Model Theory and Differential Algebraic Geometry

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Here be Dragons

Goals:

- Give a quick survey about strongly minimal sets and their geometry from a model theoretic perspective
- Describe what is know about strongly minimal sets in DCF–the theory of differentially closed fields with one derivation and characteristic zero.
- This perspective gives us a strong dividing line between parts of differential algebraic geometry that behave like algebraic geometry and parts that do not.

Strongly Minimal Sets

General Setting: Fix a language $\mathcal{L}$ and $\mathcal{L}$-theory T and work in $\mathbb{M}$ a universal domain for T. For example, $\mathcal{L} = \{+, \cdot, \delta, 0, 1\}$, $T =$DCF, $\mathbb{K}$ a universal differentially closed field

Definition

$X \subseteq \mathbb{M}^n$ is *definable* if there is an $\mathcal{L}$-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $\mathbf{b} \in \mathbb{M}^m$ such that $X = \{\mathbf{a} \in \mathbb{M}^m : \phi(\mathbf{a}, \mathbf{b})\}$

Example: $X = \{a \in \mathbb{M} : \forall w \exists v \; w^2 + vw + bv = a\}$.

In DCF, definable = Kolchin-constructible.

Definition

A definable set $X \subseteq \mathbb{M}^n$ is *strongly minimal* if X is infinite and for every definable $Y \subset X$ either Y or $X \setminus Y$ is finite.

Examples of Strongly Minimal Sets

Definition

A definable set $X \subseteq \mathbb{M}^n$ is *strongly minimal* if X is infinite and for every definable $Y \subset X$ either Y or $X \setminus Y$ is finite.

- ACF: K an algebraically closed field and $X \subseteq K^n$ an irreducible algebraic curve ($\pm$ finitely many points).
- DCF: $\mathbb{K}$ differentially closed, C the field of constants.
- Equality: $\mathbb{M}$ an infinite set with no structure and $X = \mathbb{M}$.
- Successor: $\mathbb{M}$ an infinite set $f : \mathbb{M} \to \mathbb{M}$ a bijection with no finite orbits, $X = \mathbb{M}$.
- DAG: $\mathbb{M}$ a torsion free divisible abelian group, $X \subseteq \mathbb{M}^n$ a translate of a one-dimensional subspace defined over $\mathbb{Q}$.

Model Theoretic Algebraic Closure

Definition

If $a \in \mathbb{M}$, $B \subset \mathbb{M}$, a is *algebraic* over B if there is an $\mathcal{L}$-formula $\phi(x, y_1, \ldots, y_m)$ and $\mathbf{b} \in B^m$ such that $\phi(a, \mathbf{b})$ and $\{x \in \mathbb{M} : \phi(x, \mathbf{b})\}$ is finite.
Let $\mathrm{cl}(B) = \{a : a \text{ algebraic over } B\}$.

- ACF: $\mathrm{cl}(A)=$ algebraic closure of field generated by A.
- DCF: $\mathrm{cl}(A)=$algebraic closure of differential field generated by A.
- equality: $\mathrm{cl}(A) = A$.
- Successor: $\mathrm{cl}(A) = \bigcup_{a \in A}$ orbit of a
- DAG: $\mathrm{cl}(A) = \mathrm{span}_{\mathbb{Q}}(A)$.

Combinatorial Geometry of Strongly Minimal Sets

Definition

A strongly minimal set X is *trivial* if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{a\})$ for all $A \subseteq X$.

Equality and Successor are trivial

Definition

A strongly minimal set X is *modular* if $c \in \mathrm{cl}(B \cup \{a\})$, then $c \in \mathrm{cl}(b, a)$ for some $b \in B$, for all $a \in X$, $B \subseteq X$.

DAG is non-trivial modular: If $c = \sum m_i b_i + na$ where $m_i, n \in \mathbb{Q}$, then $c = b + na$ where $b = \sum m_i b_i$.

ACF is non-modular

▸ Skip Families of Curves

Families of Curves

Let X be strongly minimal. Suppose $L \subset X \times X \times \mathbb{M}^k$ is definable. For $\mathbf{a} \in \mathbb{M}^k$ let $l_{\mathbf{a}} = \{(x, y) \in X^2 : (x, y, \mathbf{a}) \in L\}$ and assume each $l_{\mathbf{a}}$ is strongly minimal.

Suppose K is ACF, let $L = \{(x, y, a, b) : y = ax + b\}$, the family of non-vertical lines is a two-dimensional family of strongly minimal sets.

If G is a DAG $L = \{(x, y, a) : y = mx + a\}$, $m \in \mathbb{Q}$ is a one dimensional family.

Theorem (Zilber)

A strongly minimal set X is non-modular iff there is a family of curves of dimension at least two.

When are two strongly minimal sets "the same"?

Definition

Two strongly minimal sets X and Y are *non-orthogonal* ($X \not\perp Y$) if there is a definable $R \subseteq X \times Y$ such that $\{y \in Y : (x,y) \in R\}$ is non-empty finite for all but finitely many $x \in X$.

Idea: "non-orthogonal"= intimately related, "orthogonal"=not related.

In ACF: If X is a curve there is $\rho : X \to K$ rational so $X \not\perp K$.

In DCF: If X and Y are strongly minimal sets defined over a differentially closed field K, then $X \perp Y$ if and only if for $\mathbf{a} \in X(\mathbb{K}) \setminus X(K)$, $Y(K\langle \mathbf{a} \rangle^{\mathrm{dcl}}) = Y(K)$–i.e., adding points to X does not force us to add points to Y.

Zilber's Principle

Zilber's Principle: Complexity of the combinatorial geometry is an avatar of algebraic structure.

trivial strongly minimal sets have no infinite definable groups

Theorem (Hrushovski)

If X is a nontrivial modular strongly minimal set, there is an interpretable modular strongly minimal group G such that $X \not\perp G$.

Theorem (Hrushovski-Pillay)

If G is a group interpretable in a modular strongly minimal set, then any definable subset of G^n is a finite Boolean combination of cosets of definable subgroups.

Zilber Conjectured that non-modular strongly minimal sets only occur in the presence of an algebraically closed field, but Hrushovski refuted this in general.

Strongly Minimal Sets in DCF–Early Results

- The field of constants C is non-locally modular
- There are many trivial strongly minimal sets

Theorem (Rosenlicht/Kolchin/Shelah)

The differential equation $y' = y^3 - y^2$. Defines a trivial strongly minimal set. If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are distinct solutions with $a_i, b_i \neq 0, 1$, then there is an automorphism σ of $\mathbb{K}$ with $\sigma(a_i) = b_i$.

Zilber's Principle for DCF

Theorem (Hrushovski-Sokolovic)

If $X \subseteq \mathbb{K}^n$ is strongly minimal and non-locally modular, then $X \not\perp C$.

The original proof used the high powered model theoretic machinery of *Zariski Geometries* developed by Hrushovski and Zilber.

This was later given a more elementary conceptual proof by Pillay and Ziegler.

The Modular Classification

Nontrivial Modular Strongly Minimal Sets in DCF

Where do we look for nontrivial modular strongly minimal sets?

- By Hrushovski's result we should look for a modular strongly minimal group G.
- By a result of Pillay's we may assume that $G \subseteq H$ where H is an algebraic group.
- By strong minimality we may assume that H is commutative and has no proper algebraic subgroups.
- If H is $\mathbb{G}_a$, G must be a finite dimensional C-vector space so $G \not\perp C$.
- If H is $\mathbb{G}_m$ or a simple abelian variety defined over C, either
  - $G \subseteq H(C)$ and $G \not\perp C$, or
  - $G \cap H(C)$ is finite. In this case let $l : H \rightarrow \mathbb{K}^d$ be the logarithmic derivative. Then $G \not\perp l(G)$ is a a finite dimensional C vector space and $G \not\perp C$.

Nontrivial Strongly Minimal Sets in DCF

Theorem (Hrushovski-Sokolovic)

- *If A is a simple abelian variety that is not isomorphic to one defined over C and $A^\sharp$ is the Kolchin-closure of the torsion points, then $A^\sharp$ is a modular strongly minimal set.*
- *If X is a modular strongly minimal set then there is A as above such that $X \not\perp A^\sharp$.*
- *$A_0^\sharp \not\perp A_1^\sharp$ if and only if A_0 and A_1 are isogenous.*

The key tool is the Buium-Manin homomorphism, a differential algebraic $\mu : A \to \mathbb{K}^n$ such that $\ker(\mu) = A^\sharp$ and the result that $A^\sharp$ is Zariski dense and has no proper proper infinite differential algebraic subgroups.

▸ The Trivial

Diophantine Applications

The strongly minimal sets $A^\sharp$ play a fundamental role in Buium's and Hrushovski's proofs of the Mordell-Lang Conjecture for function fields in characteristic 0.

Corollary

If A is a simple abelian variety not isomorphic to a variety defined over C with $\dim(A) \geq 2$ and $X \subset A$ is a curve, then X contains only finitely many torsion points.

Proof Since $X \cap A^\sharp$ is infinite and $A^\sharp$ is strongly minimal, $X \cap A^\sharp$ is cofinite in $A^\sharp$ and hence Zariski dense in A, a contradiction.

Trivial Pursuits

So far there is no good theory of the trivial strongly minimal sets.

Look for examples:

- Rosenlicht, Kolchin, Shelah style examples: $y' = f(y)$, f a rational function over C. We can determine triviality by studying the partial fraction decomposition of $1/f$. Generically trivial.
- Hrushovski-Itai: For X a curve of genus at least 2 defined over C there is a trivial $Y \subset X$ such that $\mathbb{K}(X) = \mathbb{K}\langle Y \rangle$.
- Nagloo-Pillay: Generic Painlevé equations.
  For example, if $\alpha \in C$ is transcendental over $\mathbb{Q}$, then $P_{II}(\alpha)$ is strongly minimal and trivial where $P_{II}(\alpha)$ is $y'' = 2y^3 + ty + \alpha$.

If we have any sufficiently rich family of definable sets is a generic set trivial strongly minimal?
For example: sets $f(y, y', y'') = 0$ where f is a generic degree d polynomial over C?

Trivial Pursuits

Is there any structure theory for trivial strongly minimal sets?

Conjecture If X is a trivial strongly minimal set and $A \subset X$ is finite, then $\mathrm{cl}(A)$ is finite.

Hrushovski has proved this when X has transcendence degree 1.
One tool of his proof is of independent interest.

Theorem (Hrushovski)

Suppose V is an irreducible Kolchin closed set of transcendence degree 2 defined over C such that there are infinitely many irreducible Kolchin closed $X \subset V$ of transcendence degree 1 defined over C. Then there is a nontrivial differential rational $f : V \to C$, in which case $\{f^{-1}(c) : c \in C\}$ is a family of Kolchin closed subsets.