture.

0.10 The Issaacs-Navarro Conjecture for A_5

We compute the character degrees with GAP as follows.

```
1  gap> G:=AlternatingGroup(5);
2  Alt( [ 1 .. 5 ] )
3  gap> P:=Normalizer(G,SylowSubgroup(G,2));
4  Group([ (1,3)(2,4), (1,2)(3,4), (2,3,4) ])
5  gap> Q:=Normalizer(G,SylowSubgroup(G,3));
6  Group([ (1,2,3), (2,3)(4,5) ])
7  gap> R:=Normalizer(G,SylowSubgroup(G,5));
8  Group([ (1,2,3,4,5), (2,5)(3,4) ])
9  gap> CharacterDegrees(G);
10 [ [ 1, 1 ], [ 3, 2 ], [ 4, 1 ], [ 5, 1 ] ]
11 gap> CharacterDegrees(P);
12 [ [ 1, 3 ], [ 3, 1 ] ]
13 gap> CharacterDegrees(Q);
14 [ [ 1, 2 ], [ 2, 1 ] ]
15 gap> CharacterDegrees(R);
16 [ [ 1, 2 ], [ 2, 2 ] ]
```

For example, the output on line 12 tells us that the Sylow 2-subgroup has three characters of degree 1 and one character of degree 3. For $p = 2$, we have

$$M_1(G) = 4 = M_1(P),$$

for $p = 3$ we have

$$M_1(G) = 3 = M_1(Q),$$

and for $p = 5$ we have

$$M_1(G) = 2 = M_1(Q),$$
$$M_2(G) = 2 = M_2(Q)$$

satisfying the Issaacs-Navarro Conjecture.

Homework 8

1. *Find the p-blocks of G for $p = 2, 3, 5$.*

2. *Construct the isotypies between $B_3(G)$ and $B_3(N_G(P_3))$ and between $B_7(G)$ and $B_7(N_G(P_7))$, where P_3 and P_7 are Sylow 3- and 7-subgroups.*

3. *In constructing isptypies, a sign is attached to an irreducible character in $B_p(G)$. Is there any connection with the sign which appears in Issaacs-Navarro?*

4. *Let $p = 3$. Find*

   (a) the Brauer characters

   (b) the decomposition numbers

   (c) the Cartan invariants

   (d) the dimension of the PIMs.

   and repeat for $p = 7$.

We have the following table for $GL(3,2)$.

order of element	1	2	3	4	7	7
class size	1	21	56	42	24	24
χ_1	1	1	1	1	1	1
χ_2	6	2	0	0	-1	-1
χ_3	7	-1	1	-1	0	0
χ_4	8	0	-1	0	1	1
χ_5	3	-1	0	1	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
χ_6	3	-1	0	1	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$

We therefore have the following table for $Z\left(\overline{A}\right)$

class size	1	21	56	42	24	24
ω_1	1	21	56	42	24	24
ω_2	1	7	0	0	-4	-4
ω_3	1	-3	8	-6	0	0
ω_4	1	0	-7	0	3	3
ω_5	1	-7	0	14	$-4+4i\sqrt{7}$	$-4-4i\sqrt{7}$
ω_6	1	-7	0	14	$-4-4i\sqrt{7}$	$-4+4i\sqrt{7}$

Reducing this mod $2R$, $3R$, and $7R$, we have the following.

class size	1	21	56	42	24	24
ω_1	1	1	0	0	0	0
ω_2	1	1	0	0	0	0
ω_3	1	1	0	0	0	0
ω_4	1	0	1	0	1	1
ω_5	1	1	0	0	0	0
ω_6	1	1	0	0	0	0
ω_1	1	0	2	0	0	0
ω_2	1	1	0	0	2	2
ω_3	1	0	2	0	0	0
ω_4	1	0	2	0	0	0
ω_5	1	2	0	2	♣	♣
ω_6	1	2	0	2	♣	♣
ω_1	1	0	0	0	3	3
ω_2	1	0	0	0	3	3
ω_3	1	4	1	1	0	0
ω_4	1	0	0	0	3	3
ω_5	1	0	0	0	♣	♣
ω_6	1	0	0	0	♣	♣

From the table mod 2, we see that the 2-blocks of G are $\{\chi_1, \chi_2, \chi_3, \chi_5, \chi_6\}$ and $\{\chi_4\}$. Some of the elements of the table mod 3 have not been reduced. From the other elements, however, we can see that $\{\chi_1, \chi_3, \chi_4\}$ and $\{\chi_2\}$ form 3-blocks. Since 3 divides $\chi_5(1) = \chi_6(1) = 3$, we have that χ_5 and χ_6 mod 3 are the only irreducible Brauer characters in their blocks. This means that $\{\chi_5\}$ and $\{\chi_6\}$ are the remaining 3-blocks.

Next, we compute the 7-blocks. Since $7 \in \mathfrak{p}$ by assumption, we have that

$$-7 = \sqrt{-7}\sqrt{-7} \in \mathfrak{p},$$

but since $\mathfrak{p}$ is prime, this means that $\sqrt{-7} \in \mathfrak{p}$. Thus,

$$24 - \left(-4 + 4\sqrt{-7}\right) = 28 - 4\sqrt{-7} \in \mathfrak{p}$$

which shows that χ_1 and χ_5 are in the same 7-block. Repeating this argument, we have that $\{\chi_1, \chi_2, \chi_4, \chi_6, \chi_7\}$ is a 7-block. This means that $\{\chi_3\}$ is the remaining 7-block.

We next look at the character tables for $N(P_3)$ and $N(P_7)$ where P_3 and P_7 are Sylow 3- and 7- subgroups.

$N(P_3) = \mathcal{S}_3$	Table			$\omega_i(C_j)$		
order of element	1	2	3			
class size	1	3	2			
ψ_1	1	1	1	1	3	2
ψ_2	1	-1	1	1	-3	2
ψ_3	2	0	-1	1	0	-1

$N(P_7)$	Character Table					$\omega_i(C_j)$				
order of element	1	3	3	7	7					
class size	1	7	7	3	3					
ψ_1	1	1	1	1	1	1	7	7	3	3
ψ_2	1	ζ	ζ^2	1	1	1	7ζ	$7\zeta^2$	3	3
ψ_3	1	ζ^2	ζ	1	1	1	$7\zeta^2$	7ζ	3	3
ψ_4	3	0	0	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$	1	0	0	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
ψ_5	3	0	0	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$	1	0	0	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$

where ω is a primitive 7th root of unity and ζ is a primitive 3rd root of unity,

Reducing $\omega_i(C_j)$ from the first table mod $3R$, we see that $B_3(N_G(P_3)) = \{\psi_1, \psi_2, \psi_3\}$. For $\omega_i(C_j)$ from the second table, we observe that

$$\frac{7}{2} \cdot 2 = 7 \in \mathfrak{p}$$
$$\frac{\sqrt{-7}}{2} \cdot 2 = \sqrt{-7} \in \mathfrak{p},$$

but since $2 \notin \mathfrak{p}$ else $7 - 2 - 2 - 2 = 1 \in \mathfrak{p}$, we must have that $\frac{7}{2} \in \mathfrak{p}$ and $\frac{\sqrt{-7}}{2} \in \mathfrak{p}$. Thus,

$$\frac{-1+\sqrt{-7}}{2} - 3 = \frac{-7+\sqrt{-7}}{2} \in \mathfrak{p}$$

which shows that χ_1 and χ_4 are in the same 7-block. Repeating this argument, we have $B_7(N_G(P_7)) = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$.

Next, we see that the maps

$$I_3 : \left\{ \begin{array}{c} \chi_1 \\ \chi_3 \\ \chi_4 \end{array} \right\} \to \left\{ \begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_3 \end{array} \right\}$$

and

$$I_7 : \left\{ \begin{array}{c} \chi_1 \\ -\chi_2 \\ \chi_4 \\ \chi_5 \\ \chi_6 \end{array} \right\} \to \left\{ \begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_3 \\ \chi_4 \\ \chi_5 \end{array} \right\}$$

preserve the character degrees mod 3 and 7 and preserve the values of the characters on the 3- and 7-elements. Then I_3 and I_7 are the isotypies.

For $p = 3$, we have that the trivial character χ_1 is a Brauer character and that χ_2, χ_5 and χ_6 are also Brauer characters since 3 divides $\chi_2(1)$, $\chi_5(1)$, and $\chi_6(1)$. We then have the decomposition matrix

		φ_1	φ_2	φ_3	φ_4	φ_5
	deg	1	·	6	3	3
χ_1	1	1	·	·	·	·
χ_3	7	m	n	·	·	·
χ_4	8	$m+1$	n	·	·	·
χ_2	6	·	·	1	·	·
χ_5	3	·	·	·	1	·
χ_6	3	·	·	·	·	1

Let m and n be the multiplicities of φ_1 and φ_2 in χ_3 as in the table. Then

$$0 = \chi_3(g) = m\varphi_1(g) + n\varphi_2(g) = m + n\varphi_2(g) = \Phi(\omega) \tag{3}$$

for $g \in G$ an element of order 7. The right hand side of (3) is a sum of 7^{th} roots of unity and is therefore some polynomial Φ in ω . Since the minimal polynomial for ω is $1 + x + x^2 + x^3 + x^4 + x^5 + x^6$, we must have either $\Phi(x) = 0$ or $\Phi(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$. Thus, we either have that $m = 0$, in which case χ_3 is an irreducible Brauer character, or else $m = 1$, (1 is the only integer in the sum) in which case $\chi_3 = \varphi_1 + \varphi_2$.

In the latter case,

so that

$$C =^t DD = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Final Thoughts on $GL(3,2)$

Let $G = GL(3,2)$, the simple group of order 168. We begin by computing the conjugacy classes of G. We do this by enumerating all the possible rational forms. There are 6 conjugacy classes.

Next, we construct the characters of G. We begin with χ_1, the trivial character. Next, we define the subgroup $P = \left\{ \begin{pmatrix} \clubsuit & \clubsuit & \clubsuit \\ 0 & \clubsuit & \clubsuit \\ 0 & \clubsuit & \clubsuit \end{pmatrix} \right\} \leq G$ and compute $\mathrm{Ind}_P^G(1)$ where 1 is the trivial character of P. To do this, we observe that P is the stabilizer of the line $\langle v_1 \rangle$ in the vector space $V = \langle v_1, v_2, v_3 \rangle$ over $\mathbb{F}_2$ so that $\mathrm{Ind}_P^G(1)(g)$ is the number of lines fixed by $g \in G$. This gives us χ_2.

Similarly, $B = \left\{ \begin{pmatrix} \clubsuit & \clubsuit & \clubsuit \\ 0 & \clubsuit & \clubsuit \\ 0 & 0 & \clubsuit \end{pmatrix} \right\} \leq G$ is the stabilizer of the flag $\langle v_1 \rangle \leq \langle v_1, v_2 \rangle$ in V so that $\mathrm{Ind}_B^G(1)(g)$ is number of flags fixed by g. This gives us χ_4.

We next construct the character of degree 7 using Harish-Chandra induction. We take

$$P = \left\{ \begin{pmatrix} \clubsuit & \clubsuit & \clubsuit \\ \clubsuit & \clubsuit & \clubsuit \\ 0 & 0 & \clubsuit \end{pmatrix} \right\} \supset L = \left\{ \begin{pmatrix} \clubsuit & \clubsuit & 0 \\ \clubsuit & \clubsuit & 0 \\ 0 & 0 & \clubsuit \end{pmatrix} \right\} \cong \mathcal{S}_3$$

We lift the sign character of $\mathcal{S}_3 \cong L$ to a character of P and induce from this a character of G. This is not extremely easy, most of the complication arising from determining how the conjugacy classes of G split in P, a computation required in the induced character formula. This can at least be accomplished by explicit computation, which I'm sorry to say is what I did. This gives us an irreducible character χ_3 of G.

We compute the remaining two characters for G by inducing two characters from a 7-Sylow subgroup $S \leq G$. Again, the difficulty is in determining how the conjugacy classes of G split in S, but again, this can at least be accomplished with an explicit computation. This produces the characters χ_5 and χ_6 of G.

The characters computed in the discussion are presented in the character table at the head of this document.

Next, we determine the Sylow 2-, 3-, and 7-subgroups P_2, P_3, and P_7 of G and their normalizers. By order considerations and other algebraic manipulations, we find that

$P_2 = N_G(P_2) \cong B$ and $P_3 \lhd N_G(P_3) \cong L \cong \mathcal{S}_3$. For P_7, we use the construction of the characters for groups of order pg to compute the character table for $N_G(P_7)$ which has order $3 \cdot 7$. The character tables for $N_G(P_3)$ and $N_G(P_7)$ are presented on pages 19 and 20 of this document.

Having the character tables for $N_G(P_2)$ $N_G(P_3)$, and $N_G(P_7)$, we were able to verify the Issaacs-Navarro conjecture for G.

Finally, in this document, we computed various p-blocks, isotypies, Brauer characters and Cartan invariants for G.