Homework 1

1. For $A := \begin{pmatrix} -8 & -10 & -1 \\ 7 & 9 & 1 \\ 3 & 2 & 0 \end{pmatrix}$, we have that $c_A(X) = (X-1)^2(X+1)$. Since $u_1 := \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ is an eigenvector corresponding with -1, we have that $U_1 := \langle u_1 \rangle$ is an A-submodule as

$$\left(\sum_{j=0}^{r} \alpha_j T^j\right)(\beta u_1) = \left(\beta \sum_{j=0}^{r} \alpha_j (-1)^j\right) u_1 \in U_1.$$

Linear algebra gives that

$$\mathcal{N}(A+1)^2 = \left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

is the generalized eigenspace for the eigenvector 1, with the first vector listed being an eigenvector for 1. Now taking

$$u_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ and } u_2 := (A+1)u_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

we have that $U_2 := \langle u_2, u_3 \rangle = \langle u_3 \rangle$ is a A-submodule as $Au_2 = u_2 \in U_2$ and $Au_3 = u_2 + u_3 \in U_2$.

Also, $\{v_1, v_2, v_3\}$ is a basis for V_1 so that $V_1 = \langle u_1 \rangle \oplus \langle u_3 \rangle = U_1 \oplus U_2$.

However, U_2 is *not* irreducible as $\langle u_2 \rangle$ is a proper A-submodule of U_2. Also U_2 is not completely reducible as the only possible compliment of $\langle u_2 \rangle$ would be $\langle u_3 \rangle$, which is not a submodule as $Au_3 = u_2 \notin \langle u_3 \rangle$. This also means that V_1 is not completely reducible.

We compute the Jordan form for A for our amusment: define

$$S_A = (u_1, u_2, u_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -4 & -1 & 1 \end{pmatrix}$$

and

$$J_A = (Au_1, Au_2, Au_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

so that $A = SJS^{-1}$.

Continuing, let $B := \begin{pmatrix} -3 & 2 & -4 \\ 4 & -1 & 4 \\ 4 & -2 & 5 \end{pmatrix}$. Then $c_B(X) = c_A(X) = (X-1)^2(X+1)$ and the Jordan form for B is $J_B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with conjugating matrix $S_B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$. In particular, this means that B has 3 independent eigenvectors

$$v_1 := \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, v_2 := \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \text{ and } v_3 := \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

so that $\langle v_1 \rangle$,$\langle v_2 \rangle$, and $\langle v_3 \rangle$ are all submodules of V_2 and $V_2 = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \langle v_3 \rangle$ so that V_2 is completely reducible. We have, however, that V_1 is *not* completely reducible so that V_2 is *not* isomorphic with V_1.

2. Let $V = \langle e_1, e_2 \cdots e_m \rangle$. We have that

$$e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n} = e_{\sigma j_1} \otimes e_{\sigma j_2} \otimes \cdots \otimes e_{\sigma j_n}$$

in $S^n V$ for any $\sigma \in S_n$ since if $\sigma = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_r$ for transposition s τ_j, then

$$e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n} = e_{\tau_r j_1} \otimes e_{\tau_r j_2} \otimes \cdots \otimes e_{\tau_r j_n} = e_{\tau_{r-1}\tau_r j_1} \otimes e_{\tau_{r-1}\tau_r j_2} \otimes \cdots \otimes e_{\tau_{r-1}\tau_r j_n} = \cdots =$$
$$e_{\tau_1 \cdots \tau_{r-1}\tau_r j_1} \otimes e_{\tau_1 \cdots \tau_{r-1}\tau_r j_2} \otimes \cdots \otimes e_{\tau_1 \cdots \tau_{r-1}\tau_r j_n} = e_{\sigma j_1} \otimes e_{\sigma j_2} \otimes \cdots \otimes e_{\sigma j_n}.$$

This shows that two tensors are equal in the symmetric product if their component vectors are permuted. We can then assume that any tensor can be expressed with its components in non-decreasing order as follows:

$$\underbrace{\overbrace{e_1 \otimes e_1 \otimes \cdots \otimes e_1}^{r_1} \otimes \overbrace{e_2 \otimes e_2 \otimes \cdots \otimes e_2}^{r_2} \otimes \cdots \overbrace{e_m \otimes e_m \otimes \cdots \otimes e_m}^{r_m}}_{n}$$

with $r_j \geq 0$. Then two tensors will be equal if they have the same number of e_1's, the same number of e_2's, etc., so that the problem is reduced to selecting $r_1, r_2 \ldots r_m$ with $r_j \geq 0$ and $r_1 + r_2 + \cdots r_m = n$.

Now we all know that there are $\binom{n+m-1}{r}$ ways to put n balls into m baskets, for, you see, the m baskets, when placed side by side, have $m-1$ interior walls, so that we only have to count the number of ways to select n of $n+m-1$ objects to be balls (and not walls), giving the formula above.

Now, returning to the symmetric product, we see that selecting $r_j \geq 0$ with $\sum_j r_j = n$ is equivalent to putting n balls into m baskets so that there will be $\binom{n+m-1}{r}$ ways to do so. Hence, $\dim S^n V = \binom{n+m-1}{r}$.

3.

$$(\bigwedge^n \rho)(e_1 \wedge e_2 \wedge \cdots \wedge e_n) = \rho e_1 \wedge \rho e_2 \wedge \cdots \wedge \rho e_n =$$

$$\sum_i a_{i,1} e_i \wedge \sum_i a_{i,2} e_i \wedge \cdots \wedge \sum_i a_{i,n} e_i =$$

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n})(e_1 \wedge e_2 \wedge \cdots \wedge e_n) = \det A(e_1 \wedge e_2 \wedge \cdots \wedge e_n).$$

4. If $gh = k$, then $h = g^{-1}k$ so that

$$g\left(\sum_h \alpha_h e_h\right) = \sum_h \alpha_h e_{gh} = \sum_k \alpha_{g^{-1}k} e_k.$$

We identify the vector $\sum_h \alpha_h e_h$ with the $\mathbf{C}$-valued function $\sum_h \alpha_h I_h$ and make the collection R_1 of $\mathbf{C}$-valued function into a G-module by defining

$$g\varphi(k) := \varphi(g^{-1}k)$$

for $\varphi \in R_1$ and $g, k \in G$. Then

$$g\left(\sum_h \alpha_h I_h\right)(k) = \left(\sum_h \alpha_h I_h\right)(g^{-1}k) = \alpha_{g^{-1}k}$$

so that

$$g\left(\sum_h \alpha_h I_h\right) = \sum_k \alpha_{g^{-1}k} I_k.$$

Then under the identification above, $g\left(\sum_h \alpha_h e_h\right)$ corresponds with $g\left(\sum_h \alpha_h I_h\right)$ as required.

Let R_2 be the space of functions on G made into a module by the action

$$g\varphi(k) := \varphi(kg)$$

for $\varphi \in R_2$ and $k, g \in G$.

Then

$$g\left(\sum_h \alpha_h I_h\right)(k) = \left(\sum_h \alpha_h I_h\right)(kg) = \alpha_{kg}$$

so that

$$g\left(\sum_h \alpha_h I_h\right) = \sum_k \alpha_{kg} I_k.$$

Define $\Phi : R_1 \longrightarrow R_2$ by

$$\Phi\left(\sum_h \alpha_h I_h\right) := \sum_h \alpha_{h^{-1}} I_h.$$

Then Φ is clearly a $\mathbf{C}$-vector space isomorphism. To see that Φ is a G-module homomorphism,

$$\Phi\left(g\sum_h \alpha_h I_h\right) = \Phi\left(\sum_k \alpha_{g^{-1}k} I_k\right) = \sum_k \alpha_{k^{-1}g} I_k = g\sum_h \alpha_{h^{-1}} I_h = g\Phi\left(\sum_h \alpha_h I_h\right).$$

Cheers!

Homework 10

Marcus

1. *Let V be a vector space of dimension k over $\mathbb{C}$. Show that $V^{\otimes 3}$ is the direct sum of 4 irreducible $GL(V) = GL(k, \mathbb{C})$-modules of dimensions*

$$\frac{k(k+1)(k+2)}{6}, \frac{k(k-1)(k-2)}{6}, \frac{k(k^2-1)}{3}, \frac{k(k^2-1)}{3}.$$

$\mathbb{S}_\lambda V$ is irreducible for all $\lambda \vdash 3$ with fewer than 4 parts. Assuming that $k \geq 3$, all the partitions of 3 have fewer than 4 parts. For $\lambda = 1^3$, we compute

$$\begin{aligned}
\dim \mathbb{S}_{(1^3)} V &= \prod_{1\leq i<j\leq k} \frac{\lambda_i - \lambda_j + j - i}{j-i} \\
&= \left(\frac{1-1+2-1}{2-1}\right)\left(\frac{1-1+3-1}{3-1}\right)\left(\frac{1-0+4-1}{4-1}\right)\cdots\left(\frac{1-0+k-1}{k-1}\right)\cdot \\
&\quad \left(\frac{1-1+3-2}{3-2}\right)\left(\frac{1-0+4-2}{4-2}\right)\left(\frac{1-0+5-2}{5-2}\right)\cdots\left(\frac{1-0+k-2}{k-2}\right)\cdot \\
&\quad \left(\frac{1-0+4-3}{4-3}\right)\left(\frac{1-0+5-3}{5-3}\right)\left(\frac{1-0+6-3}{6-3}\right)\cdots\left(\frac{1-0+k-3}{k-3}\right)\cdot \\
&\quad \left(\frac{0-0+5-4}{5-4}\right)\left(\frac{0-0+6-4}{6-4}\right)\left(\frac{0-0+7-4}{7-4}\right)\cdots\left(\frac{0-0+k-4}{k-4}\right)\cdot \\
&\quad \cdots\left(\frac{0-0+k-(k-1)}{k-(k-1)}\right) \\
&= 1\cdot 1\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{k}{k-1}\cdot \\
&\quad 1\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{k-1}{k-2}\cdot \\
&\quad \frac{2}{1}\cdot\frac{3}{2}\cdots\frac{k-2}{k-3}\cdot \\
&\quad 1\cdot 1\cdots 1\cdot \\
&\quad 1 \\
&= \frac{k}{3}\cdot\frac{k-1}{2}\cdot\frac{k-2}{1} \\
&= \frac{k(k-1)(k-2)}{6}.
\end{aligned}$$

We similarly compute that

$$\dim \mathbb{S}_{(2,1)}V = \frac{k(k^2-1)}{3} \text{ and } \dim \mathbb{S}_{(3)}V = \frac{k(k-1)(k-2)}{6}$$

and that

$$V^{\otimes 3} = \mathbb{S}_{(1^3)}V \oplus \left(\mathbb{S}_{(2,1)}V\right)^{\otimes 2} \oplus \mathbb{S}_{(3)}V$$

2. *Let $k = 3$ and $d = 3$ as above. Let $\{e_1, e_2, e_3\}$ be a basis of V. Let $\lambda = 2, 1$.*

   (a) *Verify that $\mathbb{S}_\lambda V$ has dimension 8 and that is has a basis as given in the Semistandard Basis Theorem.*

   We compute that the semi-standard tableux are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\ \text{and}\ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,.$$

   Now for $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$, we have

$$\begin{aligned}
\big(e_1 \otimes e_2 \otimes e_1\big)c_\lambda &= \big(e_1 \otimes e_2 \otimes e_1\big)\big(1 + (12) - (13) - (132)\big) \\
&= \big(e_1 \otimes e_2 \otimes e_1\big) + \big(e_2 \otimes e_1 \otimes e_1\big) - \big(e_1 \otimes e_2 \otimes e_1\big) - \big(e_1 \otimes e_1 \otimes e_2\big) \\
&= \big(e_2 \otimes e_1 \otimes e_1\big) - \big(e_1 \otimes e_1 \otimes e_2\big).
\end{aligned}$$

   Similarly, we have

$$\begin{aligned}
\big(e_1 \otimes e_3 \otimes e_1\big)c_\lambda &= \big(e_3 \otimes e_1 \otimes e_1\big) - \big(e_1 \otimes e_1 \otimes e_3\big), \\
\big(e_1 \otimes e_2 \otimes e_2\big)c_\lambda &= \big(e_1 \otimes e_2 \otimes e_2\big) - \big(e_2 \otimes e_2 \otimes e_1\big), \\
\big(e_1 \otimes e_3 \otimes e_2\big)c_\lambda &= \big(e_1 \otimes e_3 \otimes e_2\big) + \big(e_3 \otimes e_1 \otimes e_2\big) - \big(e_2 \otimes e_3 \otimes e_1\big) - \big(e_2 \otimes e_1 \otimes e_3\big), \\
\big(e_1 \otimes e_2 \otimes e_3\big)c_\lambda &= \big(e_1 \otimes e_2 \otimes e_3\big) + \big(e_2 \otimes e_1 \otimes e_3\big) - \big(e_3 \otimes e_2 \otimes e_1\big) - \big(e_3 \otimes e_1 \otimes e_2\big), \\
\big(e_1 \otimes e_3 \otimes e_3\big)c_\lambda &= \big(e_1 \otimes e_3 \otimes e_3\big) - \big(e_3 \otimes e_3 \otimes e_1\big), \\
\big(e_2 \otimes e_3 \otimes e_2\big)c_\lambda &= \big(e_3 \otimes e_2 \otimes e_2\big) - \big(e_2 \otimes e_2 \otimes e_3\big), \\
&\text{and} \\
\big(e_2 \otimes e_3 \otimes e_3\big)c_\lambda &= \big(e_2 \otimes e_3 \otimes e_3\big) - \big(e_3 \otimes e_3 \otimes e_2\big)
\end{aligned}$$

   We notice that no simple tensor $e_i \otimes e_j \otimes e_k$ appears more than once as a summand of any of the 8 vectors above so that if

$$\begin{aligned}
0 &= \alpha_1\Bigg(e_2 \otimes e_1 \otimes e_1 - e_1 \otimes e_1 \otimes e_2\Bigg) + \alpha_2\Bigg(e_3 \otimes e_1 \otimes e_1 - e_1 \otimes e_1 \otimes e_3\Bigg) + \cdots \\
&= 0 \cdot \big(e_1 \otimes e_1 \otimes e_1\big) - \alpha_1\big(e_1 \otimes e_1 \otimes e_2\big) + \cdots
\end{aligned}$$

   then by the independence of $\Bigg\{e_i \otimes e_j \otimes e_k \big| 1 \leq i, j, k \leq 3\Bigg\}$, we have that $\alpha_j = 0$ for all $1 \leq j \leq 8$ so that the above vectors are independent. Thus these vectors form a basis for $\mathbb{S}_\lambda V$.

(b) *Let* $g = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}$. *Verify the character formula given in Fulton and Harris in Theorem 6.3*

We compute the character of $\mathbb{S}_V$ at $g = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}$ directly by determining the images of the basis vectors computed in the previous problem under the action of g. We have that

$$\begin{aligned} \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \cdot \big(e_2 \otimes e_1 \otimes e_1 - e_1 \otimes e_1 \otimes e_2\big) &= x_2 e_2 \otimes x_1 e_1 \otimes x_1 e_1 - x_1 e_1 \otimes x_1 e_1 \otimes x_2 e_2 \\ &= x_1^2 x_2 \big(e_2 \otimes e_1 \otimes e_1 - e_1 \otimes e_1 \otimes e_2\big) \end{aligned}$$

and similarly for the other basis elements so that the matrix corresponding with this action is

$$\begin{pmatrix} x_1^2 x_2 & 0 & \cdots \\ 0 & x_1^2 x_3 & \\ \vdots & & \ddots \end{pmatrix}$$

and the trace of this matrix is

$$x_2^2 x_3 + x_1 x_2^2 + x_3^2 x_2 + 2x_1 x_2 x_3 + x_1^2 x_2 + x_1 x_3^2 + x_1^2 x_3$$

which corresponds with the Schur polynomial I computed with Maple.

We *further* corroborate the claim that the Schur polynomials S_λ are in fact the characters of $\mathbb{S}_\lambda$ by computing the trace of $\big(g, (12)\big)$ on $V^{\otimes 3}$ and verifying that this is $\sum_{\lambda \in \Lambda(3,3)} S_\lambda(x_1, x_2, x_3)\chi_\lambda(12)$, the trace of $(g, (12)$ on $\bigoplus_{\lambda \in \Lambda(3,3)} \mathbb{S}_\lambda V \otimes V_\lambda$. To compute the trace on $V^{\otimes 3}$, we note that only the basis vectors $e_j \otimes e_j \otimes e_k$ will be fixed under the right action of (12) and that in this situation,

$$\begin{aligned} \left(\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, (12) \right) \cdot e_j \otimes e_j \otimes e_k &= \left(\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \cdot e_j \otimes e_j \otimes e_k \right) \cdot (12) \\ &= (x_j e_j \otimes x_j e_j \otimes x_k e_k) \cdot (12) \\ &= x_j e_j \otimes x_j e_j \otimes x_k e_k \\ &= x_j^2 x_k \big(e_j \otimes e_j \otimes e_k\big). \end{aligned}$$

where $\cdot$ generically represents all actions involved. Thus,

$$\begin{aligned}
\text{Trace of } \left(\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, (12) \right) \text{ on } V^{\otimes 3} &= \operatorname{Tr} \begin{pmatrix} x_1^3 & 0 & 0 & 0 & \cdots \\ 0 & x_1^2 x_2 & 0 & 0 & \cdots \\ 0 & 0 & x_1^2 x_3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & & & & \\ 0 & 0 & 0 & x_1^2 x_2 & \end{pmatrix} \\
&= \sum_{1 \le j,k \le 3} x_j^2 x_k \\
&= (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2)
\end{aligned}$$

Maple gives that

$$\begin{aligned}
S_{1^3}(x_1, x_2, x_3) &= x_1 x_2 x_3 \\
S_{2,1}(x_1, x_2, x_e) &= x_2^2 x_3 + x_1 x_2^2 + x_3^2 x_2 + 2x_1 x_2 x_3 + x_1^2 x_2 + x_1 x_3^2 + x_1^2 x_3 \\
S_3(x_1, x_2, x_3) &= x_2^3 + x_2^2 x_3 + x_1 x_2^2 + x_3^2 x_2 + x_1 x_2 x_3 + x_1^2 x_2 + x_3^3 + x_1 x_3^2 + x_1^2 x_3 + x_1^3.
\end{aligned}$$

Substituting these polynomials for $\chi_{\mathbb{S}_{1^3} V}(x_1, x_2, x_3)$, $\chi_{\mathbb{S}_{2,1} V}(x_1, x_2, x_3)$, and $\chi_{\mathbb{S}_3 V}(x_1, x_2, x_3)$, we have

$$\begin{aligned}
&\text{Trace of } \left(\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, (12) \right) \\
&\qquad \text{on } \bigoplus_{\lambda \in \Lambda(3,3)} \mathbb{S}_\lambda V \otimes V_\lambda = \sum_{\lambda \in \Lambda(3,3)} \chi_{\mathbb{S}_\lambda V}(x_1, x_2, x_3) \chi_\lambda(12) \\
&\qquad = \chi_{\mathbb{S}_{1^3} V}(x_1, x_2, x_3) \chi_{1^3}(12) + \chi_{\mathbb{S}_{2,1} V}(x_1, x_2, x_3) \chi_{2,1}(12) \\
&\qquad\quad + \chi_{\mathbb{S}_3 V}(x_1, x_2, x_3) \chi_3(12) \\
&\qquad = \chi_{\mathbb{S}_{1^3} V}(x_1, x_2, x_3) \cdot (-1) + \chi_{\mathbb{S}_{2,1} V}(x_1, x_2, x_3) \chi_{2,1}(12) \cdot 0 \\
&\qquad\quad + \chi_{\mathbb{S}_3 V}(x_1, x_2, x_3) \chi_3(12) \cdot 1 \\
&\qquad = x_2^3 + x_2^2 x_3 + x_1 x_2^2 + x_3^2 x_2 \\
&\qquad\quad + x_1^2 x_2 + x_3^3 + x_1 x_3^2 + x_1^2 x_3 + x_1^3 \\
&\qquad = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2) \\
&\qquad = \text{Trace of } \left(\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, (12) \right) \text{ on } V^{\otimes 3}
\end{aligned}$$

Homework 2

2. Let W be some representation of $D_4 = \langle r, s | r^4 = s^2 = 1, srs = r^3 \rangle$. When we allow the abelian subgroup $\langle r \rangle \leq D_4$ to act on W, we have by Maschke that $W = \bigoplus_{j=1}^m V_j$ for irreducible $\langle r \rangle$-submodules V_j, but by Schur, $V_j = \langle v_j \rangle$ for eigenvectors v_j of r. Moreover, the eigenvalue for v_j is i^{k_j}, $k_j = 0, 1, 2, 3$.

1. Now suppose v_j is an eigenvector for r with eigenvalue i. Then

$$rsv_j = sr^3 v_j = si^3 v_j = -isv_j$$

so that sv_j is also an eigenvector for r with eigenvalue $-i$. Then v_j and sv_j are independent, having different eigenvalues, so that $W_1 = \langle v_j, sv_j \rangle$ is a D_4-submodule of W.

Now W_1 is irreducible since any proper subspace $\langle \alpha v_j + \beta sv_j \rangle \leq W_1$, is *not* D_4-stable as $s(\alpha v_j + \beta sv_j) = \beta v_j + \alpha sv_j \in \langle \alpha v_j + \beta sv_j \rangle$ so that $\alpha = \pm\beta$ and we can assume $\alpha = 1$ and $\beta = \pm 1$, but $r(v_j \pm sv_j) = iv_j \mp isv_j \notin \langle v_j \pm sv_j \rangle$.

Thus W_1 is a irreducible two-dimensional D_4-module.

2. If v_j is an eigenvector for r with eigenvalue -1, then by a similar argument, we have that sv_j is also an eigenvector for r with eigenvalue -1.

   (a) If v_j and sv_j are *not* independent, then $sv_j = cv_j$ for some constant $c \in \mathbf{C}$. Then applying s to both sides,

   $$v_j = csv_j = c^2 v_j$$

   so that $c = \pm 1$.

      i. If $sv_j = v_j$, then let $W_2 = \langle v_j \rangle$. W_2 is an irreducible D_4-module.

      ii. If $sv_j = -v_j$, then let $W_3 = \langle v_j \rangle$. W_3 is also an irreducible D_4-module.

   (b) If v_j and sv_j *are* independent, then $\langle v_j, sv_j \rangle$ is a D_4-submodule, but is not irreducible as $\langle v_j + sv_j \rangle$ and $\langle v_j - sv_j \rangle$ are proper D_4-submodules of $\langle v, sv_j \rangle$ with $\langle v_j + sv_j \rangle \oplus \langle v_j - sv_j \rangle = \langle v_j, sv_j \rangle$. These modules are isomorphic with W_2 and W_3 above.

3. If v_j is an eigenvector for r with eigenvalue 1, then as above, sv_j is also an eigenvector for r with eigenvalue 1.

   (a) If v_j and sv_j are *not* independent, then as above, $sv_j = \pm v_j$

      i. If $sv_j = v_j$, then $W_4 = \langle v_j \rangle$ is an irreducible submodule and is isomorphic with the trivial module.

      ii. If $sv_j = -v_j$, then $W_5 = \langle v_j \rangle$ is a irreducible submodule.

   (b) If v_j and sv_j *are* independent, then $\langle v_j, sv_j \rangle$ is a D_4-submodule, but is not irreducible as $\langle v_j + sv_j \rangle$ and $\langle v_j - sv_j \rangle$ are proper D_4-submodules of $\langle v, sv_j \rangle$ with $\langle v_j + sv_j \rangle \oplus \langle v_j - sv_j \rangle = \langle v_j, sv_j \rangle$. These modules are isomorphic with W_4 and W_5 above.

We have thus computed that the only irreducible D_4-modules are W_1, which is two-dimensional, and W_2, W_3, W_4, and W_5, which are one-dimensional.

1.11 S^2V has basis $\{\alpha^2, \alpha\beta, \beta^2\}$ with

$$\tau\alpha^2 = \tau\alpha \cdot \tau\alpha = \omega\alpha \cdot \omega\alpha = \omega^2\alpha^2.$$

We similarly compute

$$\begin{array}{ll} \tau\alpha^2 = \omega^2\alpha^2 & \sigma\alpha^2 = \beta^2 \\ \tau\alpha\beta = \alpha\beta & \sigma\alpha\beta = \alpha\beta \\ \tau\beta^2 = \omega\beta^2 & \sigma\beta^2 = \alpha^2. \end{array}$$

We have, by arguments similar to those given in problem 2 above, that $\langle\alpha\beta\rangle$ and $\langle\alpha^2, \beta^2\rangle$ are irreducible submodules and

$$S^2V = \langle\alpha\beta\rangle \oplus \langle\alpha^2, \beta^2\rangle \cong U \oplus V.$$

Similarly, $\{\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3\}$ is a basis for S^3V. We compute

$$\begin{array}{ll} \tau\alpha^3 = \alpha^3 & \sigma\alpha^3 = \beta^3 \\ \tau\alpha^2\beta = \omega\alpha^2\beta & \sigma\alpha^2\beta = \alpha\beta^2 \\ \tau\alpha\beta^2 = \omega^2\alpha\beta^2 & \sigma\alpha\beta^2 = \alpha^2\beta \\ \tau\beta^3 = \beta^3 & \sigma\beta^3 = \alpha^3. \end{array}$$

Then

$$S^3V = \langle\alpha^2\beta, \alpha\beta^2\rangle \oplus \langle\alpha^3 + \beta^3\rangle \oplus \langle\alpha^3 - \beta^3\rangle.$$

1.12 This approach is extremely direct. You will hate every word of it. We use matrices to compute the eigenvectors for τ.

	Eigenvector for τ	Eigenvalue
v_1	$1 + \tau + \tau^2$	1
v_2	$\sigma + \sigma\tau + \sigma\tau^2$	1
v_3	$1 + \omega^2\tau + \omega\tau^2$	ω
v_4	$\sigma + \omega\sigma\tau + \omega^2\sigma\tau^2$	ω
v_5	$1 + \omega\tau + \omega^2\tau^2$	ω^2
v_6	$\sigma + \omega^2\sigma\tau + \omega\sigma\tau^2$	ω^2

and note that v_3 and $\sigma v_3 = v_6$ are independent, being eigenvectors for τ which correspond with different eigenvalues. Similarly, v_4 and $\sigma v_4 = v_5$ are independent. Moreover, $\langle v_3, v_6\rangle$ and $\langle v_4, v_5\rangle$ are irreducible as demonstrated in problem 2 above. Moreover,

$$\langle v_3, v_6\rangle \cong \langle v_4, v_5\rangle \cong V$$

where V is the standard module.

Finally, we can all see that v_1 and $\sigma v_1 = v_2$ are independent. However, $\langle v_1, v_2\rangle$ is *not* irreducible as $\langle v_1 + v_2\rangle \oplus \langle v_1 - v_2\rangle = \langle v_1, v_2\rangle$. Moreover,

$$\langle v_1 + v_2\rangle \cong U \text{ and } \langle v_1 - v_2\rangle \cong U'$$

where U is the trivial module and U' is the alternating module.

Hence, we have that $R \cong U \oplus U' \oplus V^2$.

Homework 3

1. (a) Let $V_j := \langle e_1, e_2 \dots e_j \rangle$ where $e_j = (0, 0 \dots 1 \dots 0)$ with the 1 in the jth position. Then V_j is a B-submodule. Now as $V_j/V_{j-1} = \langle e_j \rangle + V_{j-1} \cong \langle e_j \rangle$, we have that V_j/V_{j-1} is irreducible and that

$$V \geq V_{n-1} \geq V_{n-1} \geq \cdots \geq V_2 \geq V_1 \geq 0$$

Write ρ_j for the representation of B induced by V_j/V_{j-1}. Then for $M \in B$ we have

$$M \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{j-1} \\ v_j \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \spadesuit \\ \spadesuit \\ \vdots \\ \spadesuit \\ m_{j,j} v_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

so that $\rho_j(M) : v_j e_j + V_{j-1} \mapsto m_{j,j} v_j e_j + V_{j-1}$. Thus, M acts as multiplication by $m_{j,j}$.

(b) Write $E_{i,j}$ for the matrix with 1 in the (i, j) position and 0's elsewhere. Define $B_j = \langle E_{1,j}, E_{2,j} \dots E_{j,j} \rangle$. Then B_j is a B-submodule with $B = \bigoplus_{j=1}^{n} B_j$.

Write ρ_k for the representation of B induced by B_k. To describe ρ_k, it suffices to describe the transformations $\rho_k(E_{i,j})$ for $E_{i,j}$, $i \leq j$, the generators of B. Now in general, left multiplication by $E_{i,j}$ replaces the the jth row with the ith row and clobbers everything else. Then for $M \in B_k$, where only the kth column is non-zero, we have $E_{i,j} M = m_{j,k} E_{i,k}$, that is, $\rho_k(E_{i,j})$ is the transformation which extracts the (j, k) entry and moves it to position (i, k), and clobbers everything else.

2. (a) Let $\rho_g : h \mapsto gh$ for all $h \in G$. Then $\rho_a : a^j \mapsto a^{j+1}$ and this transformation has matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & & & 1 & 0 \end{pmatrix}$$

with respect to the basis G of KG.

(b) Computation gives that the characteristic polynomial for A is $c_A(X) = (X - 1)^p$. To find the Jordan form of A, we locate a basis for $\mathbb{C}^n$ consisting of $(A-1)$-strings. There are at least two good ways to do this.

First, we know by the theorem in Professor Radford's book about the number o $(A - 1)$-strings that there are

$$\text{Rank}(A - 1)^{p+1} + \text{Rank}(A - 1)^{p-1} - 2\text{Rank}(A - 1)^p = \text{Rank}(A - 1)^{p-1}$$

$(A-1)$-strings of length p in any basis of $(A-1)$-strings. But the range of $(A-1)^{p-1}$ is the nullspace of $A-1$ as $(A-1)\left[(A-1)^{p-1}\mathbf{v}\right] = (A-1)^p\mathbf{v} = 0$ by Caley-Hamilton. Thus, the nullity of $A-1$ is 1 as

$$\mathcal{N}(A-1) = \mathcal{N}\begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \\ 1 & -1 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & & & 1 & -1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle$$

so that there exits an $(A-1)$-string of length p.
Alternately, if we know that $(A-1)^{p-1} \neq 0$ for some other reason, then we can produce an $(A-1)$-string of length p by defining $v_n := e_j$, where e_j is a standard basis element of $\mathbb{C}^p$ and j is the column of $(A-1)^{p-1}$ which has a non-zero element. Define also $v_k := (A-1)v_{k-1}$. Then $\{v_k\}$ forms an $(A-1)$-string of length p.
Now, to show that $(A-1)^{p-1}$ is not 0, we note that if

$$0 = (A-1)^{p-1} = \sum_{j=0}^{p-1} \binom{p-1}{j} A^j I^{p-1-j} = \sum_{j=0}^{p-1} \binom{p-1}{j} A^j,$$

then applying this transformation to the group-algebra element 1, we have

$$0 = \sum_{j=0}^{p-1} \binom{p-1}{j} a^j,$$

but since $\{a_j\}$ forms a basis for $\mathbb{C}G$, we have that $\binom{p-1}{j} = 0$ for all j, a contradiction.
Incidently, taking $v_n := e_1$ and $v_k = (A-1)^{k-1}e_1$, we can easily compute v_k for all $1 \leq k \leq p$, since v_k is the first column of

$$(A-1)^{k-1} = \sum_{j=0}^{k-1} \binom{k-1}{j} A^j$$

and we can easily compute powers of A. For example, the first column of $(A-1)^2 = A^2 - 2A + I$ is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Note also that $v_p = (A-1)^{p-1} = \sum_{j=0}^{p-1} \binom{p-1}{j} A^j \neq 0$ as the first columns of $I, A, A^2, \ldots A^{p-1}$ have entries in different positions and the $\binom{p-1}{j}$ are not all zero in $\mathbb{F}_p$.

This basis of $\mathbb{C}^p$ yields the basis $\{1, 1-2a+a^2, -1+3a-3a^2+a^4\ldots\}$ of $\mathbb{C}G$. Now taking one of the $(A-1)$-strings generated above as a basis, we have that

$$A \sim \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ 0 & & & & 1 \end{pmatrix}$$

so that defining $V_j = \langle e_1, e_2 \cdots e_j \rangle$ as in the previous problem, we have that V_j are submodules and that

$$V \geq V_{p-1} \geq V_{p-1} \geq \cdots \geq V_2 \geq V_1 \geq 0$$

with $V_j/V_{j-1} \cong \langle e_j \rangle$ irreducible.

3. (a) Let $V = \langle v_1, v_2 \rangle$ be the two-dimensional irreducible $\mathbb{C}G$-module with $sv_1 = v_2$, $sv_2 = v_1$, $rv_1 = iv_1$, and $rv_2 = -iv_2$. Then V induces a representation ρ of $\mathbb{C}G$ on V defined by

$$\rho\left(\sum_j \alpha_j g_j\right)(v) = \left(\sum_j \alpha_j g_j\right) \cdot v$$

where $\cdot$ represents the action of group ring on the module described above. In particular, this means that

$$\rho : r \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } \rho : s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis $\{v_1, v_2\}$ of V. Using this information, we compute

$$\rho\left(-\frac{1}{2}ir - \frac{1}{2}r^2\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\rho\left(\frac{1}{2}isr + \frac{1}{2}s\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and similarly for the other two generators of $M_2(\mathbb{C})$, so that ρ is surjective.

(b) The procedure employed in this computation is justified by the following:

Lemma. *Let $N \lhd G$. If $\sum_{gN\in G/N} \alpha_{gN} gN$ is idempotent in $\mathbb{C}(G/N)$, then $\frac{1}{|N|}\sum_{g\in G} \alpha_{\pi(g)} g$ is idempotent in $\mathbb{C}G$ where $\pi(g) = gN$ is the cannonical projection of G onto G/N.*

Proof.

Since

$$\sum_{gN\in G/N} \alpha_{gN} gN = \left(\sum_{gN\in G/N} \alpha_{gN} gN\right)^2 = \sum_{gN\in G/N}\left(\sum_{h_1Nh_2N=gN} \alpha_{h_1N}\alpha_{h_2N}\right) gN,$$

we have that for each gN,

$$\alpha_{gN} = \sum_{h_1Nh_2N=gN} \alpha_{h_1N}\alpha_{h_2N}.$$

But for each fixed pair h_1N, h_2N with $h_1h_2N = gN$, we have that

$$h_{1,j}Nh_{2,k}N = gN$$

for all $|N|$ of the elements $h_{1,j} \in h_1N$ and for all $|N|$ of the elements $h_{2,j} \in h_2N$. But of the $|N|^2$ pairs $h_{1,j}, h_{2,k}$, exactly $|N|$ of them satisfy $h_{1,j}h_{2,k} = g$ and the product $\alpha_{\pi(h_{1,j})}\alpha_{\pi(h_{2,k})} = \alpha_{h_1N}\alpha_{h_2N}$ remains fixed for each such pair. Thus

$$\sum_{h_1h_2=g} \alpha_{\pi(h_1)}\alpha_{\pi(h_h)} = |N| \sum_{h_1Nh_2N=gN} \alpha_{h_1N}\alpha_{h_2N} = |N|\alpha_{gN}$$

so that

$$\left(\frac{1}{|N|}\sum_{g\in G}\alpha_{\pi g}g\right)^2 = \frac{1}{|N|^2}\sum_{g\in G}\left(\sum_{h_1h_2=g}\alpha_{\pi(h_1)}\alpha_{\pi(h_h)}\right)g = \frac{1}{|N|}\sum_{g\in G}\alpha_{\pi g}g.$$

Now getting down to buisness, we take the normal subgroup $\langle r\rangle$ of D_8. Then $D_8/\langle r\rangle \cong \mathbb{Z}_2$ and the idempotents for the quotient are

$$\widetilde{e}_1 = \frac{1}{2}\Big(\langle r\rangle + s\langle r\rangle\Big) \text{ and } \widetilde{e}_2 = \frac{1}{2}\Big(\langle r\rangle - s\langle r\rangle\Big)$$

Lifting as in the lemma above, we have

$$e_1 := \frac{1}{8}\left(1 + r + r^2 + r^3 + s + sr + sr^2 + sr^3\right)$$

and

$$e_2 := \frac{1}{8}\left(1 + r + r^2 + r^3 - s - sr - sr^2 - sr^3\right).$$

Repeating the above procedure for the subgroups $\langle s, r^2\rangle$ and $\langle sr, r^2\rangle$, we generate

$$e_3 := \frac{1}{8}\left(1 - r + r^2 - r^3 + s - sr + sr^2 - sr^3\right)$$

and

$$e_4 := \frac{1}{8}\left(1 - r + r^2 - r^3 - s + sr - sr^2 + sr^3\right).$$

Finally, to construct the fifth idempotent f, note that since $\sum_{j=1}^{5} e_j = 1$ by the definition of the e_j, we have

$$1 - \sum_{j=1}^{4} e_j = \frac{1}{2}\left(1 - r^2\right).$$

(a) Subject to further consideration.

(b) i. Take v to be an eigenvector for a with eigenvalue i. Then as in the proceeding homework, the computation

$$a(bv) = ba^3v = -i(bv)$$

shows both that bv is an eigenvector for a with eigenvalue $-i$ and that v and bv are independent. This together with the observation that

$$b(bv) = a^2v = -v$$

proves that $W := \langle v, bv\rangle$ is a $\mathbb{C}G$ submodule. W is irreducible since if $\langle \alpha v + \beta bv\rangle$ is any proper subspace, then

$$a(\alpha v + \beta bv) = i\alpha v - i\beta bv \notin \langle \alpha v + \beta bv\rangle$$

so that $\langle \alpha v + \beta bv\rangle$ is not G-stable. Now if ρ is the representation induced by this module, then by the above computations,

$$\rho(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } \rho(b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

ii. If ρ is a one-dimensional representation, then $\rho(a^4) = \rho(1) = (1)$ which we will write as 1, so that $\rho(a) = i^j$, $j = 0, 1, 2$, or 3.
If $\rho(a) = i$, then since

$$\rho(b)^{-1}i\rho(b) = \rho(b^{-1})\rho(a)\rho(b) = \rho(b^{-1}ab) = \rho(a^{-1}) = \rho(a)^{-1} = -i,$$

we have that

$$i\rho(b) = \rho(b)(-i) = -i\rho(b)$$

which is impossible, so we conclude that $\rho(a) \neq i$. Similarly, $\rho(a) \neq -i$.
This means that $\rho(a) = \pm 1$ and in both cases, $\rho(b)$ is easily seen to be ± 1.

iii. Knowing that

$$\frac{1}{2}\Big(\langle a\rangle + b\langle a\rangle\Big) \text{ and } \frac{1}{2}\Big(\langle a\rangle - b\langle a\rangle\Big),$$

are idempotent in $\mathbb{C}(Q_8/\langle a\rangle)$, we have that

$$e_1 := \frac{1}{8}\Big(1 + a + a^2 + a^3 + b + ba + ba^2 + ba^3\Big)$$

and

$$e_2 := \frac{1}{8}\Big(1 + a + a^2 + a^3 - b - ba - ba^2 - ba^3\Big)$$

are idempotent in $\mathbb{C}Q_8$. By replacing $\langle a\rangle$ in the above argument with $\langle b\rangle$ and $\langle ba\rangle$, we similarly generate

$$e_3 := \frac{1}{8}\Big(1 - a + a^2 - a^3 + b - ba + ba^2 - ba^3\Big)$$

and

$$e_4 := \frac{1}{8}\Big(1 - a + a^2 - a^3 - b + ba - ba^2 + ba^3\Big).$$

Finally, as $\sum_{j=1}^{5} e_j = 1$, we have that

$$e_5 = \frac{1}{2}\Big(1 - a^2\Big).$$

Multiplication (not shown here for sanity) confirms that $e_j^2 = e_j$ for all j and that $e_j e_k = 0$ for $e_j \neq e_k$ so that the e_j are mutually orthogonal.

iv. Assume that ρ is a representation of G over $\mathbb{R}$ and assume also (why?) that

$$\rho(a) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now if

$$\rho(b) = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

for $x, y, z, w \in \mathbb{R}$, we compute

$$\rho(ab) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} -z & -w \\ x & y \end{pmatrix}$$

and

$$\rho(ba^{-1}) = \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -y & x \\ -w & z \end{pmatrix}.$$

Then, as $\rho(ab) = \rho(ba^{-1})$, we must have $x = -w$ and $y = z$. Then write

$$\rho(b) = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

and note that since

$$\begin{pmatrix} x^2 + y^2 & 2xy \\ 2xy & x^2 + y^2 \end{pmatrix} = \rho(b^2) = \rho(a^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

we have that $x^2 + y^2 = -1$ which contradicts $x, y \in \mathbb{R}$.

(c) i. Since

$$(x - x_1)(x - x_2)\cdots(x - x_n) = x^n - E_1 x^{n-1} + E_2 x^{n-2} + \cdots + (-1)^n E_n,$$

we have all of the following:

$$\begin{aligned}
0 &= (x_1 - x_1)(x_1 - x_2)\cdots(x_1 - x_n) = x_1^n - E_1 x_1^{n-1} + E_2 x_1^{n-2} + \cdots + (-1)^n E_n \\
0 &= (x_2 - x_1)(x_2 - x_2)\cdots(x_2 - x_n) = x_2^n - E_1 x_2^{n-1} + E_2 x_2^{n-2} + \cdots + (-1)^n E_n \\
&\vdots \\
0 &= (x_n - x_1)(x_n - x_2)\cdots(x_n - x_n) = x_n^n - E_1 x_n^{n-1} + E_2 x_n^{n-2} + \cdots + (-1)^n E_n
\end{aligned}$$

so that adding we have

$$0 = p_n - p_{n-1}E_1 + p_{n-2}E_2 + \cdots + (-1)^n nE_n$$

which is the nth equation.
Now for the $(n-1)$st equation, factor out the x_j:

$$\begin{aligned}
0 &= x_1\left(x_1^{n-1} - E_1x_1^{n-2} + E_2x_1^{n-3} + \cdots + (-1)^{n-1}E_{n-1} + \frac{1}{x_1}(-1)^nE_n\right)\\
0 &= x_2\left(x_2^{n-1} - E_1x_2^{n-2} + E_2x_2^{n-3} + \cdots + (-1)^{n-1}E_{n-1} + \frac{1}{x_2}(-1)^nE_n\right)\\
&\vdots\\
0 &= x_n\left(x_n^{n-1} - E_1x_n^{n-2} + E_2x_n^{n-3} + \cdots + (-1)^{n-1}E_{n-1} + \frac{1}{x_n}(-1)^nE_n\right)
\end{aligned}$$

but as $x_j \neq 0$, begin indeterminants, we have

$$\begin{aligned}
0 &= x_1^{n-1} - E_1x_1^{n-2} + E_2x_1^{n-3} + \cdots + (-1)^{n-1}E_{n-1} + \frac{1}{x_1}(-1)^nE_n\\
0 &= x_2^{n-1} - E_1x_2^{n-2} + E_2x_2^{n-3} + \cdots + (-1)^{n-1}E_{n-1} + \frac{1}{x_2}(-1)^nE_n\\
&\vdots\\
0 &= x_n^{n-1} - E_1x_n^{n-2} + E_2x_n^{n-3} + \cdots + (-1)^{n-1}E_{n-1} + \frac{1}{x_n}(-1)^nE_n.
\end{aligned}$$

Note that the sum of the last terms of each equation is

$$(-1)^nE_n\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right) = (-1)^nE_n\frac{x_2x_3\cdots x_n + x_1x_3\cdots x_n + \cdots x_1x_2\cdots x_{n-1}}{x_1x_2\cdots x_n}$$

$$= (-1)^nE_n\frac{E_{n-1}}{E_n} = (-1)^nE_{n-1}.$$

Now, adding the equations,

$$\begin{aligned}
0 &= p_{n-1} - p_{n-2}E_1 + p_{n-2}E_2 + \cdots + (-1)^{n-1}nE_{n-1} + (-1)^nE_{n-1}\\
&= p_{n-1} - p_{n-2}E_1 + p_{n-2}E_2 + \cdots + (-1)^{n-1}E_{n-1}(n-1),
\end{aligned}$$

which is the $(n-1)$st equation. Similarly for the others.

ii. Suppose that we are given some character χ of G of dimension n and let g be some fixed element of G. Write $x_1, x_2 \ldots x_n$ for the eigenvalues of g. Then $p_1 = \chi(g)$, $p_2 = \chi(g^2)$, etc, since, for example

$$\chi(g) = \mathrm{tr}(\rho(g)) = \mathrm{tr}(S\rho(g)S^{-1}) = \mathrm{tr}(J) = \sum_{j=1}^{n} x_j = p_1$$

where J is the Jordan form for $\rho(g)$. Thus, the p_j are known for $j = 1, 2, \ldots n$. We can then inductively determine the E_j. For example, $E_1 = p_1$, $E_2 = \frac{1}{2}(E_1 p_1 - p_2)$, etcetera. Then, the characteristic polynomial for g is given by

$$c_g(X) = (X - x_1)(X - x_2)\cdots(X - x_n) = X^n - E_1 X^{n-1} + E_2 X^{n-2} + \cdots + (-1)^n E_n$$

so that the eigenvalues x_j are determined provided we can find the roots of this polynomial, and that's not my problem.

Homework 5

0. We begin with the trivial and alternating characters

	1	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
χ_U	1	1	1	1	1	1	1
$\chi_{U'}$	1	-1	1	1	-1	-1	1

We directly compute the character for

$$V = \left\{ \sum_j \alpha_j v_j \,\middle|\, \sum_j \alpha_j = 0 \right\} = \langle v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 - v_5 \rangle$$

where v_j are a basis for the permutation module. For example, to compute $\chi_V(12)$, we compute that

$$\begin{gathered} (12)(v_1 - v_2) = -(v_1 - v_2) \\ (12)(v_2 - v_3) = v_1 - v_3 = (v_1 - v_2) + (v_2 - v_3) \\ (12)(v_3 - v_4) = v_3 - v_4 \\ (12)(v_4 - v_5) = v_4 - v_5 \end{gathered}$$

so that

$$\rho_V(12) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and so $\chi_V(12) = 2$. Similarly for the other elements. The character for $V' := U' \otimes V$ is given by multiplication.

Next, we compute the character for $\bigwedge^2 V$ directly. A basis for $\bigwedge^2 V$ is

$$\begin{aligned} v_1 - v_2 \wedge v_2 - v_3 &= v_1 \wedge v_2 - v_1 \wedge v_3 + v_2 \wedge v_3 \\ v_1 - v_2 \wedge v_3 - v_4 &= v_1 \wedge v_3 - v_2 \wedge v_3 - v_1 \wedge v_4 + v_2 \wedge v_4 \\ v_1 - v_2 \wedge v_4 - v_5 &= v_1 \wedge v_4 - v_2 \wedge v_4 - v_1 \wedge v_5 + v_2 \wedge v_5 \\ v_2 - v_3 \wedge v_3 - v_4 &= v_2 \wedge v_3 - v_2 \wedge v_4 + v_3 \wedge v_4 \\ v_2 - v_3 \wedge v_4 - v_5 &= v_2 \wedge v_4 - v_3 \wedge v_4 - v_2 \wedge v_5 + v_3 \wedge v_5 \\ v_3 - v_4 \wedge v_4 - v_5 &= v_3 \wedge v_4 - v_3 \wedge v_5 + v_4 \wedge v_5 \end{aligned}$$

Now to compute the value of the character at (12), we compute

$$\begin{aligned}
(12)(v_1 \wedge v_2 - v_1 \wedge v_3 + v_2 \wedge v_3) &= -(v_1 \wedge v_2 - v_1 \wedge v_3 + v_2 \wedge v_3)\\
(12)(v_1 \wedge v_3 - v_2 \wedge v_3 - v_1 \wedge v_4 + v_2 \wedge v_4) &= -(v_1 \wedge v_3 - v_2 \wedge v_3 - v_1 \wedge v_4 + v_2 \wedge v_4)\\
(12)(v_1 \wedge v_4 - v_2 \wedge v_4 - v_1 \wedge v_5 + v_2 \wedge v_5) &= -(v_1 \wedge v_4 - v_2 \wedge v_4 - v_1 \wedge v_5 + v_2 \wedge v_5)\\
(12)(v_2 \wedge v_3 - v_2 \wedge v_4 + v_3 \wedge v_4) &= v_2 \wedge v_3 - v_2 \wedge v_4 + v_3 \wedge v_4 +\\
&\quad v_1 \wedge v_3 - v_2 \wedge v_3 - v_1 \wedge v_4 + v_2 \wedge v_4\\
(12)(v_2 \wedge v_4 - v_3 \wedge v_4 - v_2 \wedge v_5 + v_3 \wedge v_5) &= v_2 \wedge v_4 - v_3 \wedge v_4 - v_2 \wedge v_5 + v_3 \wedge v_5 +\\
&\quad v_1 \wedge v_4 - v_2 \wedge v_4 - v_1 \wedge v_5 + v_2 \wedge v_5\\
(12)(v_3 \wedge v_4 - v_3 \wedge v_5 + v_4 \wedge v_5) &= v_3 \wedge v_4 - v_3 \wedge v_5 + v_4 \wedge v_5
\end{aligned}$$

so that

$$\rho_{\bigwedge^2 V}(12) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and so $\chi_{\bigwedge^2 V}(12) = 0$. Similarly for the other elements. Compiling this information, we have

S_5	1	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
χ_U	1	1	1	1	1	1	1
$\chi_{U'}$	1	-1	1	1	-1	-1	1
χ_V	4	2	0	1	-1	0	-1
$\chi_{V'}$	4	-2	0	1	1	0	-1
$\chi_{\bigwedge^2 V}$	6	0	-2	0	0	0	1

We also verify that $\chi_{\bigwedge^2 V}$ is irreducible as

$$\langle \chi_{\bigwedge^2 V}, \chi_{\bigwedge^2 V} \rangle = 1.$$

We recall a portion of the character table for S_4:

S_4	1	(12)	(12)(34)	(123)	(1234)
χ_U	1	1	1	1	1
χ_V	3	1	-1	0	-1

Let $g_1 = 1$, $g_2 = (15)$, $g_3 = (25)$, $g_4 = (35)$, and $g_5 = (45)$ be representatives of the cosets of S_4 in S_5. Then the character of S_5 induced by the character χ_V of S_4 evaluated at (12) is

$$\begin{aligned}
\mathrm{Ind}_{S_4}^{S_5}\chi_V(12) &= \sum_j \chi_V\left(g_j^{-1} g g_j\right)\\
&= \chi_V(1(12)1) + \chi_V(15)(12)(15) + \chi_V(25)(12)(25) + \chi_V(35)(12)(35) + \chi_V(45)(12)\\
&= \chi_V(12) + \chi_V(25) + \chi_V(15) + \chi_V(12) + \chi_V(12)\\
&= 1 + 0 + 0 + 1 + 1\\
&= 3.
\end{aligned}$$

Similarly for the other elements. Then the induced character together with the character for $\bigwedge^2 V$ are

	1	10	15	20	20	30	24
S_5	1	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
$\chi_{\bigwedge^2 V}$	6	0	-2	0	0	0	1
$\chi_{\mathrm{Ind}_{S_4}^{S_5}\chi_V}$	15	3	-1	0	0	-1	0.

Then

$$\langle \chi_{\bigwedge^2 V}, \chi_{\mathrm{Ind}_{S_4}^{S_5}\chi_V} \rangle = \frac{1}{120}\sum_j \chi_{\bigwedge^2 V}(g_j)\overline{\chi_{\mathrm{Ind}_{S_4}^{S_5}\chi_V}(g_j)} = 1$$

where here the g_j are the elements S_5. Similarly,

$$\begin{aligned}
\langle \chi_U, \chi_{\mathrm{Ind}_{S_4}^{S_5}\chi_V} \rangle &= 0 \\
\langle \chi_{U'}, \chi_{\mathrm{Ind}_{S_4}^{S_5}\chi_V} \rangle &= 0 \\
\langle \chi_V, \chi_{\mathrm{Ind}_{S_4}^{S_5}\chi_V} \rangle &= 1 \\
\langle \chi_{V'}, \chi_{\mathrm{Ind}_{S_4}^{S_5}\chi_V} \rangle &= 0
\end{aligned}$$

Now let m and n be the degrees of the two remaining representations. Then

$$120 = \sum_j (\chi_j(1))^2 = 70 + m^2 + n^2$$

and since m and n both divide 120, trial and error gives that $m = n = 5$. Hence, from the inner product argument above, we have

$$\chi_{\mathrm{Ind}_{S_4}^{S_5}} = \chi_V + \chi_{\bigwedge^2 V} + \chi_W$$

where W is one of the remaining modules of degree 5. Then subtraction gives

S_5	1	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
χ_V	4	2	0	1	-1	0	-1
$\chi_{\bigwedge^2 V}$	6	0	-2	0	0	0	1
$\chi_{\mathrm{Ind}_{S_4}^{S_5}\chi_V}$	15	3	-1	0	0	-1	0
χ_W	5	1	1	-1	1	-1	0

so that the second orthogonality relation gives

	1	10	15	20	20	30	24
S_5	1	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
χ_U	1	1	1	1	1	1	1
$\chi_{U'}$	1	-1	1	1	-1	-1	1
χ_V	4	2	0	1	-1	0	-1
$\chi_{V'}$	4	-2	0	1	1	0	-1
$\chi_{\bigwedge^2 V}$	6	0	-2	0	0	0	1
χ_W	5	1	1	-1	1	-1	0
$\chi_{W'}$	5	-1	1	-1	-1	1	0

2. We first note that if V is one-dimensional vector space over $\mathbb{C}$ and $\rho : G \longrightarrow \mathrm{GL}(V) \cong \mathbb{C}^*$ is a representation of G over V, then since $G/\ker\rho \cong \mathrm{im}\rho \leq \mathbb{C}$ is abelian, we must have that $G' \leq \ker\rho$. We'll keep in mind during the following discussion that this means that ρ maps all the elements of each coset of G' to the same number.

Now since G/G' is abelian, all of the representations of G/G' are one-dimensional and there are exactly $[G : G']$ of them since there are $[G : G']$ elements in G/G', all of which are singleton conjugacy classes. They all lift to different one-dimensional characters of G so that G has at least $[G : G']$ one-dimensional characters.

Now if ρ is a one-dimensional character of G, then as we observed above, ρ is constant on the cosets of G' so that $\widetilde{\rho} : gG' \mapsto \rho(g)$ well-defined homomorphism $\widetilde{\rho} : G/G' \longrightarrow \mathbb{C}^*$ and hence is a one-dimensional representation of G/G'. Hence, there can be no more than $[G : G']$ one-dimensional representations of G. This shows that there are exactly $[G : G']$ one-dimensional representations of G.

Now since G' is normal, we have

$$G' = \bigcap \left\{ \ker\rho_j \,\middle|\, G' \leq \ker\rho_j \right\}$$

so that G' can be determined from the character table by taking those elements which have character 1 for all one-dimensional characters. Of course, the other elements in those conjugacy classes are also in G' as G' is normal.

3. We compute that

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

and that

$$\begin{pmatrix} 1 & -x & xz-y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & b-az-xc \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

so that conjugation changes only the $(1,3)$ entry. Moreover, whenever $a \neq 0$ or $c \neq 0$, we see that the conjugacy class of $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ contains three elements since we can select the conjugating matrix $\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$ with x and z such that $b - az - xc = 0, 1,$ or 2. This gives the following conjugacy classes and sizes, where $\begin{matrix} a & * \\ & c \end{matrix}$ represents the conjugacy class containing $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$.

	1	1	1	3	3	3	3	3	3	3	3
G	$\begin{matrix}0 & 0\\ & 0\end{matrix}$	$\begin{matrix}0 & 1\\ & 0\end{matrix}$	$\begin{matrix}0 & 2\\ & 0\end{matrix}$	$\begin{matrix}1 & *\\ & 0\end{matrix}$	$\begin{matrix}2 & *\\ & 0\end{matrix}$	$\begin{matrix}0 & *\\ & 1\end{matrix}$	$\begin{matrix}1 & *\\ & 1\end{matrix}$	$\begin{matrix}2 & *\\ & 1\end{matrix}$	$\begin{matrix}0 & *\\ & 2\end{matrix}$	$\begin{matrix}1 & *\\ & 2\end{matrix}$	$\begin{matrix}2 & *\\ & 2\end{matrix}$

The conjugacy classes of G happen to be the same sets as the cosets of G' in G with the exception that the three central elements comprise the trivial coset G' but are in singleton conjugacy classes.

We next show that $G/G' \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Define $\varphi : \mathbb{Z}_3 \times \mathbb{Z}_3 \longrightarrow G/G'$ by

$$\varphi(x, y) = \begin{pmatrix} 1 & x & * \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} G'.$$

Then φ is easily seen to be a well-defined, surjective homomorphism so that $G/G' \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

The computation of the character table for $\mathbb{Z}_3 \times \mathbb{Z}_3$ is based on the following procedure. If ρ_1, ρ_2 are representations of G on V, then $\rho_1 \times \rho_2$ defined

$$(\rho_1 \times \rho_2)(g_1, g_2) : v_1 \otimes v_2 \mapsto (\rho_1 g_1)v_1 \otimes (\rho_2 g_2)v_2$$

is a representation of $G \times G$ on $V \otimes V$, where v_j is a basis for V. Then $(\rho_1 \times \rho_2)(g_1, g_2)$ has matrix

$$\begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots \\ a_{2,1}B & a_{2,2}B & \cdots \\ \vdots & & \end{pmatrix}$$

with respect to v_j where A is the matrix for $\rho_1 g_1$ and B is the matrix for $\rho_2 g_2$. Thus, the character for $\rho_1 \times \rho_2$ at (g_1, g_2) is $\chi_1 g_1 \chi_2 g_2$. The details of all the above assertions are available upon request.

We recall the character table for $\mathbb{Z}_3$:

$\mathbb{Z}_3$	0	1	2
χ_0	1	1	1
χ_1	1	ω	ω^2
χ_2	1	ω^2	ω

Using these characters, we construct the characters for $\mathbb{Z}_3 \times \mathbb{Z}_3$. For example, the character for $\chi_1 \times \chi_0$ at $(1, 0)$ is $\chi_1(1)\chi_0(0) = \omega \cdot 1 = \omega$ and similarly,

$\mathbb{Z}_3 \times \mathbb{Z}_3$	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$
$\chi_1 \times \chi_0$	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2

This produces 9 one-dimensional characters for $\mathbb{Z}_3 \times \mathbb{Z}_3 \cong G/G'$ which we lift to characters for G, keeping in mind the trivial coset G', which corresponds under the isomorphism above with $(0,0)$, splits into three conjugacy classes, each with the same values for each character. Writing $\chi_{i,j}$ for the character lifted from $\chi_i \times \chi_j$, we have

	1	1	1	3	3	3	3	3	3	3	3
G	$\begin{smallmatrix}0&0\\&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\&0\end{smallmatrix}$	$\begin{smallmatrix}0&2\\&0\end{smallmatrix}$	$\begin{smallmatrix}1&*\\&0\end{smallmatrix}$	$\begin{smallmatrix}2&*\\&0\end{smallmatrix}$	$\begin{smallmatrix}0&*\\&1\end{smallmatrix}$	$\begin{smallmatrix}1&*\\&1\end{smallmatrix}$	$\begin{smallmatrix}2&*\\&1\end{smallmatrix}$	$\begin{smallmatrix}0&*\\&2\end{smallmatrix}$	$\begin{smallmatrix}1&*\\&2\end{smallmatrix}$	$\begin{smallmatrix}2&*\\&2\end{smallmatrix}$
$\chi_{0,0}$	1	1	1	1	1	1	1	1	1	1	1
$\chi_{1,0}$	1	1	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2
$\chi_{2,0}$	1	1	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω
$\chi_{0,1}$	1	1	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
$\chi_{1,1}$	1	1	1	ω	ω^2	ω	ω^2	1	ω^2	1	ω
$\chi_{2,1}$	1	1	1	ω^2	ω	ω	1	ω^2	ω^2	ω	1
$\chi_{0,2}$	1	1	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
$\chi_{1,2}$	1	1	1	ω	ω^2	ω^2	1	ω	ω	ω^2	1
$\chi_{2,2}$	1	1	1	ω^2	ω	ω^2	ω	1	ω	1	ω^2

Now consider the subgroup

$$H := \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \leq G$$

We compute the character $\mathrm{Ind}_H^G(\psi)$ of G where ψ is the character of H given by

H	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
ψ	1	ω	ω^2

For example,

$$\begin{aligned} \mathrm{Ind}_H^G(\psi) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \left| C_G \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| \sum_j \frac{\psi(g_j)}{|C_H g_j|} \\ &= 27\frac{1}{3} \\ &= 9 \end{aligned}$$

where $\mathcal{C}$ is the conjugacy class of G containing $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and g_j are representatives of the conjugacy classes $\mathcal{D}_j$ of H with $\bigcup_j \mathcal{D}_j = \mathcal{C} \cap H$. In this case, $\mathcal{C} \cap H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = g_1$. Similarly for the other elements. We then have that

G	$\begin{smallmatrix}0&0\\&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\&0\end{smallmatrix}$	$\begin{smallmatrix}0&2\\&0\end{smallmatrix}$	$\begin{smallmatrix}1&*\\&0\end{smallmatrix}$	$\begin{smallmatrix}2&*\\&0\end{smallmatrix}$	$\begin{smallmatrix}0&*\\&1\end{smallmatrix}$	$\begin{smallmatrix}1&*\\&1\end{smallmatrix}$	$\begin{smallmatrix}2&*\\&1\end{smallmatrix}$	$\begin{smallmatrix}0&*\\&2\end{smallmatrix}$	$\begin{smallmatrix}1&*\\&2\end{smallmatrix}$	$\begin{smallmatrix}2&*\\&2\end{smallmatrix}$
$\mathrm{Ind}_H^G\psi$	9	0	0	9ω	$9\omega^2$	0	0	0	0	0	0

We compute that

$$9 = \langle \mathrm{Ind}_H^G \psi, \mathrm{Ind}_H^G \psi \rangle = \sum_j \langle \mathrm{Ind}_H^G, \chi_j \rangle^2$$

where χ_j are the irreducible characters of G. This is only possible if $\langle \mathrm{Ind}_H^G, \chi_j \rangle = 3$ for some j or if $\mathrm{Ind}_H^G, \chi_j \rangle = 1$ for 9 different j's. In the latter case, we must have $\langle \mathrm{Ind}_H^G, \chi_j \rangle = 1$ for the 9 characters χ_j already computed as the remaining two have degree 3. We eliminate this possibility since $\mathrm{Ind}_H^G \psi \neq \sum_{j=1}^{9} \chi_j$. Hence, $\mathrm{Ind}_H^G \psi = 3\chi_j$ for some new character χ_j. The conjugate of this character is also an irreducible character so that we have

	1	1	1	3	3	3	3	3	3	3	3
G	$\begin{smallmatrix}0 & 0\\ & 0\end{smallmatrix}$	$\begin{smallmatrix}0 & 1\\ & 0\end{smallmatrix}$	$\begin{smallmatrix}0 & 2\\ & 0\end{smallmatrix}$	$\begin{smallmatrix}1 & *\\ & 0\end{smallmatrix}$	$\begin{smallmatrix}2 & *\\ & 0\end{smallmatrix}$	$\begin{smallmatrix}0 & *\\ & 1\end{smallmatrix}$	$\begin{smallmatrix}1 & *\\ & 1\end{smallmatrix}$	$\begin{smallmatrix}2 & *\\ & 1\end{smallmatrix}$	$\begin{smallmatrix}0 & *\\ & 2\end{smallmatrix}$	$\begin{smallmatrix}1 & *\\ & 2\end{smallmatrix}$	$\begin{smallmatrix}2 & *\\ & 2\end{smallmatrix}$
$\chi_{0,0}$	1	1	1	1	1	1	1	1	1	1	1
$\chi_{1,0}$	1	1	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2
$\chi_{2,0}$	1	1	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω
$\chi_{0,1}$	1	1	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
$\chi_{1,1}$	1	1	1	ω	ω^2	ω	ω^2	1	ω^2	1	ω
$\chi_{2,1}$	1	1	1	ω^2	ω	ω	1	ω^2	ω^2	ω	1
$\chi_{0,2}$	1	1	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
$\chi_{1,2}$	1	1	1	ω	ω^2	ω^2	1	ω	ω	ω^2	1
$\chi_{2,2}$	1	1	1	ω^2	ω	ω^2	ω	1	ω	1	ω^2
$\mathrm{Ind}_H^G \psi$	3	0	0	3ω	$3\omega^2$	0	0	0	0	0	0
$\mathrm{Ind}_H^G \psi$	3	0	0	$3\omega^2$	3ω	0	0	0	0	0	0

The Mackey Problem

(a) Note that

$$\begin{aligned}\left\langle \psi_1^{(12)}, \psi_1^{(12)} \right\rangle &= \sum_g \psi_1\big((12)g(12)\big)\overline{\psi_1\big((12)g(12)\big)} \\ &= \sum_g \psi_1(g)\overline{\psi_1(g)} \\ &= \langle \psi_1, \psi_1 \rangle \\ &= 1\end{aligned}$$

so that $\psi_1^{(12)}$ is an irreducible character of H.
We have that $G = H1H \cup H(12)H$ so that $D = \{1, (12)\}$ may be taken as a set of double coset representatives. Then

$$\begin{aligned}\langle \mathrm{Ind}\psi_1 \mathrm{Ind}\psi_2 \rangle &= \left\langle \psi_1^1\big|_{1H1\cap H}, \psi_2\big|_{1H1\cap H} \right\rangle + \left\langle \psi_1^{(12)}\big|_{(12)H(12)\cap H}, \psi_2\big|_{(12)H(12)\cap H} \right\rangle \\ &= \langle \psi_1, \psi_2 \rangle + \left\langle \psi_1^{(12)}, \psi_2 \right\rangle\end{aligned}$$

Suppose now that $\psi_1 = \psi_1^{(12)}$. Then

$$\langle \mathrm{Ind}\psi_1 \mathrm{Ind}\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle + \langle \psi_1, \psi_2 \rangle = \begin{cases} 2 & \text{if } \psi_1 = \psi_2 \\ 0 & \text{if } \psi_1 \neq \psi_2 \end{cases}$$

and if $\psi_1 \neq \psi_1^{(12)}$

$$\langle \mathrm{Ind}\psi_1, \mathrm{Ind}\psi_2 \rangle = \left\langle \psi_1^{(12)}, \psi_2 \right\rangle + \langle \psi_1, \psi_2 \rangle = \begin{cases} 1 & \text{if } \psi_1^{(12)} = \psi_2 \text{ or } \psi_1 = \psi_2 \\ 0 & \text{if } \psi_1^{(12)} \neq \psi_2 \text{ and } \psi_1 \neq \psi_2 \end{cases}.$$

(b) Let χ be an irreducible character of G and consider Res χ. We have that no irreducible character ψ of H can appear in Res U more than 2 times since if $\langle \mathrm{Res}\ \chi, \psi \rangle > 2$, then by Frobenius, $\langle \chi, \mathrm{Ind}\ \psi \rangle > 2$, that is, χ appears more than 2 times in Ind ψ. This means that $\{$number of constituents of Ind $\psi\} = \langle \mathrm{Ind}\ \psi, \mathrm{Ind}\ \psi \rangle > 2$, which contradicts part (a).
Now suppose that ψ is a constituent of Res χ. Then, of course, we have $\deg \psi \leq \deg \mathrm{Res}\ \chi = \deg \chi$. However, in this situation, we also have that χ is a constituent of Ind ψ so that

$$\deg \chi \leq \deg \mathrm{Ind}\ \psi = [G : H] \deg \psi = 2 \deg \psi.$$

Compiling this information, we have

$$\deg \psi \leq \deg \chi \leq 2 \deg \psi$$

and this inequality is only satisfied when $\deg \psi = \deg \chi$ or when $\deg \psi = \frac{1}{2} \deg \chi$. This shows that Res χ can have no more than two different consituents.

(c) Restricting the characters of G and taking inner products, we have that the following characters are irreducible.

	1	20	15	12	12
A_5	1	(123)	(12)(34)	(12345)	(21345)
Res U	1	1	1	1	1
Res V	4	1	0	-1	-1
Res W	5	-1	1	0	0

Also, Res U' = Res U, Res V' = Res V, and Res W' = Res W provide no new information. We have, however, that the two remaining characters both have degree 3 since $\sum_j (\chi_j(1))^2 = 60$.

Restricting $\bigwedge^2 V$,

A_5	1	(123)	(12)(34)	(12345)	(21345)
Res $\bigwedge^2 V$	6	0	-2	1	1

we find that $\langle \text{Res } \bigwedge^2 V, \text{Res } \bigwedge^2 V, \rangle = 2$ so that by part (b), Res $\bigwedge^2 V$ is the sum of two irreducible characters Y and Z.

We observe that these two characters must be different since if Res $\bigwedge^2 V = 2Y$, then

A_5	1	(123)	(12)(34)	(12345)	(21345)
Y	3	0	-1	$\frac{1}{2}$	$\frac{1}{2}$

and $\langle Y, Y\rangle = \frac{1}{2}$, which contradicts the assumption that Y is irreducible.

We next observe that $\langle \text{Res } \bigwedge^2 V, \text{Res } U\rangle = 0$, $\langle \text{Res } \bigwedge^2 V, \text{Res } V\rangle = 0$, and $\langle \text{Res } \bigwedge^2 V, \text{Res } U\rangle$
0. These lead to the system

$$\begin{aligned} 20y_1 + 15y_2 + 12y_3 + 12y_4 &= 3 \\ 20y_1 - 12y_3 - 12y_4 &= -12 \\ -20y_1 + 15y_2 &= -15 \end{aligned}$$

where y_j are the remaining values of Y:

	1	20	15	12	12
A_5	1	(123)	(12)(34)	(12345)	(21345)
Res U	1	1	1	1	1
Res V	4	1	0	-1	-1
Res W	5	-1	1	0	0
Y	3	y_1	y_2	y_3	y_4

This system reduces to

$$\begin{aligned} y_1 &= 0 \\ y_2 &= -1 \\ y_3 + y_4 &= 1 \end{aligned}$$

Thus, Y so far is the following.

	1	20	15	12	12
A_5	1	(123)	(12)(34)	(12345)	(21345)
Y	3	0	-1	y_3	$1-y_3$

Now since $1 = \langle Y, Y\rangle = \frac{1}{60}\left(9 + 15 + 12y_3^2 + 12(1-y_3)^2\right)$, we have that $0 = y_3^2 - y_3 - 1 = 0$. The quadratic formula gives $y_3 = \dfrac{1 \pm \sqrt{5}}{2}$. We can take either value for y_3. Thus we have

	1	20	15	12	12
A_5	1	(123)	(12)(34)	(12345)	(21345)
Res U	1	1	1	1	1
Res V	4	1	0	-1	-1
Res W	5	-1	1	0	0
Y	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
Z	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

The Prelim Problem.

Let P be a non-abelian group of order p^3. Forevermore I will write g_j for the representatives of the non-central conjugacy classes of whatever group is under discussion.

We determine the size of $Z(P)$. $|Z(P)| \neq p^3$ since P is non-abelian.

The class equation gives

$$p^3 = |Z(P)| + \sum_j \frac{p^3}{|C_G(g_j)|}.$$

Now $C_G(g_j)$ contains at least g_j and the identity element, and these are different, so $|C_G(g)| \geq p$. But as g_j is non-central, we have $|C_G(g_j)| \neq p^3$ so that $|C_G(g_j)| = p$ or p^2. This means that p divides $\sum_j \frac{p^3}{|C_G(g_j)|}$. We then have that p divides $|Z(P)| = p^3 - \sum_j \frac{p^3}{|C_G(g_j)|}$. This eliminates the possibility that $|Z(P)| = 1$. (The same argument shows that we have $|Z(G)| \neq 1$ for any group G of order p^n. We use this observation later.)

Now if $|Z(P)| = p^2$, then $C_G(g_j)$ contains at least g_j and all p^2 central elements so that $|C_G(g_j)| = p^3$, a contradiction since $g_j \notin Z(P)$.

We conclude then that $|Z(P)| = p$. Moreover, $|C_G(g_j)|$ includes at least g_j and the p central elements, so $|C_G(g_j)| > p$, but as $g_j \notin Z(P)$, we have $|C_G(g_j)| \neq p^3$ so that $|C_G(g_j)| = p^2$. Thus, each non-central conjugacy class contains exactly p elements. From the class equation, we have

$$p^3 = |Z(P)| + \sum_j \frac{p^3}{C_G(g_j)} = p + rp$$

yielding $r = p^2 - 1$ non-central conjugacy classes.

Next, we deduce that any group G of order p^2 is abelian. If $|Z(G)| = p$ then $|C_G(g_j)| = p^2$, since it contains at least g_j and all p central elements. This contradicts $g_j \notin Z(G)$. As noted above, $|Z(G)| \neq 1$. Hence $Z(G) = G$ and G is abelian.

This means that $P/Z(P)$ is abelian as $|P/Z(P)| = p^2$. We then have that $P' \leq Z(P)$. But as $P' \neq 1$ since P is non-abelian, we have that $P' = Z(P)$.

The Dummit and Foote Problem

(a) i. By the prelim discussion above, we have that the commutator subgroup of P has order p so that by the last homework, P has $[P : P'] = p^2$ characters of degree 1.

ii. The degrees of the characters of P divide p^3. G can have no irreducible character χ_{j_0} of degree p^2 as then

$$p^3 = \sum_j (\chi_j(1))^2 > (\chi_{j_0}(1))^2 = p^4,$$

where χ_j are the irreducible characters of P, and this is a contradiction. Similarly, P can have no irreducible character of degree p^3. Hence, all the characters of P have degree 1 or p. Let r be the number of characters of degree p. Then

$$p^3 = \sum_j (\chi_j(1))^2 = p^2 \cdot 1^2 + r \cdot p^2 = p^2(1 + r)$$

so that P has $r = p - 1$ characters of degree p.

iii. The conjugacy classes were determined above. Now suppose that $g_1 \neq 1$ and that $g_1 = hg_2h^{-1}$. Then $g_1 g_2^{-1} = hg_2h^{-1}g_2^{-1} \in Z(P)$. This means that

$$g_1 = (g_1 g_2^{-1})g_2 = g_2(g_1 g_2^{-1}) \in g_2 Z(P).$$

This shows that whenever g_1 and g_2 are non-central, conjugate elements, they are in the same coset of $Z(P)$.
Conversely, if g_1 and g_2 are in the same coset...

iv. Let ρ be an irreducible representation of degree p and consider $\ker \rho$. Of course, $|\ker \rho| \neq p^3$ else $\rho = p\rho_U$ is not irreducible. If $|\ker \rho| = p$ or p^2, then $|G/\ker \rho| = p^2$ or p, so that $G/\ker \rho$ is abelian. Now

$$\widetilde{\rho} : G/\ker \rho \longrightarrow GL(V) \text{ defined } \widetilde{\rho}(g \ker \rho) = \rho(g)$$

is a representation of degree p on $G/\ker \rho$. Now,

$$1 = \langle \rho, \rho \rangle = \frac{1}{|G|} \sum_{g \in G} \rho(g)\overline{\rho(g)} = \frac{1}{|G|} \sum_{g \ker \rho \in G/\ker \rho} |\ker \rho| \rho(g)\overline{\rho(g)}$$

$$= \frac{1}{|G/\ker\rho|} \sum_{g\ker\rho \in G/\ker\rho} \widetilde{\rho}(g\ker\rho)\overline{\widetilde{\rho}(g\ker\rho)} = \langle \widetilde{\rho}, \widetilde{\rho} \rangle$$

so that $\widetilde{\rho}$ is irreducible. However, since $G/\ker\rho$ is abelian, it can have no irreducible representations of degree p, a contradiction. This shows that $|\ker\rho| = 1$ so that ρ is faithful.

v. Let $z \in Z(P)$ and let V be an irreducible $\mathbb{C}P$ module of degree p. Then the map $\varphi_z : V \longrightarrow V$ defined $\varphi_z(v) = z \cdot v$ is a $\mathbb{C}P$ homomorphism. Let λ be such that $\det(\varphi_z - \lambda I) = 0$. Then $\varphi_z - \lambda I$ is also a $\mathbb{C}P$ homomorphism and $\ker(\varphi - \lambda I) \neq 0$. But as V is irreducible, we have that $\ker(\varphi - \lambda I) = V$ so that $z \cdot v = \lambda v$ for all $v \in V$. Moreover, since $z^p = 1$, we have $v = z^p \cdot v = \lambda^p v$ so that λ is a pth root of unity. Also, it follows that $\chi_V(z) = p\lambda$. Note also that if $z \in Z(P)$ but $z \neq 1$, then z generates $Z(P)$ and we have $z^j \cdot v = \lambda^j v$ so that $\chi_V(z^j) = p\lambda^j$, and so X_V is determined for all the elements of $Z(P)$. Now if $g \in P \backslash Z(P)$, we have that the conjugacy class containing g is

$$\left\{g, gz, gz^2, \cdots, gz^{p-1}\right\}$$

by the observation above that the non-identity cosets of $Z(P)$ are conjugacy classes of P. In particular, this means that $\chi_V(gz^j) = \chi_V(g)$ for all j since χ_V is constant on conjugacy classes. However, since z^j acts as scalar multiplication by λ^j, we have that $\chi_V(gz^j) = \lambda^j \chi_V(g)$ for all j. Thus,

$$0 = \left(\sum_j \lambda^j\right) \chi_V(g) = \sum_j \chi_V(gz^j) = p\chi_V(g)$$

so that $\chi_V(g) = 0$. This determines the character for V

vi. Let ρ_1 and ρ_2 be irreducible representations of P of degree p. As indicated above, if $\rho_1(z) = \rho_2(z)$ for any $z \in Z(P)$ other than 1, then ρ_1 and ρ_2 agree on all of $Z(P)$, and since ρ_j vanishes outside of $Z(P)$, we have that ρ_1 and ρ_2 agree on all of P. Hence, if $\rho_1 \neq \rho_2$, then $\rho_1(z) \neq \rho_2(z)$ for all $z \in Z(P)\backslash\{1\}$. Hence, the $p-1$ representations of degree p can be produced by assigning one of the pth roots of unity other than 1 to some $z \in Z(P)$ other than 1.

The Group of order 42.

(a) The character table for G is the following.

	1	6	7	7	7	7	7
G	1	a	b	c	d	e	f
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ζ	$\overline{\zeta}$	ζ	$\overline{\zeta}$
χ_3	1	1	1	$\overline{\zeta}$	ζ	$\overline{\zeta}$	ζ
χ_4	1	1	-1	$-\zeta$	$-\overline{\zeta}$	ζ	$\overline{\zeta}$
χ_5	1	1	-1	$-\overline{\zeta}$	$-\zeta$	$\overline{\zeta}$	ζ
χ_6	1	-1	-1	-1	1	1	1
χ_7	6	-1	0	0	0	0	0

We see from the table that $\ker\rho_1 \cap \ker\rho_2 \cap \ker\rho_3 = \mathcal{O}_1 \cup \mathcal{O}_a \cup \mathcal{O}_b$. Since $\ker\rho_j$ is a normal subgroup of G and the intersection of three normal subgroups is normal, we have that $N := \mathcal{O}_1 \cup \mathcal{O}_a \cup \mathcal{O}_b$ is a normal subgroup of order 14.

Moreover, since $|G/N| = 3$, we have that $G/N \cong \mathbb{Z}_3$ and that the cosets of N are exactly $\mathcal{O}_c \cup \mathcal{O}_e$ and $\mathcal{O}_d \cup \mathcal{O}_f$ as only this combination causes χ_1, χ_2, and χ_3 to agree with the characters of $\mathbb{Z}_3$.

(?) N is not $\mathbb{Z}_{14}$, so we must have $N \cong D_{14}$.

(b) We briefly compute the conjugacy classes of $D_{14} = \langle r, s | r^7 = s^2 = 1, rs = sr^6\rangle$. Since r^j, $j = 1\ldots 6$, commutes only with r^k, $k = 0\ldots 6$, we have $|C_{D_{14}}(r^j)| = 7$, $j = 1\ldots 6$ so that $|\mathcal{O}_{r^j}| = 2$. We explicitly compute that $\mathcal{O}_r = \{r, r^6\}$, $\mathcal{O}_{r^2} = \{r^2, r^5\}$, and $\mathcal{O}_{r^3} = \{r^3, r^4\}$. By order considerations, we must have that the G orbit of a is the union of these three N orbits.

Similarly, sr^j, $j = 0\ldots 6$ commutes only with itself and with 1 so that $|C_{D_{14}}(sr^j)| = 2$ and $|\mathcal{O}_{sr^j}| = 7$. We must have then that $\mathcal{O}_s = \{sr^j\}_{j=0}^6$ and of course $\mathcal{O}_1 = \{1\}$. The distinct restrictions then are the following.

	1	2	2	2	7
N	1	r	r^2	r^3	s
Resχ_1	1	1	1	1	1
Resχ_4	1	1	1	1	-1
Resχ_7	6	-1	-1	-1	0

We compute $\langle \text{Res }\chi_1, \text{Res }\chi_1\rangle = \langle \text{Res }\chi_4, \text{Res }\chi_4\rangle = 1$ so that Res χ_1 and Res χ_4 are irreducible.

Unfortunately, Res χ_1 and Res χ_4 are not constituents of Res χ_7 as $\langle \text{Res }\chi_1, \text{Res }\chi_7\rangle = 0$ and $\langle \text{Res }\chi_4, \text{Res }\chi_7\rangle = 0$. However, the remaining three characters of N must each have degree 2 since $\sum \chi_j(1)^2 = 14$. Now since $\langle \text{Res }\chi_7, \text{Res }\chi_7\rangle = 3$ and this number is the sum of the squares of the multiplicities of the constituents, we must have that these multiplicities are all 1 so that Res χ_7 is the sum of the three remaining characters.

(c) $n_7 = 7j + 1$ and divides 6. This forces $n_7 = 1$. $n_3 = 3j + 1$ and divides 14. Then $n_3 = 1$ or 7. Write $G/N = \{N, g_1N, g_2N\}$ for some g_1, g_2. Now $(g_1N)^3 = g_1^3N = N$ so that $g_1 \in N$. Now since $|N| = 14$, we have that $(g_1^3)^{14} = (g_1^{14})^3 = 1$. Also, we have that $g_1^{14} \neq 1$ else $(g_1N)^{14} = g_1^{14}N = N = (g_1N)^3$ which forces $g_1N = N$ since 3 and 14 are relatively prime, a contradiction. Incidentally, I stole this argument from Herstein's proof of the Cauchy theorem, which I enjoyed immensely. This shows that $|g_1^{14}| = 3$. Similarly, $|g_2^{14}| = 3$ and $g_1^{14} \neq g_2^{14}$ since $g_1 \neq g_2$, being representatives of different cosets. Thus there are at least 2 elements of order 3 so that $n_3 \neq 1$. This forces $n_3 = 7$.

Finally, we have $n_2 = 2j + 1$ and divides 21. Now since N itself has 7 elements of order 2, namely sr^j for $j = 0 \ldots 6$, we have that G has at least 7 such elements, and these are the elements in $\mathcal{O}_b$ as adduced above. There can be no other elements of order 2 since χ_2 is ζ or $\overline{\zeta}$ on elements of $\mathcal{O}_c$, $\mathcal{O}_d$, $\mathcal{O}_e$, and $\mathcal{O}_f$ and this is not possible if any of these elements has order 2.

(d) As indicated above, a and b have orders 7 and 2 respectively. $\chi_2(c) = \chi_3(d) = \zeta$ so that 3 divides the orders of these elements. However, $\chi_6(c) = \chi_6(d) = -1$, a second root of unity, so 2 divides the orders of c and d also. Obviously, these elements don't have order 42, so we are left with 6 as the only possible order for c and d.

By elimination, e and f must have order 3 to account for the 14 elements of order 3 in the 7 Sylow 3-subgroups.

Next, let P be a Sylow 7 subgroup and Q_j, $j = 1 \ldots 7$ the Sylow 3 subgroups. Then since $P \trianglelefteq G$, we have that $PQ_j \leq G$ with $|PQ_j| = 21$. PQ_j has index 2 and so is normal in G. We have that $Q_j \leq PQ_j$, but we also have that since the Q_j are conjugate,

$$Q_k = gQ_jg^{-1} \leq gPQ_jg^{-1} = PQ_j$$

so that all the Q_k lie in PQ_j. Also, the 14 elements of order 3 in the 7 Sylow 3-subgroups together with the 6 elements of order 7 from the 7-subgroup and the identity constitute all the elements of PQ_j.

(a) Define $\varphi(aa_\lambda b_\lambda) = aa_\lambda b_\lambda a_\lambda$. Then $\varphi : Aa_\lambda b_\lambda \longrightarrow Ab_\lambda a_\lambda$ as $aa_\lambda \in A$. Moreover, φ is a A-module homomorphism as

$$\begin{aligned}
\varphi(a_1a_\lambda b_\lambda + a_2a_\lambda b_\lambda) &= \varphi((a_1 + a_2)a_\lambda b_\lambda) \\
&= (a_1 + a_2)a_\lambda b_\lambda a_\lambda \\
&= a_1a_\lambda b_\lambda a_\lambda + a_2a_\lambda b_\lambda a_\lambda \\
&= \varphi(a_1a_\lambda b_\lambda) + \varphi(a_2a_\lambda b_\lambda)
\end{aligned}$$

and

$$\varphi(a_1a_2a_\lambda b_\lambda) = a_1a_2a_\lambda b_\lambda a_\lambda = a_1\varphi(a_2a_\lambda b_\lambda).$$

Similarly, the map $\psi : Ab_\lambda a_\lambda \longrightarrow Aa_\lambda b_\lambda$ defined $\psi(ab_\lambda a_\lambda) = ab_\lambda a_\lambda b_\lambda$ is an A-module homomorphism. Moreover, φ and ψ are almost two sided inverses as

$$\psi \circ \varphi(aa_\lambda b_\lambda) = aa_\lambda b_\lambda a_\lambda b_\lambda = ac_\lambda^2 = adc_\lambda = daa_\lambda b_\lambda$$

and

$$\varphi \circ \psi(ab_\lambda a_\lambda) = ab_\lambda a_\lambda b_\lambda a_\lambda = ac_\lambda^2 = adc_\lambda = dab_\lambda a_\lambda$$

where $d \in \mathbb{C}$ is such that $c_\lambda^2 = dc_\lambda$. Incidentally, the same d works for $b_\lambda a_\lambda$ as

$$a_\lambda b_\lambda a_\lambda b_\lambda a_\lambda = (a_\lambda b_\lambda a_\lambda b_\lambda)a_\lambda = (da_\lambda b_\lambda)a_\lambda = a_\lambda(db_\lambda a_\lambda)$$

so that by by right cancelation, $b_\lambda a_\lambda b_\lambda a_\lambda = db_\lambda a_\lambda$.

We then have that φ is surjective since for any $a \in A$ we have $\varphi(d^{-1}\psi(a)) = a$ and φ is injective since whenever $\varphi(a_1) = \varphi(a_2)$ we have $\psi(\varphi(a_1)) = \psi(\varphi(a_2))$ so that $a_1 = a_2$. Thus, φ is an isomorphism and $Aa_\lambda b_\lambda \cong Ab_\lambda a_\lambda$

(b) The map $\varphi : Aa_\lambda \longrightarrow Aa_\lambda b_\lambda$ given by $\varphi(aa_\lambda) = aa_\lambda b_\lambda$ is clearly surjective. Hence, A_λ is the image of Aa_λ under φ.

(c) We have

$$\begin{aligned}
Ab_{\lambda'} &\cong \operatorname{Ind}_{Q_{\lambda'}}^G \widetilde{U}' \\
&= \operatorname{Ind}_{P_\lambda}^G \widetilde{U}' \\
&= \operatorname{Ind}_{P_\lambda}^G \left(\widetilde{U}' \otimes U\right) \\
&= \operatorname{Ind}_{P_\lambda}^G \left(\operatorname{Res}_{P_\lambda}^G U' \otimes U\right) \\
&= U' \otimes \operatorname{Ind}_{P_\lambda}^G U \\
&\cong U' \otimes Aa_\lambda.
\end{aligned}$$

where $\widetilde{U}'$ is the alternating module for $Q_{\lambda'} = P_\lambda$, U' is the alternating module for G, and U is the trivial module for P_λ. By a parallel argument, we have that $Aa_{\lambda'} \cong U' \otimes Ab_\lambda$.

Now by part 2, we have that the image of $Aa_{\lambda'}$ under right multiplication by $b_{\lambda'}$ is $V_{\lambda'}$. We want to apply this same map to $Ab_\lambda \otimes U'$, but first we note that

$$\begin{aligned}
(u \otimes ab_\lambda)b_{\lambda'} &= (u \otimes ab_\lambda)\sum_j (\operatorname{sgn} p_j)p_j \\
&= \sum_j (u \otimes ab_\lambda)(\operatorname{sgn} p_j)p_j \\
&= \sum_j u(\operatorname{sgn} p_j)p_j \otimes ab_\lambda(\operatorname{sgn} p_j)p_j \\
&= \sum_j up_j \otimes ab_\lambda p_j \\
&= \sum_j (u \otimes ab_\lambda)p_j \\
&= (u \otimes ab_\lambda)\sum_j p_j \\
&= (u \otimes ab_\lambda)a_\lambda
\end{aligned}$$

so that in $U' \otimes Ab_\lambda$, right mulfftiplication by $b_{\lambda'}$ is the same as right multiplication by a_λ. Again, by part 2, we have that the image of $U' \otimes Ab_\lambda$ under right multiplication by a_λ is $U' \otimes V_\lambda$. Now since $U' \otimes Ab_\lambda$ and $Aa_{\lambda'}$ are isomorphic, we have that their images under right multiplication by $b_{\lambda'}$ are isomorphic. Hence, we have that $V\lambda' \cong V_\lambda \otimes U'$.

Homework 8

15. *Prove that $\mathbb{Q}(\varphi)$ is a finite extension*

Let $\mathcal{B}$ be a fixed basis of some vector space V of dimension m over F. Now $\mathbb{Q}(\varphi) = \mathbb{Q}(a_{1,1}^{g_1}, a_{1,2}^{g_1} \ldots)$ where the $a_{i,j}^{g_k}$ are the i,j entries of the matrix $\varphi(g_k)$ with respect to $\mathcal{B}$. Then $(\varphi)/\mathbb{Q}$ is a finite extension of since there are no more than $m^2|G|$ different $a_{i,j}^{g_k}$, each of them algebraic over $\mathbb{Q}$, being in F.

16. *Prove that φ^σ is a representation and that the character of φ^σ is $\psi^\sigma := \sigma(\psi(g))$ where ψ is the character of φ.*

Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be the matricies $\varphi(g)$ and $\varphi(h)$ with respect to some fixed basis. Then $\varphi^\sigma(g) = (\sigma(a_{i,j}))$ and $\varphi^\sigma(h) = (\sigma(b_{i,j}))$. Now

$$\begin{aligned}
i,j \text{ entry of } \varphi^\sigma(g)\varphi^\sigma(h) &= \sum_k \sigma(a_{i,k})\sigma(b_{k,j}) \\
&= \sigma\left(\sum_k a_{i,k}b_{k,j}\right) \\
&= \sigma\Big(i,j \text{ entry of } \varphi(g)\varphi(h)\Big) \\
&= \sigma\Big(i,j \text{ entry of } \varphi(gh)\Big) \\
&= i,j \text{ entry of } \varphi^\sigma(gh).
\end{aligned}$$

This shows that φ^σ is a homomorphism. Next, we compute the character of φ^σ.

$$\begin{aligned}
\operatorname{tr}(\varphi^\sigma(g)) &= \sum_j \sigma(a_{j,j}) \\
&= \sigma\left(\sum_k (a_{j,j})\right) \\
&= \sigma\left(\operatorname{tr}(\varphi(g))\right) \\
&= \sigma\left(\psi(g)\right)
\end{aligned}$$

17. *Show that φ is irreducible if and only if φ^σ is irreducible.*

We note that

$$\begin{aligned}
\overline{\sigma(\varphi(g))} &= \overline{\sigma(a+bi)} \\
&= \overline{\sigma(a)+\sigma(b)i} \\
&= \sigma(a)-\sigma(b)i \\
&= \sigma(a-bi) \\
&= \sigma(\overline{a+bi}) \\
&= \sigma(\overline{\varphi(g)})
\end{aligned}$$

where $\overline{\cdot}$ denotes complex conjugation. Then

$$\begin{aligned}
\langle \varphi^\sigma, \varphi^\sigma \rangle &= \frac{1}{|G|}\sum_g \varphi^\sigma(g)\overline{\varphi^\sigma(g)} \\
&= \frac{1}{|G|}\sum_g \sigma(\varphi(g))\overline{\sigma(\varphi(g))} \\
&= \frac{1}{|G|}\sum_g \sigma(\varphi(g))\sigma(\overline{\varphi(g)}) \\
&= \sigma\left(\frac{1}{|G|}\sum_g \varphi(g)\overline{\varphi(g)}\right) \\
&= \sigma\left(\langle \varphi, \varphi \rangle\right)
\end{aligned}$$

and we have that $\langle \varphi^\sigma, \varphi^\sigma \rangle = 1$ if and only if $\langle \varphi, \varphi \rangle = 1$ since automorphisms map the element 1 to itself.

18. *Prove that $\mathbb{Q}(\psi) \subset \mathbb{Q}(\epsilon)$ where $\mathbb{Q}(\psi)$ is the extension of $\mathbb{Q}$ generated by $\psi(g)$, $g \in G$, and ϵ is an nth root of unity where $n = |G|$. Deduce that $\mathbb{Q}(\psi)$ is a Galois extension of $\mathbb{Q}$ with abelian Galois group.*

$\psi(g)$ is a sum of eigenvalues of $\varphi(g)$, which are all nth root of unity. This means that

$$\psi(g) \in \mathbb{Q}(\epsilon)$$

for all g so that

$$\mathbb{Q}(\psi) = \mathbb{Q}(\psi(g_1), \psi(g_2), \ldots \psi(g_n)) \subseteq \mathbb{Q}(\epsilon).$$

Since the Galois group for $\mathbb{Q}(\epsilon)/\mathbb{Q}$ is $(\mathbb{Z}/n\mathbb{Z})^\times$, we see by the Galois correspondence that $\mathbb{Q}(\varphi)$, being a subfield of $\mathbb{Q}(\epsilon)$ containing $\mathbb{Q}$, corresponds with some subgroup H of $(\mathbb{Z}/n\mathbb{Z})^\times$. Now H is normal since $(\mathbb{Z}/n\mathbb{Z})^\times$ is abelian so that $\mathbb{Q}(\psi)/\mathbb{Q}$ is Galois.

19. *Let $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$ be defined $\sigma_a(\epsilon) = \epsilon^a$ and show that $\psi^{\sigma_a}(g) = \psi(g^a)$ for all $g \in G$.*

Let φ be a representation of G on some vector space V of dimension m. Let φ have character ψ and let $\mathcal{B}$ be a fixed basis of V with respect to which $\varphi(g) = (b_{i,j})$ with $b_{i,j} = 0$ for $i \neq j$ and $b_{j,j}$ a kth root of unity where $k = |g|$. Write $b_{j,j} = \epsilon^c$. Then

$$\sigma_a(b_{j,j}) = \sigma_a(\epsilon^c) = (\sigma_a(\epsilon))^c = (\epsilon^a)^c = \epsilon^{ca} = (b_{j,j})^a.$$

Also note that with respect to $\mathcal{B}$, we have that $\varphi(g^a) = (b_{i,j}^a)$ since φ is diagonal. Then

$$\psi^{\sigma_a}(g) = \sum_{j=1}^{m} \sigma_a(b_{j,j}) = \sum_{j=1}^{m} (b_{j,j}^a) = \psi(g^a)$$

20. *If $g \in G$ is conjugate with g^a for all a with $(a, n) = 1$, then $\psi(g) \in \mathbb{Q}$ for all characters ψ of G.*

We have

$$\psi(g) = \psi(g^a) = \psi^{\sigma_a}(g) = \sum_{j=1}^{m} \sigma_a(b_{j,j}) = \sigma_a\left(\sum_{j=1}^{m} b_{j,j}\right) = \sigma_a(\psi(g))$$

for all characters ψ and all $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q}) = \left\{\sigma_a \middle| (a, n) = 1\right\}$. By definition of the Galois group, since $\psi(g) \in \mathbb{Q}(\epsilon)$ is fixed by all $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$, we have that $\psi(g) \in \mathbb{Q}$ for all ψ.

21. *For $g \in G$ fixed, g is conjugate with g^a for all $(a, |G|) = 1$ iff g is conjugate with g^a for all $(a, |g|) = 1$.*

Suppose g is conjugate with g^a for all $(a, |g|) = 1$ and let a be such that $(a, |G|) = 1$. Then we certainly have that $(a, |g|) = 1$ so that g is conjugate with g^a by assumption.

Conversely, suppose g is conjugate with g^a for all $(a, |G|) = 1$ and let a be such that $(a, m) = 1$ where $m = |g|$. Let l be such that $n = lm$ and let k be such that $l = dk$ where $d = (l, m)$. We then have that $n = dkm$ and $(m, k) = 1$. Write $1 = rm + sk$ for some $r, s \in \mathbb{Z}$ and define $x = ask + rm$. Then

$$x \equiv_m ask + rm \equiv_m ask \equiv_m a$$

since $sk \equiv_m rm + sk \equiv_m 1$.

This equivalence implies that m divides $x - a$ so that $my = x - a$ for some $y \in \mathbb{Z}$. Then

$$g^x g^{-a} = g^{x-a} = g^{my} = 1$$

so that $g^a = g^x$. We similarly have that

$$x \equiv_k ask + rm \equiv_k rm \equiv_k 1$$

since $rm \equiv_k rm + sk \equiv_k 1$.

This equivalence imples that $k|x-1$. Now if some prime p divides k, then p divides $x-1$ so that p cannot divide x. This shows that $(k,x)=1$.

Finally, we want to show that $(x,n)=1$. Indeed, if some prime p were a common factor of both x and $n=dkm$, then by the preceeding paragraph, p would not be a factor of k. Now since $x \equiv_m a$, we have that m divides $x-a$. If p were a factor of m, then since p is a factor of x, we would have that p would be a factor of a, a contradiction since $(a,m)=1$. Finally, p cannot be a factor of $d=(l,m)$ as then it would have to be a factor of m. Hence, n and x can have no common factors, that is, $(n,x)=1$. By assumption, we then have that g is congruent with $g^x=g^a$.

22. *Every character of S_n is rational valued.*

Let $g \in S_n$ and let $g=\tau_1\tau_2\cdots\tau_m$ where τ_j are disjoint n_j-cycles. Then $|g|=lcm\ n_j$ and if a is such that $(a, lcm\ n_j)=1$, then we must have that $(a,n_j)=1$ for all j, for if $(a,n_{j_0})\neq 1$ for some j_0, then we have

$$(a,n_{j_0}) \,\Big|\, n_{j_0} \,\Big|\, lcm\ n_j,$$

but we have also that $(a,n_{j_0}) \,\Big|\, a$ so that (a,n_{j_0}) is a common divisor of both $lcm\ n_j$ and a. This means that $(a, lcm\ n_j)>1$, a contradiction. We must have that $(a,n_j)=1$ for all j. Then the ath power of each n_j-cycle is also an n_j-cycle so that g^a has the same cycle type as g. Thus, g^a is conjugate with g. Thus, we've shown that g is conjugate with g^a for all $(a,|g|)=1$ and for all $g\in G$. Thus, by 21, we have that g^a is conjugate with g for all $(a,|G|)=1$. Thus, by 20, we have that $\psi(g)\in\mathbb{Q}$ for all $g\in G$ and all characters ψ of G.

Homework 9

(a) *Show that $|G|=q(q-1)(q^2-1))$.*

We count the number of bases $\left\{\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}\right\}$ of $\mathbb{F}_q^2$ since these are in bijection with the elements of $GL(2,\mathbb{F}_q)$.

Now there are q choices for a and q choices for b except that $\begin{pmatrix}a\\b\end{pmatrix}$ should not be $\begin{pmatrix}0\\0\end{pmatrix}$ so that we have q^2-1 ways to select $\begin{pmatrix}a\\b\end{pmatrix}$. $\begin{pmatrix}c\\d\end{pmatrix}$ can be anything not in the span of $\begin{pmatrix}a\\b\end{pmatrix}$. The span of $\begin{pmatrix}a\\b\end{pmatrix}$ contains q different vectors, including the forbidden $\begin{pmatrix}0\\0\end{pmatrix}$ so we have q^2-q choices for $\begin{pmatrix}c\\d\end{pmatrix}$. This gives a grand total of $(q^2-1)(q^2-q)$ different bases for $\mathbb{F}_q^2$.

(b) *Let $B=\begin{pmatrix}\bigstar & \bigstar\\ 0 & \bigstar\end{pmatrix}\leq G$, $T=\begin{pmatrix}\bigstar & 0\\ 0 & \bigstar\end{pmatrix}\leq G$, and $U=\begin{pmatrix}1 & \bigstar\\ 0 & 1\end{pmatrix}\leq G$. Show that B is the semidirect product $B=T\ltimes U$.*

We have that $B = TU$ since

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & ba^{-1} \\ 0 & 1 \end{pmatrix} \in TU.$$

Next, we see that $T \leq N_B(U)$ since

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} 1 & abc \\ 0 & 0 \end{pmatrix} \in U.$$

Of course, we have that $U \leq N_B(U)$ so that $B = TU \leq N_B(U)$ so that $U \trianglelefteq B$. We also have that $T \cap U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so that $B = TU = T \ltimes U$.

Now $B/U = TU/U \cong T/(T \cap U) \cong T/\{1\} \cong T$ by the isomorphism theorems. T is abelian so that T has $|T| = (q-1)^2$ characters, all of which are of degree 1. These all lift to distinct characters of of B so that B has at least $(q-1)^2$ characters of degree one.

To show that B has exactly $(q-1)^2$ characters of degree one, we will show that $U = B'$, the commutator subgroup of B, so that we will have exactly $[B : B'] = [B : U] = (q-1)^2$ characters of degree one.

Indeed, we have $B' \leq U$ since $B/U \cong T$ is abelian. Imagine now that B' is strictly contained in U so that B/B' is isomorphic with a subgroup $\widetilde{T}$ of B containing T which has order strictly larger than $|T|$. Let

$$a := \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in \widetilde{T} \backslash T.$$

Then x and z are non-zero else a is not invertible, and y is non-zero else $a \in T$. Then

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} zy^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} x^{-1}yz^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix} \in \left\langle \begin{pmatrix} \bigstar & 0 \\ 0 & \bigstar \end{pmatrix}, \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right\rangle,$$

the subgroup of B generated by a and the elements of T, but this means that

$$U = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \leq \left\langle \begin{pmatrix} \bigstar & 0 \\ 0 & \bigstar \end{pmatrix}, \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right\rangle,$$

so that

$$B = TU \leq \left\langle \begin{pmatrix} \bigstar & 0 \\ 0 & \bigstar \end{pmatrix}, \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right\rangle \leq \widetilde{T}$$

so that $\widetilde{T} = B$. But this is impossible since B is non-abelian. Thus, we must have that $T = \widetilde{T}$ and $U = B'$.

Now to compute the characters of T, we note that $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times \cong \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$. The characters of $\mathbb{Z}_{q-1}$ correspond to the assignment of a $(q-1)$st root of unity to a generator of $\mathbb{Z}_{q-1}$, and the characters of $\mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$ correspond with the product of two characters of $\mathbb{Z}_{q-1}$. Then we can index the characters of T as follows

$$\psi_{i,j} = \begin{cases} (a,1) & \mapsto & \zeta^i \\ (1,b) & \mapsto & \zeta^j \end{cases}$$

where 0 is the identity element of $\mathbb{Z}_{q-1}$, a and b are generators of $\mathbb{Z}_{q-1}$, and ζ is a primative $(q-1)$st root of unity. Or more compactly, taking $a = b = 1$ we have

$$\psi_{i,j}(x,y) = \zeta^{xi+yj}$$

(c) *Show that $B\backslash G/B$ has representatives B and BwB where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. If $c = 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B$. If $c \neq 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\in B}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & -c^{-1}d \\ 0 & 1 \end{pmatrix}}_{\in B} \in BwB.$$

(d) *Let ψ be a character of B of degree 1 constructed as in (2). Analyze $\langle \mathrm{Ind}_B^G(\psi), \mathrm{Ind}_B^G(\psi)\rangle$ by Mackey's Theorem and describe when $\mathrm{Ind}_B^G(\psi)$ is irreducible. How many irreducible characters of G do you get in this way?*

We have that $w^{-1}Bw = wBw$ is the subgroup of lower-triangular matricies since the w on the left swaps the rows and the w on the right swaps the columns. Thus, $x^{-1}Bw \cap B$ is exactly T, the subgroup of diagonal matricies. Also, we have $\psi^w(b) = \psi^w(w^{-1}wbw^{-1}w) = \psi(wbw^{-1})$. Then by Mackey,

$$\begin{aligned} \left\langle \mathrm{Ind}_B^G(\psi), \mathrm{Ind}_B^G(\psi)\right\rangle &= \sum_{x\in 1,w} \left\langle \psi^x\Big|_{x^{-1}Bx\cap B}, \psi\Big|_{x^{-1}Bx\cap B}\right\rangle_{x^{-1}Bx\cap B} \\ &= \langle \psi,\psi\rangle_B + \left\langle \psi^w\Big|_T, \psi\Big|_T\right\rangle_T \end{aligned}$$

Now $\left\langle \psi^w\big|_T, \psi\big|_T\right\rangle_T = \langle \psi^w,\psi\rangle_B$ since ψ is lifted from some character of T. We see then that $\langle\psi,\psi\rangle_B + \langle\psi^w,\psi\rangle_B = 1$ when $\langle\psi^w,\psi\rangle_B = 0$, that is, when ψ and ψ^w are different characters of T. Using the notation of (2), we have that $\psi_{i,j} = \psi_{i,j}^w = \psi_{j,i}$ when $i = j$. Hence, we can produce $(q-1)^2 - (q-1)$ different characters of G in this manner.