Richy's Talk on Clifford Theory

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This talk is on Clifford Theory from the point of view of characters. The reference for this talk is Isaacs' *Character Theory*, chapter 6.

Let $H \leq G$ and take $\chi \in \mathrm{Irr}(G)$. We can't say much about χ_H in general, but when $H \lhd G$, the situation changes for the better.

Let ϑ be a class function on H. Define Let $x \in G$ and for $h \in H$ define

$$\vartheta^x(h) = \vartheta\left(xhx^{-1}\right).$$

Lemma 1 *Let φ, ϑ be class functions on H, let χ be a class function on G, and let $x, y \in G$. Then we have the following.*

1. *ϑ^x is a class function.*
2. *$(\vartheta^x)^y = \vartheta^{xy}$.*
3. *$\langle \varphi, \vartheta \rangle = \langle \varphi^x, \vartheta^x \rangle$.*
4. *$\langle \chi_H, \vartheta \rangle = \langle \chi_H, \vartheta^x \rangle$.*
5. *ϑ^x is a character iff ϑ is.*

Proof. To see 4, observe that $(\chi_H)^x = \chi_H$ since χ is a class function on G and use 3. The remaining parts are left as an exercise. ■

We can think of G as permuting $\mathrm{Irr}(H)$ by $g : \vartheta \to \vartheta^g$.

Theorem 1 *(Clifford) Let $H \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Suppose that $\vartheta \in \mathrm{Irr}(H)$ is such that ϑ is a constituent of $\chi|_H$. Let $\vartheta_1, \vartheta_2, \ldots \vartheta_t$ be the distinct G-conjugates of ϑ. Then*

$$\chi_H = e \sum_{i=1}^{t} \vartheta_j$$

where $e = \langle \chi_H, \vartheta \rangle$.

Proof. Observe that the ϑ_j are irreducible since ϑ is. We have

$$\left(\vartheta^G\right)_H(h) = \frac{1}{|H|} \sum_{x \in G} \dot{\vartheta}\left(xhx^{-1}\right) = \frac{1}{|H|} \sum_{x \in G} \dot{\vartheta}^x(h).$$

Thus,

$$|H| \left(\vartheta^G\right)_H = \sum_{x \in G} \vartheta^x.$$

Now if $\lambda \in \mathrm{Irr}(H)$ with $\lambda \notin \{\vartheta_j\}$, then $\left\langle \left(\vartheta^G\right)_H, \lambda \right\rangle = 0$ so that $\langle \chi_H, \lambda \rangle = 0$ by Frobenius and because ϑ is a constituent of χ. Then

$$\chi_H = \sum_{j=1}^{t} \langle \chi_H, \vartheta_j \rangle \vartheta_j,$$

but by Lemma 1, we have $\langle \chi_H, \vartheta_j \rangle = \langle \chi_H, \vartheta \rangle$ for all j so that

$$\chi_H = e \sum_{j=1}^{t} \vartheta_j$$

for $e = \langle \chi_H, \vartheta \rangle$. ■

Definition 1 *Let $H \lhd G$ and let $\vartheta \in \mathrm{Irr}(H)$. Define*

$$T = \{x \in G : \vartheta^x = \vartheta\}.$$

T is called the inertia group *of ϑ.*

The inertia group is the stabilizer of the action of G on $\mathrm{Irr}(G)$. Thus, $[G : T] = t$. This also implies that t divides $[G : H]$ since $H \leq T$.

Theorem 2 *(Clifford) Let $H \lhd G$, $\vartheta \in \mathrm{Irr}(H)$. Define*

$$A = \{\psi \in \mathrm{Irr}(T) : \langle \psi_H, \vartheta \rangle \neq 0\}$$
$$B = \{\chi \in \mathrm{Irr}(G) : \langle \chi_H, \vartheta \rangle \neq 0\}$$

Then

1. *If $\psi \in A$, then $\psi^G \in B$.*
2. *The map $A \to B$ given by $\psi \to \psi^G$ is a bijection.*
3. *If $\psi^G = \chi$, then $\psi \in A$ is the unique irreducible constituent of χ_T that lies in A.*
4. *If $\psi^G = \chi$ with $\psi \in A$, then $\langle \psi_H, \vartheta \rangle = \langle \chi_H, \vartheta \rangle$.*

Proof. Let $\psi \in A$ and choose $\chi \in \mathrm{Irr}(G)$ such that $\langle \psi^G, \chi \rangle \neq 0$. By Frobenius, we have

$$\langle \psi, \chi_T \rangle = \langle \psi^G, \chi \rangle \neq 0$$

so that ψ is a constituent of χ_T. Restricting to H, we have

$$0 < \langle \psi_H, \vartheta \rangle \leq \langle \chi_H, \vartheta \rangle. \tag{1}$$

This shows that $\chi \in B$.

Look at the distinct G conjugates $\vartheta_1, \ldots, \vartheta_t$, where $t = [G : T]$. By Theorem 1, we have

$$\chi_H = e \sum_{j=1}^{t} \vartheta_j$$

where $e = \langle \chi_H, \vartheta \rangle$. Using Theorem 1 again, we have

$$\psi_H = f \sum_{x \in T} \vartheta^x = f\vartheta$$

since T stabilizes ϑ where $f = \langle \psi_H, \vartheta \rangle$.

We have

$$\chi(1) = \chi_H(1) = e\sum_{j=1}^{t} \vartheta_j(1) = et\vartheta(1). \tag{2}$$

But since χ is a constituent of ψ^G, we have

$$\chi(1) \leq \psi^G(1) = t\psi(1) = t\psi_H(1) = tf\vartheta(1) \leq te\vartheta(1). \tag{3}$$

Combining (2) and (3), we have

$$et\vartheta(1) = \chi(1) \leq \psi^G(1) \leq te\vartheta(1)$$

so we have equality all the way through. This means that $\chi(1) = \psi^G(1)$ so that $\chi = \psi^G$ and 1 follows.

Additionally, this means $e = f$ so that 4 follows. To see 3, notice that if we have $\psi_1 \in A$ with $\psi_1 \neq \psi$ and $\langle \psi_1, \chi_T \rangle \neq 0$, then we have

$$e = \langle \chi_H, \vartheta \rangle \geq \langle (\psi + \psi_1)_H, \vartheta \rangle = \langle \psi_H, \vartheta \rangle + \langle (\psi_1)_H, \vartheta \rangle > 0 + \langle \psi_H, \vartheta \rangle = f,$$

a contradiction and we have 3. Observe also that 3 is the statement that $\psi \to \psi^G$ is injective, since if we have $\psi_1 \in A$ with $\psi_1^G = \chi$, then $\psi_1 = \psi$.

To see that this map is surjective, observe that if $\vartheta \in B$, then there must exist $\psi \in \mathrm{Irr}(T)$ with $\langle \psi, \chi_T \rangle \neq 0$. But then $\langle \psi_H, \vartheta \rangle \neq 0$ so that $\psi \in A$ and χ is a constituent of ψ^G, so $\chi = \psi^G$. ■

Application 1 *(Ito's Theorem) If $A \triangle G$ with A abelian. Then $\chi(1)$ divides $[G : A]$ for all $\chi \in \mathrm{Irr}(G)$.*

Suppose $N \triangleleft G$ and let $\vartheta \in \mathrm{Irr}(N)$. Then

$$\vartheta^G = \sum e_j \chi_j$$

for $\chi_j \in \mathrm{Irr}(G)$ and $e_j \geq 0$. By Clifford Theory, we have

$$(\chi_j)_N = e_j \sum_{k=1}^{t} \vartheta_k$$

where $\vartheta_1, \ldots, \vartheta_t$ are the G-conjugates of ϑ. Then

$$\vartheta^T = \sum e_j \psi_j$$

where $\psi_j^G = \chi_j$ by the bijection above.

Assume that $T = G$, that is, ϑ is invariant in G. Hence,

$$\begin{aligned} \vartheta^G &= \sum e_j \chi_j \\ (\chi_j)_N &= e_j \vartheta \\ \chi_j(1) &= e_j \vartheta_j(1). \end{aligned}$$

Additionally, we have

$$[G : N]\,\vartheta(1) = \vartheta^G(1) = \sum e_j \chi_j(1) = \sum e_j^2 \vartheta(1).$$

Thus,

$$[G : N] = \sum e_j^2.$$

It can be proved using projective representations that $e_j | [G : N]$. If it were true that $e_1 = 1$, then $(\chi_1)_N = \vartheta$ and in this situation, ϑ is called *extensible to* χ.

Theorem 3 *(Gallagher) If there exists* $\{e_j\}$ *such that* $e_1 = 1$*, then the* $\{e_j\}$ *are exactly the character degrees of* G/N*.*

Note that if N is abelian, then $e_j = \chi_j(1)$, and also by Ito's theorem, $e_j | [G : N]$.