Steve Smith's Talk on Groups of Lie Type

This is an introduction to Groups of Lie Type. In particular, we discuss the uniform construction of these groups by Chevalley. A reference for this topic is Carter's book *Simple Groups of Lie Type*. Groups of Lie type have been a main theme in finite group theory for all of Smith's career.

The outline for this talk is the following.

1. What we hope to unify,
2. The model case: Lie Algebras over $\mathbb{C}$,
3. What from the model case we hope to mimic for finite fields in place of $\mathbb{C}$,
4. How Chevalley et al. *did* mimic this construction, and
5. other directions including representation theory of these groups.

"Most" of the finite simple groups are of Lie Type. The classification of FSG's is the following.

1. A_n,
2. Groups of Lie Type
3. Sporadic

The groups of Lie type consist of 13 infinite families of matrix algebras over finite fields. The topic essentially started in analysis.

The simple Lie groups over $\mathbb{C}$ have been classified/unified. Their names are A_n, B_n, C_n, D_n, (the classical) and G_2, F_4, E_{10}, E_7, and E_8 (the exceptional). For example, $A_n = M_n(\mathbb{C})$. B_n corresponds to certain orthogonal, C_n corresponds to the symplectic, D_n corresponds to other orthogonal groups, and the remaining groups are exceptional.

Analogously, for finite fields, you replace the $\mathbb{C}$ with $\mathbb{F}_q$ in the classical groups. The problem we're considering involves determining whether analogs of the exceptional groups exist for finite fields and if so, to unify/classify them.

Next, we look at the Model Case of the Lie Groups/Lie Algebras over $\mathbb{C}$. A good reference for this topic is Varadarajen's book. This topic arises in analysis since a Lie Group over $\mathbb{C}$ is a group together with a metric topology. Over a finite field, the metric topology is discrete. The connection between Lie Groups and Lie Algebras is given by the exponential map.

For example, consider $\mathrm{O}(3,\mathbb{R})$, the 3×3 orthogonal matrices over $\mathbb{R}$, that is, the rotations. A one-parameter family for this collection is the set of rotations $R_\theta(x)$ about the x-axis

$$R_\theta(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

Then

$$R_\theta(x) \xrightarrow{\frac{d}{d\theta}} R_\theta(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \\ 0 & -\cos\theta & -\sin\theta \end{pmatrix} \xrightarrow{|_{\theta=0}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and the exponential map is the inverse of this operation.

Over the finite field $\bar{\mathbb{F}_p}$, the algebraic closure of $\mathbb{F}_p$, we can use the Zariski topology to mimic the metric topology.

Using this, we develop root systems and eventually the classification.

Next we consider what we hope to mimic for finite fields. Certain aspects of classical groups involving the metric won't work with the Zariski topology. The operation $\frac{d}{d\theta}$ won't work, but the exponential map can be used as

$$\exp(M) = I + M + \frac{M^2}{2} + \frac{M^3}{3!} + \dots \tag{1}$$

If M is nilpotent, this sum is finite and we don't have to consider limits.

Now if F is a finite field, than F has characteristic p for some finite prime p. To make (1) work in characteristic p, we can make M integral and take

$$\exp(M) \otimes_{\mathbb{Z}/p} F.$$

This process is known as *producing a Chevalley basis.*

Next, we consider the actual implementation by Chevalley. From the theory of simple Lie algebras $\mathcal{G}$ over $\mathbb{C}$, there is a Cartan decomposition

$$\mathcal{G} = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^-$$

where the parts represent the positive roots, the Cartan subgroup, and the negative roots

$$\mathcal{N}^+ = \begin{pmatrix} 0 & \clubsuit & \clubsuit \\ 0 & 0 & \clubsuit \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{H} = \begin{pmatrix} \clubsuit & 0 & 0 \\ 0 & \clubsuit & 0 \\ 0 & 0 & \clubsuit \end{pmatrix} \qquad \mathcal{N}^- = \begin{pmatrix} 0 & 0 & 0 \\ \clubsuit & 0 & 0 \\ \clubsuit & \clubsuit & 0 \end{pmatrix}.$$

In Lie theory, we have the Lie brackets $[a,b] = ab - ba$ defined using multiplication in the algebra. We use this to define an adjoint operation $\mathrm{Ad}(x)$. This produces nilpotent matrices. This doesn't work for matrices in $\mathcal{H}$.

This summarizes Chevalley's construction.

Let $\mathcal{U} = \exp(\mathrm{ad}(\mathcal{N}^+))$, $\mathcal{V} = \exp(\mathrm{ad}(\mathcal{N}^-))$, and define $G = \langle U, V \rangle \subset \mathrm{GL}(\mathcal{G}_2)$. This is the Chevalley group of type G over $\mathbb{Z}$.

The representation theory of the Chevalley groups over characteristic p involves aspects of highest weight theory. Over characteristic 0.

The groups of Lie Type are the Chevelly groups and the twisted groups. The twisted groups are certain fixed point subgroups under certain automorphisms. Much of this is due to Steinberg. Consider $\bar{\mathbb{F}_p}$ and let $\mathcal{G}$ be a type of Lie algebra over $\mathbb{C}$. We construct the Chevalley group of type $\mathcal{G}$ over $\bar{\mathbb{F}_p}$. Denote by $F : \bar{\mathbb{F}_p} \to \bar{\mathbb{F}_p}$ the map $x \to x^p$ (the Frobenius map). Then $\mathrm{fix}_{F:\bar{\mathbb{F}_p}}(F) = \mathbb{F}_p$.

Given an automorphism of the Dynkin's diagram for the type and a suitable field, we look at the set of fixed points under the automorphism and this is the Chevalley construction for the twisted groups.